Dissipativity Inequalities for Continuous-Time Descriptor Systems with Applications to Synthesis of Control Gains

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Abstract

This paper is concerned with a KYP-type result for descriptor systems. Matrix inequalities are shown that provide necessary and sufficient conditions of dissipativity of descriptor systems, without such restriction on the realization of descriptor systems as in many of previous results. Furthermore, LMI conditions are presented for synthesis of control gains to attain dissipativity of feedback systems represented in the descriptor form.

Key words: Descriptor systems, dissipativity, LMIs

1 Introduction

It has been well understood that the descriptor form provides system representations that are more natural and general than state-space systems (See e.g., [4]). The descriptor form is useful to represent such as mechanical systems, electric circuits, interconnected systems, parameter-varying systems and so on. Among considerable number of basic notions of dynamical systems generalized to descriptor systems, dissipativity, including positive and bounded realness, is one of the most important properties and plays crucial roles in various problems of analysis and synthesis of control systems.

For linear time-invariant systems, Kalman-Popov-Yakuvobich (KYP) Lemma and its related results give characterization of positive or bounded realness in terms of the state-space realization [1,13,7]. The KYP lemma is valid with

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an arbitrary state-space realization that is controllable and has no poles on the imaginary axis. Those results have been generalized to descriptor systems and several matrix equations and inequalities have been proposed to give a criterion for positive or bounded realness [9,6,12,11,8,14,2]. However, for continuous-time systems, most of the existing results require a certain assumption or restriction on the realization of descriptor systems in addition to such as regularity on the imaginary axis and controllability (See Subsection 2.2).

In this paper, we propose a new matrix inequality condition that is necessary and sufficient for dissipativity of descriptor systems [5]. Unlike previous results, the proposed criterion only assumes on a descriptor realization so that the system pencil is regular on the imaginary axis and its finite dynamics is controllable, as the counterpart of the assumptions in the KYP lemma for state-space systems. Moreover, our result is applicable to any quadratic supply rates, not only to ones related to positive or bounded realness. A criterion for the nonstrict dissipativity inequality is given with assuming that the descriptor system is impulse-free. We also provide a criterion for strict dissipativity inequality with admissibility [1], where the criterion implies that the descriptor system has no impulsive modes.

Next, we utilize the new condition for synthesis of feedback gains for descriptor systems to attain dissipativity and admissibility of the closed-loop system. Since the proposed matrix inequality does not have a form for which widely-known change-of-variables technique is directly applicable, we provide an equivalent LMI condition for dissipativity through which we derive LMIs for synthesis. Numerical examples are presented to illustrate the new dissipativity criteria and the LMIs for synthesis.

Notation. Let $\mathbf{R}$ and $\mathbf{C}$ denote the sets of real and complex numbers, respectively. For a matrix X, we denote by X^{-1}, X^{T}, $X^{-\mathsf{T}}$ and X^* the inverse, the transpose, the inverse of the transpose and the conjugate transpose of X, respectively. $\mathbf{He}X$ stands for $X + X^{\mathsf{T}}$. For a symmetric matrix represented blockwise, offdiagonal blocks are abbreviated with '$*$', such as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & * \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}.$$

[1] A generalization of internal stability to descriptor systems.

2 Preliminaries

2.1 Dissipativity of descriptor systems

Consider the following descriptor system:

$$\begin{cases} E\dot{x} = Ax + Bw, \\ \quad z = Cx + Dw, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^m$ is the input and $z \in \mathbf{R}^p$ is the output of the system. Let $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}E = r \leq n$.

Definition 1 *(1) The pencil $sE - A$ is* regular *if* $\det(sE - A)$ *is not identically zero.*

(2) Suppose that $sE - A$ is regular. The exponential modes *of $sE - A$ are the finite eigenvalues of $sE - A$, namely, $s \in \mathbf{C}$ such that* $\det(sE - A) = 0$.

