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On Connectedness in Nested Row-Column Designs

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Some theorems related to row-treatment and column-treatment connectedness of a commutative nested row-column design are provided with one example.

Keywords: Nested row-column design; treatment-connected; row-treatment connected; column-treatment connected.

1. Introduction

Preece (1967) introduced nested balanced incomplete block designs in which within each block of an incomplete block design another set of incomplete blocks is nested. Caliński and Kageyama (1996, 2000) studied combinatorial and statistical properties of block designs. Singh and Dey (1979) considered block designs with nested rows and columns such that within each block the row × column classification is orthogonal, and presented their analysis. These designs are useful when it is desired to eliminate heterogeneity in two directions within each block or set and are viewed as a generalization of lattice square designs. The above-mentioned authors have also discussed efficiency-balanced and variance-balanced nested row-column designs and have constructed some designs of this kind.

In this paper, the concept of commutative nested row-column designs is introduced. This concept is an extension of the concepts developed by Pal and Katyal (1988) and Katyal and Pal (1991). Two theorems on connectedness for such designs are proved.

2. Nested row-column designs

Consider a design with v treatments and s blocks, each block contains $pq(\leq v)$ treatments and is classified into p rows and q columns. Such designs are called block designs with nested rows and columns [or nested row-column (NRC) designs].

Hereinafter, we follow the notations used in Singh and Dey (1979). For ready reference, some notations are presented below. With respect to an NRC design let $\boldsymbol{N}$ be the $v \times s$ incidence matrix of the block vs. treatment classification, $\boldsymbol{N}_{1j}$ be the $v \times p$ incidence matrix of treatments vs. rows in the jth block and $\boldsymbol{N}_{2j}$ be the $v \times q$ incidence matrix of treatments vs. columns in the jth block ($j = 1, ..., s$). Then $\boldsymbol{N}_1(v \times ps)$ and $\boldsymbol{N}_2(v \times qs)$ are defined as

$$\boldsymbol{N}_1 = (\boldsymbol{N}_{11} : \cdots : \boldsymbol{N}_{1s}), \qquad \boldsymbol{N}_2 = (\boldsymbol{N}_{21} : \cdots : \boldsymbol{N}_{2s}).$$

Also, let $\boldsymbol{r} = (r_1, ..., r_v)'$ be the $v \times 1$ vector of replications of treatments and let $\boldsymbol{r}^{\delta} = \mathrm{diag}\{r_1, ..., r_v\}$ and then $\boldsymbol{r}^{-\delta} = \mathrm{diag}\{1/r_1, ..., 1/r_v\}$. Finally, let $\boldsymbol{T} = (T_1, ..., T_v)'$ of size $v \times 1$, $\boldsymbol{S} = (S_1, ..., S_s)'$ of size $s \times 1$, $\boldsymbol{R} = (\boldsymbol{R}'_1, ..., \boldsymbol{R}'_s)'$ of size $ps \times 1$, $\boldsymbol{C} = (\boldsymbol{C}'_1, ..., \boldsymbol{C}'_s)'$ of size $qs \times 1$, $\boldsymbol{R}_j = (R_{j1}, ..., R_{jp})'$ of size $p \times 1$, $\boldsymbol{C}_j = (C_{j1}, ..., C_{jq})'$ of size $q \times 1$, where

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T_i = the total yield for the ith treatment, $i = 1, ..., v$,
S_j = the total yield for the jth block, $j = 1, ..., s$,
R_{jm} = the total yield for the mth row in the jth block, $m = 1, ..., p$,
$C_{j\ell}$ = the total yield for the ℓth column in the jth block, $\ell = 1, ..., q$.

In what follows, we consider only those NRC designs in which treatment vs. block classifications in the two-way design (ignoring other classifications) are orthogonal.