(3) Let a vector v_1 satisfy $Ev_1 = 0$. Then the infinite eigenvalues associated with the generalized eigenvectors v_k satisfying $Ev_k = Av_{k-1}$, $k = 2, 3, 4, \ldots$ are impulsive modes *of (E, A).*

(4) The descriptor system (1) is impulse-free *if the pencil $sE - A$ is regular and has no impulsive modes.*

(5) The descriptor system (1) is said to be admissible *if the pencil $sE - A$ is regular, impulse-free and has no unstable exponential modes.*

Let $S = S^{\mathsf{T}} \in \mathbf{R}^{(m+p)\times(m+p)}$ and consider the following quadratic form of (w, z):

$$s(w, z) = \begin{bmatrix} w \\ z \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} w \\ z \end{bmatrix}, \tag{2}$$

which defines a *supply rate.*

Definition 2 *The descriptor system (1) is said to be* dissipative *with respect to the supply rate $s(\cdot, \cdot)$ if the pencil $sE - A$ is regular, the descriptor system (1) have no impulsive modes and for any $T \geq 0$ and for any $w \in L_2[0, T]$ it holds that*

$$\int_0^T s(w(t), z(t))\mathrm{d}t \leq 0 \tag{3}$$

provided $x(0) = 0$.

The time-domain condition (3) is equivalent to the following frequency-domain condition:

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^* S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} \leq 0, \ \forall \omega \in \mathbf{R} \cup \{\infty\}, \tag{4}$$

where $G(s) = C(sE - A)^{-1}B + D$. By setting

$$M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}, \tag{5}$$

the inequality (4) is written as

$$\begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix} \leq 0, \tag{6}$$

which we consider in the next subsection.

2.2 A KYP-type lemma for descriptor systems

For every regular pencil $sE - A$, there exist regular matrices L, R with which $sE - A$ is transformed into the Weierstrass canonical form as follows:

$$L^{\mathsf{T}}(sE - A)R = \begin{bmatrix} sI - A_1 & 0 \\ 0 & s\Lambda - I \end{bmatrix}, \tag{7}$$

where $A_1 \in \mathbf{R}^{\bar{r}\times\bar{r}}$, $\Lambda \in \mathbf{R}^{(n-\bar{r})\times(n-\bar{r})}$ and Λ is nilpotent. Let $L^{\mathsf{T}}B = \begin{bmatrix} B_1^{\mathsf{T}} & B_2^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ with $B_1 \in \mathbf{R}^{\bar{r}\times m}$. We say that (E, A, B) is *finite dynamics controllable* if (A_1, B_1) is controllable in the sense of state-space systems.

Consider the inequality (6), without assuming that M has the structure of (5).

Theorem 3 *Suppose that the following assumptions (1°)–(3°) hold.*

(1°) $\det(j\omega E - A) \neq 0, \quad \forall \omega \in \mathbf{R}$.

(2°) $\lim_{\omega\to\infty}(j\omega E - A)^{-1}$ *exists.*
(3°) (E, A, B) *is finite-dynamics controllable.*

Then the two conditions below are equivalent:

(i) *For any* $\omega \in \mathbf{R} \cup \{\infty\}$*, it holds that*

$$\begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix} \leq 0. \tag{8}$$

(ii) *There exist matrices* $X \in \mathbf{R}^{n\times n}$ *and* $W \in \mathbf{R}^{n\times m}$ *that satisfy the following matrix equations and inequality:*

$$\begin{cases} E^\mathsf{T} X = X^\mathsf{T} E, \quad E^\mathsf{T} W = 0, \\ M + \mathbf{He} \begin{bmatrix} X^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \leq 0. \end{cases} \tag{9}$$

Proof. Let L and R be regular matrices for which $sE - A$ is transformed to the Weierstrass form (7). Since $\lim_{\omega\to\infty}(j\omega E - A)^{-1}$ exists, it holds that $\Lambda = 0$ and $\bar{r} = r$. Then we have that

$$(j\omega E - A)^{-1}B = R \begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ -B_2 \end{bmatrix} \tag{10}$$

and that (A_1, B_1) is controllable. Let

$$\tilde{M} = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} M \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}.$$

Then the inequality (8) is expressed as

$$\begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}^\mathsf{T} \tilde{M} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ I \end{bmatrix} \leq 0.$$

From the KYP-Lemma for state-space systems [7], the above inequality holds for all $\omega \in \mathbf{R} \cup \{\infty\}$ if and only if there exists a matrix $P = P^\mathsf{T} \in \mathbf{R}^{r\times r}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \tilde{M} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} PA_1 + A_1^{\mathsf{T}} P & PB_1 \\ B_1^{\mathsf{T}} P & 0 \end{bmatrix} \leq 0 \tag{11}$$

holds. By the elimination lemma, (11) is equivalent that for some matrices F, G, H it holds that

$$\tilde{M} + \begin{bmatrix} PA_1 + A_1^{\mathsf{T}} P & 0 & PB_1 \\ 0 & 0 & 0 \\ B_1^{\mathsf{T}} P & 0 & 0 \end{bmatrix} + \mathbf{He} \begin{bmatrix} F \\ G \\ H \end{bmatrix} \begin{bmatrix} 0 & I & B_2 \end{bmatrix} \leq 0.$$

This inequality is rewritten as

$$\tilde{M} + \mathbf{He} \begin{bmatrix} P & F \\ 0 & G \\ 0 & H \end{bmatrix} \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \end{bmatrix} \leq 0,$$

which implies that the inequality in (9) holds. This is seen by setting

$$\begin{bmatrix} X^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} R^{-\mathsf{T}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & F \\ 0 & G \\ 0 & H \end{bmatrix} L^{\mathsf{T}} \tag{12}$$

and performing congruent transformation. Also it is easy to see that the equality conditions in (9) hold if and only if X and W have the form of (12).

Remark 4 *If $E = I$, we have $W = 0$, $X = X^{\mathsf{T}}$ from (9) and the inequality condition (9) reduces to that for state-space systems [7].*

Remark 5 *The result of Theorem 3 gives a necessary and sufficient condition for the dissipativity inequality (8) to hold for $\omega \in \mathbf{R} \cup \{\infty\}$. A generalization of the KYP lemma for descriptor systems with bounded regions of ω has been discussed in [3]. On the other hand, LMIs for the dissipativity inequality in (4) for positive realness required to hold for $\forall \omega \in \mathbf{R}$ and not for $\omega = \infty$ are provided in [2]. The results there can handle impulsive positive real descriptor systems.*

There have been proposed several criteria for positive or bounded realness of descriptor systems [9,6,12,11,8,14,2]. Those results require certain assumptions on the realization of descriptor systems other than such as finite-dynamics controllability. In results for bounded realness or H_∞-norm condition $\|G\|_\infty < \gamma$, the formal feedthrough term D is assumed to be zero [6] or to satisfy $\|D\| < \gamma$ [9]. Similarly, an LMI criterion has been derived in [14] that gives a necessary and sufficient condition for extended strict positive realness with assuming $D + D^\mathsf{T} > 0$. Also in [2] an LMI condition has been proposed that is sufficient for positive realness, but not necessary unless the realization of D satisfies an additional inequality condition. However, since $D \in \mathbf{R}^{p\times m}$ can be chosen arbitrarily with preserving the assumptions of Theorem 3 in the descriptor realization, restrictions on the realization of D such as $\|D\| < \gamma$ are never essential for positive or bounded realness of descriptor systems; see Subsection 2.3.

Below are corollaries of Theorem 3 for strict dissipativity.

Corollary 6 *Suppose that the assumptions (1°) and (2°) hold. Then the following two conditions are equivalent:*

(i) For any $\omega \in \mathbf{R} \cup \{\infty\}$, it holds that

$$\begin{bmatrix} (j\omega E - A)^{-1} B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1} B \\ I \end{bmatrix} < 0. \tag{13}$$

(ii) There exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ satisfying

$$\begin{cases} E^\mathsf{T} X = X^\mathsf{T} E, \quad E^\mathsf{T} W = 0, \\ M + \mathbf{He} \begin{bmatrix} X^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} A \ B \end{bmatrix} < 0. \end{cases} \tag{14}$$

Proof. The corollary is proved in the same way as Theorem 3 with replacing inequalities '$\leq$' with '$<$'.

The following corollary gives a matrix inequality condition for dissipativity as well as admissibility, without assuming (1°)–(3°) of Theorem 3.

Corollary 7 *Consider partition of M as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\mathsf{T} & M_{22} \end{bmatrix}, \quad M_{11} \in \mathbf{R}^{n\times n}$$

and suppose that $M_{11} \geq 0$. Then the descriptor system (1) is admissible and *satisfies (13) if and only if the matrix equations and inequality (14) hold as well as* $E^\mathsf{T} X \geq 0$.

Proof. For the proof of admissibility, see [10,6].

Remark 8 *The* $(1,1)$*-block of (14) is* $A^\mathsf{T} X + X^\mathsf{T} A + M_{11} < 0$*. Hence the inequality (14) together with* $M_{11} \geq 0$ *implies that* X *is nonsingular.*

For convenience, we show LMI conditions about *positive realness* and H_∞ *norm condition* of descriptor systems *with* admissibility. Denote $G(s) = C(sE - A)^{-1}B + D$ and let $m = p$. The descriptor system (1) is *extended strictly positive real* (*ESPR* in short) if $G(s)$ is analytic in $\{s \in \mathbf{C} : \mathrm{Re}(s) \geq 0\}$ and satisfies $G(j\omega) + G^*(j\omega) > 0$ for $\forall \omega \in \mathbf{R} \cup \{\infty\}$[14].