For such NRC designs, we have

$$\boldsymbol{N} = \boldsymbol{r}\boldsymbol{1}'/s.$$

Then $\boldsymbol{NN}' = \boldsymbol{rr}'/s$. Putting these values in the well-known matrix $\boldsymbol{M}_0$ (see, for example, Calinski and Kageyama, 1996, for the definition), we get,

$$\boldsymbol{M}_0 = \boldsymbol{r}^{-\delta}\{q^{-1}\boldsymbol{N}_1\boldsymbol{N}_1' + p^{-1}\boldsymbol{N}_2\boldsymbol{N}_2' - (pq)^{-1}\boldsymbol{NN}' - (spq)^{-1}\boldsymbol{rr}'\},$$

or

$$\boldsymbol{M}_0 = \boldsymbol{r}^{-\delta}\{q^{-1}\boldsymbol{N}_1\boldsymbol{N}_1' + p^{-1}\boldsymbol{N}_2\boldsymbol{N}_2 - 2(spq)^{-1}\boldsymbol{rr}'\}.$$

Finally, $\boldsymbol{M}_0$ can be written as,

$$\boldsymbol{M}_0 = \boldsymbol{M}_{01} + \boldsymbol{M}_{02},$$

where

$$\boldsymbol{M}_{01} = \boldsymbol{r}^{-\delta}\{q^{-1}\boldsymbol{N}_1\boldsymbol{N}_1' - (spq)^{-1}\boldsymbol{rr}'\}$$

and

$$\boldsymbol{M}_{02} = \boldsymbol{r}^{-\delta}\{p^{-1}\boldsymbol{N}_2\boldsymbol{N}_2' - (spq)^{-1}\boldsymbol{rr}'\}.$$

Next, we introduce some definitions related to NRC designs and thereafter present some theorems (with proofs).

Definition 2.1. An NRC design is said to be commutative if $\boldsymbol{M}_{01}\boldsymbol{M}_{02} = \boldsymbol{M}_{02}\boldsymbol{M}_{01}$.

It can be seen that a balanced incomplete block design with nested rows and columns (Singh and Dey, 1979) is a commutative NRC design.

Definition 2.2. An NRC design is said to be row-treatment connected when in the two-way design considering rows as blocks (over all the sets) all the independent treatment effect contrasts are estimable in the two-way design, that is, rank$(\boldsymbol{M}_{01}) = v - 1$.

Definition 2.3. An NRC design is said to be column-treatment connected when in the two-way design considering columns as blocks (over all the sets) all the independent treatment effect contrasts are estimable in the two-way design, that is, rank$(\boldsymbol{M}_{02}) = v - 1$.

Definition 2.4. An NRC design is said to be treatment-connected when all the independent treatment effect contrasts are estimable in the four-way design, that is, rank$(\boldsymbol{M}_0) = v - 1$.

Theorem 2.1. Row-treatment and column-treatment connectedness of a commutative NRC design together do not imply the treatment-connectedness of the design.

Proof. For an NRC design, we have $\boldsymbol{M}_0 = \boldsymbol{M}_{01} + \boldsymbol{M}_{02}$. Furthermore, as the design is commutative we have $\boldsymbol{M}_{01}\boldsymbol{M}_{02} = \boldsymbol{M}_{02}\boldsymbol{M}_{01}$. Let μ_1 and μ_2 be the losses of information for a particular treatment effect contrast, $\boldsymbol{s}$, related to the matrices $\boldsymbol{M}_{01}$ and $\boldsymbol{M}_{02}$, respectively. Then, since M_{01} and M_{02} commute and the design is both row-treatment and column-treatment connected, we have

$$M_{01}s = \mu_1 s,\ 0 \le \mu_1 < 1;\ M_{02}s = \mu_2 s,\ 0 \le \mu_2 < 1.$$

That is,

$$(M_{01} + M_{02})s = (\mu_1 + \mu_2)s.$$

Now, in a row-treatment and column-treatment connected design $\mu_1 + \mu_2 = \mu$ with the condition $0 \le \mu \le 1$, hence the above μ_1 and μ_2 must satisfy the following conditions,

$$0 \le \mu_1 < 1,\ 0 \le \mu_2 < 1,\ \mu_1 + \mu_2 \le 1.$$

Furthermore, using $M_0 = M_{01} + M_{02}$,

$$M_0 s = \mu s,\ 0 \le \mu \le 1.$$

But, if $\mu < 1$ for all independent treatment effect contrasts, then the design is treatment-connected, while if $\mu = 1$ for any treatment effect contrast, then the design is not treatment-connected. Hence the nested design satisfying the conditions of the theorem may or may not be treatment-connected. □

Example. Consider the following NRC design with $v = 4$, $s = 5$, $p = 2$ and $q = 2$:

1	2	2	3	3	4	4	4	4	1
3	4	4	4	4	1	1	2	2	3

It can be checked that this unequal-replicated commutative NRC design is row-treatment connected, column-treatment connected and also treatment-connected.

Theorem 2.2. In an NRC design satisfying the conditions, $N = r\mathbf{1}'/s$ and $N_1' r^{-\delta} N_2 = \mathbf{1}\mathbf{1}'/s$, the estimators of ρ and χ follow the relationship $\mathrm{Cov}(\hat{\rho}, \hat{\chi}) = \mathbf{0}$, the symbols having the usual significance (see proof).

Proof. For an NRC design, $\mathbf{1}'\rho_j = 0$ and $\mathbf{1}'\chi_j = 0$ for all $j = 1, ..., s$. Recalling the notations (Singh and Dey, 1979), the usual normal equations for estimation of different effects reduce to,

$$S = pq\mu\mathbf{1} + N'\tau + pq\psi, \tag{2.1}$$

$$R = q\mu\mathbf{1} + N_1'\tau + qA + q\rho, \tag{2.2}$$

$$C = p\mu\mathbf{1} + N_2'\tau + pB + p\chi, \tag{2.3}$$

$$T = \mu r + r^{\delta}\tau + N\psi + N_1\rho + N_2\chi, \tag{2.4}$$

where τ is a $v \times 1$ vector of treatment effects, ψ is an $s \times 1$ vector of block effects, ρ is a $ps \times 1$ vector of row effects in blocks, i.e., $\rho = (\rho_{11}, ..., \rho_{1p}, \rho_{21}, ..., \rho_{2p}, \cdots, \rho_{s1}, ..., \rho_{sp})' = (\rho_1', \rho_2', ..., \rho_s')'$ with ρ_{jm} being an effect of the mth row in jth block, χ is a $qs \times 1$ vector of column effects in blocks, i.e., $\chi = (\chi_{11}, ..., \chi_{1q}, \chi_{21}, ..., \chi_{2q}, \cdots, \chi_{s1}, ..., \chi_{sq})' = (\chi_1', \chi_2', ..., \chi_s')'$ with $\chi_{j\ell}$ being an effect of the ℓth column in jth block, the matrices A and B being defined as

$$A = \begin{bmatrix} \psi_1 \mathbf{1}_p \\ \vdots \\ \psi_s \mathbf{1}_p \end{bmatrix}, B = \begin{bmatrix} \psi_1 \mathbf{1}_q \\ \vdots \\ \psi_1 \mathbf{1}_q \end{bmatrix}.$$

Pre-multiplying both sides of the equation (2.4) by $\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}$ and using the conditions mentioned in the theorem, we get

$$\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{T} = q\mu\boldsymbol{1} + \boldsymbol{N}_1'\boldsymbol{\tau} + \boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_1\boldsymbol{\rho}. \tag{2.5}$$

Subtracting (2.5) from (2.2), it follows that

$$\boldsymbol{R} - \boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{T} = q\boldsymbol{A} + (q - \boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_1)\boldsymbol{\rho}. \tag{2.6}$$