Corollary 9 *Suppose* $m = p$*. The descriptor system (1) is admissible and ESPR if and only if there exist matrices* $X \in \mathbf{R}^{n\times n}$ *and* $W \in \mathbf{R}^{n\times m}$ *satisfying the following LMI condition:*

$$\begin{cases} E^\mathsf{T} X = X^\mathsf{T} E \geq 0, \quad E^\mathsf{T} W = 0, \\ \begin{bmatrix} A^\mathsf{T} X + X^\mathsf{T} A & A^\mathsf{T} W + X^\mathsf{T} B - C^\mathsf{T} \\ * & W^\mathsf{T} B + B^\mathsf{T} W - D - D^\mathsf{T} \end{bmatrix} < 0. \end{cases} \tag{15}$$

This LMI condition, yielded by setting $S = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$ in Theorem 3, is a necessary and sufficient condition for admissibility and extended strictly positive realness, including impulse-free property and the dissipativity condition at $s = \infty$. It has been shown in [2] that the LMI condition

$$E^\mathsf{T} X = X^\mathsf{T} E \geq 0, \qquad \begin{bmatrix} A^\mathsf{T} X + X^\mathsf{T} A & X^\mathsf{T} B - C \\ B^\mathsf{T} X - C^\mathsf{T} & -D - D^\mathsf{T} \end{bmatrix} \leq 0 \tag{16}$$

is sufficient for $G(s)$ to be *positive real*, where $G(s)$ is positive real if it is analytic in $\mathbf{C}_+ := \{s \in \mathbf{C} : \mathrm{Re}(s) > 0\}$ and satisfies $G(s) + G^*(s) \geq 0$ for $s \in \mathbf{C}_+$. The LMI condition (16) is also necessary for positive realness, provided that $D + D \geq M_0 + M_0$, where M_0 is the 0-th coefficient of expansion of $G(s)$ about $s = \infty$: $G(s) = \sum_{i=-\infty}^{p} M_i s^i$. Note that not-extended, not-strict positive realness is considered in [2] and impulsive modes of a positive real descriptor system can be admitted. On the other hand, Corollary 9 aims to provide an LMI condition for *admissibility* as well as extended strict positive realness so that it guarantees that a control system has no impulsive modes and no unstable exponential modes. If the descriptor system is impulse-free, it holds that $M_0 = G(\infty) = D - C_2B_2$, where $CR = [C_1 \quad C_2]$, $C_1 \in \mathbf{R}^{p\times r}$ with R in (7). Hence in general $D \neq M_0$ and the condition $D + D \geq M_0 + M_0$ is not satisfied unless the descriptor realization is chosen so that $C_2B_2 + (C_2B_2)^\mathsf{T} \geq 0$.

Corollary 10 *The descriptor system (1) is admissible and H_∞ norm from w to z is less than γ if and only if there exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ that satisfy*

$$\begin{cases} E^\mathsf{T} X = X^\mathsf{T} E \geq 0, \quad E^\mathsf{T} W = 0, \\ \begin{bmatrix} \mathbf{He} \begin{bmatrix} X^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ \begin{bmatrix} C & D \end{bmatrix} & -I \end{bmatrix} < 0. \end{cases} \tag{17}$$

This corollary is a special case of Theorem 3, with $S = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$.

Remark 11 *Sometimes it is pointed out that non-strict LMI condition $K(\xi) \geq 0$, where $K(\xi)$ is a symmetric-matrix-valued affine function of ξ, is involved with numerical singularity problems. This can be true if there exist no relatively interior-point solutions to $K(\xi) \geq 0$. The nonstrict inequality that appears in Corollary 7 and later is $E^\mathsf{T} X = X^\mathsf{T} E \geq 0$ and the other inequalities are strict. The set of relatively interior-point solutions to $E^\mathsf{T} X = X^\mathsf{T} E \geq 0$ is given by*

$$\left\{ X = L \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} R^{-1} : X_{11} = X_{11}^\mathsf{T} > 0 \right\},$$

where L and R are regular matrices such that $L^\mathsf{T} E R = \mathrm{diag}\{I_{r\times r}, 0\}$. This set is nonempty.

2.3 Numerical Examples for dissipativity inequalities

In this section, we show examples of Corollaries 9 and 10.

2.3.1 H_∞ norm

Corollary 10 gives a necessary and sufficient condition that the descriptor system (1) is admissible and its H_∞ norm is less than γ. Consider the following coefficient matrices for the descriptor system (1):

$$E = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 2\ 5\ 1-\kappa \end{bmatrix}, \quad D = \kappa,$$

where κ is a scalar. For every κ,

$$G(s) = C(sE - A)^{-1}B + D = \frac{s^2 + 7s + 5}{s^2 + 2s + 3},$$

which has constant H_∞ norm $\|G\|_\infty = 3.5551$ for any κ. In Fig. 2.3.2, values of γ are plotted for $\kappa = -10, -9, \ldots, 10$, where '$\diamond$' represents values derived by minimizing γ with respect to (17) in Corollary 10, while '+' represents values derived with setting $W = 0$ in addition to (17). It is seen by the numerical example that the proposed inequality provides the true value of H_∞ norm for every κ. Note that the LMI condition with $W = 0$, namely,

$$E^\mathsf{T} X = X^\mathsf{T} E \geq 0, \qquad \begin{bmatrix} A^\mathsf{T} X + X^\mathsf{T} A & * & * \\ B^\mathsf{T} X & -\gamma^2 I & * \\ C & D & -I \end{bmatrix} < 0, \tag{18}$$

has been proven to yield the correct H_∞ norm if $D = 0$ [6]. Actually, the optimal values of γ with LMI (18) are larger than $\|G\|_\infty$ if $|D| = |\kappa| > \|G\|_\infty$, which violates the assumption of $D = 0$.

2.3.2 Extended strict positive realness with admissibility

The above descriptor systems is ESPR. In fact

$$\mathrm{Re}G(j\omega) = \frac{\omega^4 + 6\omega^2 + 15}{(\omega^2 - 3)^2 + 4\omega^2} = 1 + \frac{4\omega^2 + 6}{(\omega^2 - 1)^2 + 8} \geq 1, \quad \forall \omega \in \mathbf{R} \cup \{\infty\}.$$

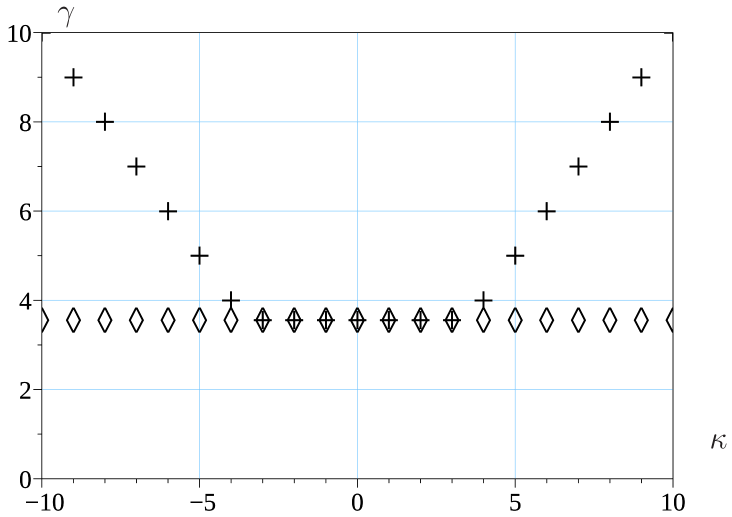

Fig. 1. Values of γ in H_∞ norm test

Thus the LMI (15) must be solvable. To examine this numerically, we solved the following minimization problem:

$$\text{Minimize } \lambda \text{ s.t. } \begin{cases} E^\mathsf{T} X = X^\mathsf{T} E \geq 0, \quad E^\mathsf{T} W = 0, \\ \begin{bmatrix} A^\mathsf{T} X + X^\mathsf{T} A & A^\mathsf{T} W + X^\mathsf{T} B - C^\mathsf{T} \\ * & W^\mathsf{T} B + B^\mathsf{T} W - D - D^\mathsf{T} \end{bmatrix} < \lambda I, \end{cases}$$

whose optimal value is less than 0 if the system is ESPR. The results with setting $\kappa = -10, -9, \ldots, 10$ are shown in Fig. 2.3.2, where again '$\diamond$' shows values derived via the above optimization problem based on the proposed LMI and '$+$' indicates values derived with setting $W = 0$. These computational results show that Corollary 9 proves that the system is ESPR for every $D = \kappa$, while the LMI with $W = 0$, which corresponds to the existing criterion [14], can conclude ESPR when $D + D^\mathsf{T} = 2\kappa > 0$, as assumed there.