Similarly, we have

$$\boldsymbol{C} - \boldsymbol{N}_2'\boldsymbol{r}^{-\delta}\boldsymbol{T} = p\boldsymbol{B} + (p - \boldsymbol{N}_2'\boldsymbol{r}^{-\delta}\boldsymbol{N}_2)\boldsymbol{\chi}. \tag{2.7}$$

Taking covariance between the elements on the left hand sides of the equations (2.6) and (2.7), we get $\mathrm{Cov}(\boldsymbol{R}- \boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{T},\ \boldsymbol{C}- \boldsymbol{N}_2'\boldsymbol{r}^{-\delta}\boldsymbol{T}) = \mathrm{Cov}(\boldsymbol{R}, \boldsymbol{C}) - \mathrm{Cov}(\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{T}, \boldsymbol{C}) - \mathrm{Cov}(\boldsymbol{R}, \boldsymbol{N}_2'\boldsymbol{r}^{-\delta}\boldsymbol{T}) + \mathrm{Cov}(\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{T}, \boldsymbol{N}_2'\boldsymbol{r}^{-\delta}\boldsymbol{T}) = \boldsymbol{0}$, since $\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_2 = \boldsymbol{1}\boldsymbol{1}'/s$. Therefore, $\mathrm{Cov}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\chi}}) = \boldsymbol{0}$. □

It may be noted that the condition $\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_2 = \boldsymbol{1}\boldsymbol{1}'/s$ can alternatively be written as $\boldsymbol{N}_1\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_2\boldsymbol{N}_2' = \boldsymbol{r}\boldsymbol{r}'/s$.

Corollary 2.1. An NRC design satisfying the conditions of Theorem 2.2 is always commutative.

Proof. It is easy to verify that under the conditions mentioned above, $\boldsymbol{M}_{01}\boldsymbol{M}_{02} = \boldsymbol{M}_{02}\boldsymbol{M}_{01}$ holds. Hence the NRC design satisfying the conditions of Theorem 2.2 is also commutative. □

Theorem 2.3. In an NRC design satisfying the conditions of Theorem 2.2, the design is treatment-connected if and only if it is both row-treatment and column-treatment connected.

Proof. In the NRC design satisfying the conditions of Theorem 2.2, the following relations hold:

$$\boldsymbol{M}_0 = \boldsymbol{M}_{01} + \boldsymbol{M}_{02} \quad \text{and} \quad \boldsymbol{M}_{01}\boldsymbol{M}_{02} = \boldsymbol{M}_{02}\boldsymbol{M}_{01}.$$

[*Sufficiency Part*]

Case (I): Let a particular treatment effect contrast, $\boldsymbol{s}_1$, be estimated from the matrix $\boldsymbol{M}_{01}$ with $\mu_1(0 < \mu_1 < 1)$ loss of information. Then $\boldsymbol{M}_{01}\boldsymbol{s}_1 = \mu_1\boldsymbol{s}_1$ which implies that $\boldsymbol{r}^{-\delta}\{\boldsymbol{N}_1\boldsymbol{N}_1'/q - \boldsymbol{r}\boldsymbol{r}'/(spq)\}\boldsymbol{s}_1 = \mu_1\boldsymbol{s}_1$. Let the same treatment effect contrast be estimated from the matrix $\boldsymbol{M}_{02}$ with μ_2 loss of information. Then $\boldsymbol{M}_{02}\boldsymbol{s}_1 = \mu_2\boldsymbol{s}_1$ which implies that $\boldsymbol{r}^{-\delta}\{\boldsymbol{N}_2\boldsymbol{N}_2'/p - \boldsymbol{r}\boldsymbol{r}'/(spq)\}\boldsymbol{s}_1 = \mu_2\boldsymbol{s}_1$.
After some algebraic manipulation,

$$0 = \mu_1\boldsymbol{r}^{-\delta}\{\boldsymbol{N}_2\boldsymbol{N}_2'/p - \boldsymbol{r}\boldsymbol{r}'/(spq)\}\boldsymbol{s}_1,$$

which implies that $\mu_2 = 0$ for $0 < \mu_1 < 1$ and $\mu = \mu_1 + 0 = \mu_1(\neq 0, < 1)$.