2.4 A Pseudo-Dual Matrix Inequality

In this subsection, assuming that the matrix M has the form of (5), we consider a certain 'dual' of the matrix inequality condition stated in Corollary 7. It will play an important role in the following section. Let us denote S in the following partitioned form

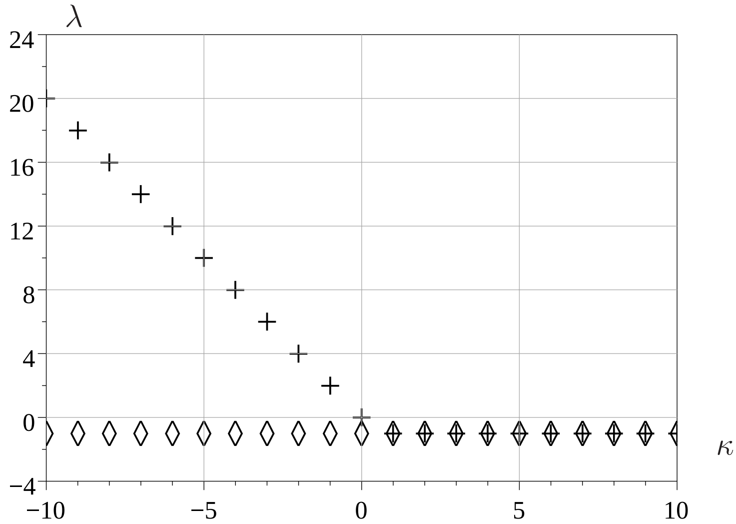

Fig. 2. Values of λ in ESPR test

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathsf{T}} & S_{22} \end{bmatrix}, \quad S_{11} \in \mathbf{R}^{m\times m},$$

according to the sizes of w, z and assume $S_{22} \geq 0$. Substituting (5) to (9) and simple manipulations yield

$$\begin{cases} E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \quad E^{\mathsf{T}}W = 0, \\ \mathbf{He}\left(\begin{bmatrix} X^{\mathsf{T}} & 0 \\ W^{\mathsf{T}} & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ S_{12} \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix}\right) \\ \qquad + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} + \begin{bmatrix} C^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} C & D \end{bmatrix} < 0. \end{cases} \tag{19}$$

Since $S_{22} \geq 0$, this matrix inequality is equivalent to an LMI of decision variables (X, W), which is affine also with respect to coefficient matrices $(E; A, B, C, D)$. Furthermore, positive semidefiniteness of S_{22} implies that X is regular, whereby we define $Y = X^{-\mathsf{T}}$, $Z = -W^{\mathsf{T}}X^{-\mathsf{T}}$. From the first and second items of (19), we derive $Y^{-1}E = E^{\mathsf{T}}Y^{-\mathsf{T}}$, $0 = -ZY^{-1}E$ and immediately $EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0$, $ZE^{\mathsf{T}} = 0$. Multiplying

$$\begin{bmatrix} X^{\mathsf{T}} & 0 \\ W^{\mathsf{T}} & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & 0 \\ Z & I \end{bmatrix} \tag{20}$$

to (19) from the left and the transpose of (20) from the right, we obtain

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \quad EZ^{\mathsf{T}} = 0, \\ \mathbf{He}\left(\begin{bmatrix} A & B \\ S_{12}C & S_{12}D \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix}\right) \\ \qquad + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} + \begin{bmatrix} Y & 0 \\ Z & I \end{bmatrix}\begin{bmatrix} C^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} C & D \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} < 0. \end{cases} \tag{21}$$

Proposition 12 *The descriptor system (1) satisfies dissipativity as well as admissibility if and only if the matrix inequality condition (21) holds for some* $Y \in \mathbf{R}^{n\times n}, Z \in \mathbf{R}^{m\times n}$.

Remark 13 *The matrix inequality (21) is utilized in the next section to obtain LMIs to compute control gains. If* $E = I$*, the inequality (21) is simplified to*

$$\begin{cases} Y = Y^{\mathsf{T}} > 0, \\ \begin{bmatrix} AY + YA^{\mathsf{T}} & * \\ B^{\mathsf{T}} + S_{12}C & S_{11} + S_{12}D + D^{\mathsf{T}}S_{12}^{\mathsf{T}} \end{bmatrix} + \begin{bmatrix} YC^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} CY^{\mathsf{T}} & D \end{bmatrix} < 0. \end{cases} \tag{22}$$

If $S_{12} = 0 \in \mathbf{R}^{m\times p}$ *or* $S_{12} = \pm I \in \mathbf{R}^{m\times m}$ *with* $m = p$ *for the latter, the inequality condition (22) gives a necessary and sufficient condition for dissipativity and internal stability of the dual state-space system, with* $G^{\mathsf{T}}(s) = B^{\mathsf{T}}(sI - A^{\mathsf{T}})^{-1}C^{\mathsf{T}} + D^{\mathsf{T}}$*. In general, the inequality condition (21) does not coincide with the dissipativity of* $G^{\mathsf{T}}(s)$.

3 Synthesis of control gains

3.1 LMI conditions

Based on the results on dissipativity analysis of descriptor systems in the previous section, we consider synthesis of a control gain to attain dissipativity and admissibility of the closed-loop descriptor system. Let us represent the plant as follows:

$$\begin{cases} E\dot{x} = Ax + B_1 w + B_2 u, \\ \quad z = C_1 x + D_{11} w + D_{12} u, \end{cases} \tag{23}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^{m_1}$ is the external input, $u \in \mathbf{R}^{m_2}$ is the control input and $z \in \mathbf{R}^{p_1}$ is the controlled output. We treat two different control laws: (i) constant-gain feedback of the dynamic part of the descriptor variable and (ii) constant-gain feedback of the descriptor variable and feedforward of the external input.

First, let $K \in \mathbf{R}^{m_2 \times n}$ and consider the following control input:

$$u = KEx, \tag{24}$$

by which all of the dynamic part of the descriptor variable is available to compute u. Applying this input to the plant (23) yields the closed-loop system as follows:

$$\begin{cases} E\dot{x} = (A + B_2 KE)x + B_1 w, \\ \quad z = (C_1 + D_{12} KE)x + D_{11} w. \end{cases} \tag{25}$$

Proposition 14 *There exists a gain K for which the closed-loop system (25) is admissible and dissipative if and only if the following LMI holds for $Y \in \mathbf{R}^{n \times n}$, $Z \in \mathbf{R}^{m \times n}$ and $\tilde{K} \in \mathbf{R}^{m_2 \times n}$:*

$$EY^\mathsf{T} = YE^\mathsf{T} \geq 0, \quad EZ^\mathsf{T} = 0, \quad \begin{bmatrix} R_{11} & R_{21}^\mathsf{T} \\ R_{21} & -I \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned} R_{11} &= \mathbf{He}\left(\begin{bmatrix} A & B_1 \\ S_{12}C_1 & S_{12}D_{11} \end{bmatrix} \begin{bmatrix} Y^\mathsf{T} & Z^\mathsf{T} \\ 0 & I \end{bmatrix} + \begin{bmatrix} B_2 \\ S_{12}D_{12} \end{bmatrix} \begin{bmatrix} \tilde{K}E^\mathsf{T} & 0 \end{bmatrix} \right) \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix}, \\ R_{21} &= T_{22}^\mathsf{T} \left(\begin{bmatrix} C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y^\mathsf{T} & Z^\mathsf{T} \\ 0 & I \end{bmatrix} + D_{12} \begin{bmatrix} \tilde{K}E^\mathsf{T} & 0 \end{bmatrix} \right) \end{aligned}$$

with T_{22} being any decomposition of $S_{22} \geq 0$ as $S_{22} = T_{22}T_{22}^\mathsf{T}$. If the LMI (26) has a solution, without loss of generality the matrix Y can be assumed to be regular[2]. One of the gains satisfying admissibility and dissipativity of the closed-loop system is given by $K = \tilde{K}Y^{-\mathsf{T}}$.

[2] See e.g., [6].

Proof. (Necessity) Substitute the expression of the closed-loop system (25) to the matrix inequality (21). Then we see a bilinear terms of KEY^T and KEZ^T. The latter vanishes since $EZ^\mathsf{T} = 0$ and by $EY^\mathsf{T} = YE^\mathsf{T}$ the former reduces to $\tilde{K}E^\mathsf{T}$, where $\tilde{K} = KY^\mathsf{T}$. Then performing Schur complement completes the proof. *(Sufficiency)* The proof of the sufficiency follows easily.