Similarly, it can be proved that if another treatment effect contrast is estimated from the matrix $\boldsymbol{M}_{02}$ with $\mu_2(0 < \mu_2 < 1)$ loss of information, then the same treatment effect contrast is estimated from the matrix $\boldsymbol{M}_{01}$ with $\mu_1 = 0$ loss of information. Thus $\mu = \mu_1 + \mu_2 = 0 + \mu_2 = \mu_2(\neq 0, < 1)$.

Case (II): If for a particular treatment effect contrast $\mu_1 = \mu_2 = 0$, then obviously, $\mu = \mu_1 + \mu_2 = 0$.

Hence all the treatment effect contrasts are estimable.

[*Necessity Part*]

For a treatment-connected NRC design, $0 \leq \mu < 1$. Also when the design satisfies the conditions of Theorem 2.2, the design must be commutative and hence $\mu = \mu_1 + \mu_2$. Therefore, $0 \leq \mu_1 + \mu_2 < 1$, which implies that $0 \leq \mu_1 < 1$ and $0 \leq \mu_2 < 1$. Hence, the design is both row-treatment and column-treatment connected. □

3. Additional remark

To set up the definition of adjusted orthogonality in the context of a row-column design, take two-factor designs, the first one being treatment vs. row design and the second one being treatment vs. column design. Now if the estimates of row effects from the first design are orthogonal to the estimates of column effects from the second design, the row-column design is said to be adjusted orthogonal (see, for example, Eccleston and Russell, 1975). More specifically, here, row effects adjusted for treatments are orthogonal to the column effects adjusted for treatments.

In the context of an NRC design, if the estimates of row effects (from row vs. treatment design) are orthogonal to the estimates of column effects (from column vs. treatment) design, then the NRC design is said to be adjusted orthogonal.

The condition $\boldsymbol{N}_1'\boldsymbol{r}^{-\delta}\boldsymbol{N}_2 = \boldsymbol{11}'/s$ (mentioned in Theorem 2.2) holds for NRC designs which are adjusted orthogonal. An NRC design is adjusted orthogonal if $\mathrm{Cov}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\chi}}) = \boldsymbol{0}$. It is seen that commutativity allows μ to be decomposed into μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$. Adjusted orthogonal NRC designs are commutative and further, if $\mu_1 \neq 0$, μ_2 must be zero, or if $\mu_2 \neq 0$, μ_1 must be zero. In general, the condition $\mu_1\mu_2 = 0$ always holds for an adjusted orthogonal NRC design.

References

Caliński, T. and Kageyama, S. (1996). Block designs: Their combinatorial and statistical properties. *Handbook of Statistics* (S. Ghosh and C. R. Rao, eds.), Vol. 13, 809-873, Elsevier Science B.V.

Caliński, T. and Kageyama, S. (2000). *Block designs: A Randomization Approach, Volume* I: *Analysis.* Lecture Notes in Statistics 150, Springer, New York.

Eccleston, J. and Russell, K. (1975). Connectedness and orthogonality in multi-factor designs, *Biometrika*, 62, 341-345.

Katyal, V. and Pal, S. (1991). Analysis and connectedness of four dimensional designs. *Jour. Ind. Soc. Agric. Statist.*, XLIII, 296-309.

Pal, P. and Katyal, V. (1988). On multi-way elimination of heterogeneity design. *Jour. Ind. Statist. Assoc.*, 26, 51-58.

Preece, D. A. (1967). Nested balanced incomplete block designs. *Biometrika*, 54, 479-486.

Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

Singh, M. and Dey, A. (1979). Block designs with nested rows and columns. *Biometrika*, 66, 321-326.