By virtue of the structure of the matrix inequality (21), the standard technique of linearizing change-of-variables for state-feedback synthesis is applicable to remove bilinear terms. This is not obvious with the original dissipativity inequality stated in Theorem 3.

Next, consider the following control input

$$u = Fx + Gw \tag{27}$$

with $F \in \mathbf{R}^{m_2 \times n}$ and $G \in \mathbf{R}^{m_2 \times m_1}$. The closed-loop system is given by

$$\begin{cases} E\dot{x} = (A + B_2F)x + (B_1 + B_2G)w, \\ \quad z = (C_1 + D_{12}F)x + (D_{11} + D_{12}G)w. \end{cases} \tag{28}$$

Proposition 15 *There exists a pair of gains (F, G) satisfying the admissibility and dissipativity of the closed-loop system if and only if there exist matrices $Y \in \mathbf{R}^{n \times n}$, $Z \in \mathbf{R}^{m \times n}$, $\tilde{F} \in \mathbf{R}^{m_2 \times n}$ and $\tilde{G} \in \mathbf{R}^{m_2 \times m_1}$ such that the following LMI holds:*

$$EY^\mathsf{T} = YE^\mathsf{T} \geq 0, \quad EZ^\mathsf{T} = 0, \quad \begin{bmatrix} R'_{11} & (R'_{21})^\mathsf{T} \\ R'_{21} & -I \end{bmatrix} < 0, \tag{29}$$

where

$$\begin{aligned} R'_{11} = \mathbf{He}&\left(\begin{bmatrix} A & B_1 \\ S_{12}C_1 & S_{12}D_{11} \end{bmatrix} \begin{bmatrix} Y^\mathsf{T} & Z^\mathsf{T} \\ 0 & I \end{bmatrix} + \begin{bmatrix} B_2 \\ S_{12}D_{12} \end{bmatrix} \begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix} \right) \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix}, \\ R'_{21} = T_{22}^\mathsf{T}&\left(\begin{bmatrix} C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y^\mathsf{T} & Z^\mathsf{T} \\ 0 & I \end{bmatrix} + D_{12} \begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix} \right). \end{aligned}$$

If the LMI (29) is solvable, the matrix Y can be assumed to be regular without *loss of generality and one of the gains satisfying the admissibility and dissipativity of the closed-loop system is given by* $F = \tilde{K}Y^{-\mathsf{T}}$, $G = \tilde{G} - \tilde{F}Y^{-\mathsf{T}}Z^{\mathsf{T}}$.

Proof. Straightforward.

3.2 Numerical examples for synthesis of control gains

Consider the following coefficient matrices for the descriptor system (1):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 3 & 1-\kappa \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} \kappa \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Note that this descriptor representation gives the same system for any κ.

First, we solved the LMI (26) resulting optimal $\gamma = 2.247$ for every κ. The solution $(Y, Z, \tilde{K})$ to (26) for, e.g., $\kappa = -10$ is

$$Y = \begin{bmatrix} 0.0456 & -0.001127 & -0.003379 \\ -0.001127 & 0.1079 & -0.03235 \\ 0 & 0 & 0.00727 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0.9178 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & -0.111 & 0 \end{bmatrix}$$

and the control input is given by $u = \begin{bmatrix} -0.0253 & -1.0307 & 0 \end{bmatrix} x$.

The second LMI (29) also yielded the same optimal value $\gamma = 2.247$, for every κ. The resulted γ coincides with the value attained by the dynamic-part feedback in this case. For $\kappa = -10$, the solution $(Y, Z, \tilde{F}, \tilde{G})$ to (29) is derived as

$$Y = \begin{bmatrix} 0.0456 & -0.001127 & -0.00423 \\ -0.001127 & 0.1079 & -0.04325 \\ 0 & 0 & 0.00882 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0.9127 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0.001641 & 0.0887 & -0.00286 \end{bmatrix}, \quad \tilde{G} = 0.00260.$$

The corresponding control law is $u = \begin{bmatrix} -0.0175 & -0.9524 & -0.3244 \end{bmatrix} x + 0.2987w$.

4 Conclusions

In this paper, we have shown new matrix inequalities that provide necessary and sufficient conditions for dissipativity of descriptor systems. Based on this result, we have proposed an LMI condition to synthesize a control gain for two types of control input to satisfy dissipativity of the closed-loop system.

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