Layer structures for the solutions to the perturbed simple pendulum problems

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Abstract

We consider the perturbed simple pendulum equation

$$\begin{aligned} -u''(t) &= \mu f(u(t)) + \lambda \sin u(t), \quad t \in I := (-T, T), \\ u(t) &> 0, \quad t \in I, \quad u(\pm T) = 0, \end{aligned}$$

where $\lambda > 0$ and $\mu \in \mathbf{R}$ are parameters. The typical example of f is $f(u) = |u|^{p-1}u$ $(p > 1)$. The purpose of this paper is to study the shape of the solutions when $\lambda \gg 1$. More precisely, by using a variational approach, we show that there exist two types of solutions: one is almost flat inside I and another is like a step function with two steps.

1 Introduction

We consider the perturbed simple pendulum equation

$$-u''(t) = \mu f(u(t)) + \lambda \sin u(t), \quad t \in I := (-T, T), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(\pm T) = 0, \tag{1.3}$$

where $T > 0$ is a constant and $\lambda > 0$, $\mu \in \mathbf{R}$ are parameters. We assume that f satisfies the following conditions.
(A.1) $f \in C^1(\mathbf{R})$, $f(-u) = -f(u)$ for $u \in \mathbf{R}$ and $f(u) > 0$ for $u > 0$.
(A.2) $f'(0) = 0$.
(A.3) $f(u)/u$ is increasing for $0 < u < \pi$.
The typical example of $f(u)$ is $f(u) = |u|^{p-1}u$ $(p > 1)$.

The purpose of this paper is to study the shape of the solutions of (1.1)–(1.3) when $\lambda \gg 1$. More precisely, by using a variational approach, we show that (1.1)–(1.3) has two

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types of solutions: one is almost flat inside I and another is like a step function with two steps. Therefore, it is shown that the structure of the solutions (1.1)–(1.3) is rich.

Linear and nonlinear multiparameter problems have been investigated intensively by many authors. We refer to [1–4, 6–9] and the references therein. In particular, one of the main topics for the nonlinear problems is to study the equations which develop layer type solutions. Indeed, concerning the layer structure of the solutions, a possible layer structure was brought out in [6, 7] for one-parameter singular perturbation problems; and it is known that the solutions with layers appear for two-parameter problems, which are different from (1.1)–(1.3) (cf. [10–12, 14]).

Recently, Shibata [13] considered (1.1)–(1.3) for the case $\mu < 0$ by means of the following constrained minimization method. Let

$$U_\beta := \{u \in H_0^1(I) : \int_I (1 - \cos u(t))dt = \beta\},$$

where $0 < \beta < 4T$ is a fixed constant and $H_0^1(I)$ is the usual real Sobolev space. Regarding $\mu < 0$ as a given parameter, consider the minimizing problem, which depends on μ:

$$\text{Minimize } \frac{1}{2}\|u'\|_2^2 - \mu \int_I F(u(t))dt \text{ under the constraint } u \in U_\beta, \tag{1.4}$$

where $F(u) := \int_0^u f(s)ds$. Then by Lagrange multiplier theorem, for a given $\mu < 0$, a unique solution triple $(\mu, \lambda(\mu), u(\mu)) \in \mathbf{R}_+^2 \times U_\beta$ was obtained, where $\lambda(\mu)$ is the Lagrange multiplier. Then the following result was obtained in [13]:

Theorem 1.0 ([13]). *Let $0 < \theta_\beta < \pi$ satisfy $\cos\theta_\beta = 1 - \beta/(2T)$. Then $u(\mu) \to \theta_\beta$ uniformly on any compact subset in I and $\lambda(\mu) \to \infty$ as $\mu \to -\infty$.*

We see from Theorem 1.0 that $u(\mu)$ is almost flat inside I and develops boundary layer as $\mu \to -\infty$. We emphasize that this asymptotic behavior of $u(\mu)$ is the most characteristic feature of the solution of two-parameter problem (1.1)–(1.3) in the following sense. Let $\mu = \mu_0 < 0$ be fixed in (1.1)–(1.3) and consider a one-parameter problem

$$-v''(t) = \mu_0 f(v(t)) + \lambda \sin v(t), \quad t \in I, \tag{1.5}$$

$$v(t) > 0, \quad t \in I, \tag{1.6}$$

$$v(\pm T) = 0. \tag{1.7}$$

Then for a given $\lambda \gg 1$, there exists a unique solution $(\lambda, v_\lambda) \in \mathbf{R}_+ \times C^2(\bar{I})$. Moreover, *if $\lambda \to \infty$, then $v_\lambda \to \pi$ locally uniformly in I* (cf. [5]). Therefore, we do not have any solution $\{v_\lambda\}$ of a one-parameter problem (1.5)–(1.7) such that $v_\lambda \to \theta_\beta(< \pi)$ as $\lambda \to \infty$.

It should be pointed out that only a flat solution has been obtained in [13], since only the case where $\mu < 0$ has been considered. Indeed, if $\mu < 0$ is assumed, then we see from [5] that the maximum norm of the solution is less than π. Therefore, by the variational method (1.4), it is impossible to treat the solution of (1.1)–(1.3) with *maximum norm larger than* π.

To treat both cases mentioned above at the same time, we adopt here another sort of variational approach. Namely, we regard $\lambda > 0$ as a given parameter here and using different type of variational approach from (1.4), we show that (1.1)–(1.3) has both *flat* and *step function type* solutions. It is shown that the maximum norm of step function type solution is bigger than π.

We now explain the variational framework used here. Let

$$M_\alpha := \{v \in H_0^1(I) : Q(v) := \int_I F(v(t))dt = 2TF(\alpha)\}, \tag{1.8}$$

where $\alpha > 0$ is a constant. Then consider the minimizing problem, which depends on $\lambda > 0$:

$$\text{Minimize } K_\lambda(v) := \frac{1}{2}\|v'\|_2^2 - \lambda \int_I (1 - \cos v(t))dt \text{ under the constraint } v \in M_\alpha. \tag{1.9}$$

Let

$$\beta(\lambda, \alpha) := \min_{v \in M_\alpha} K_\lambda(v).$$

Then by Lagrange multiplier theorem, for a given $\lambda > 0$, there exists $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}^2 \times M_\alpha$ which satisfies (1.1)–(1.3) with $K_\lambda(u_\lambda) = \beta(\lambda, \alpha)$, where $\mu(\lambda)$, which is called the *variational eigenvalue*, is the Lagrange multiplier.

Now we state our results.

Theorem 1.1. *Let $0 < \alpha < \pi$ be fixed. Then*
(a) $\mu(\lambda) < 0$ for $\lambda \gg 1$.
(b) $u_\lambda \to \alpha$ locally uniformly on I as $\lambda \to \infty$.
(c) $\mu(\lambda) = -C_1\lambda + o(\lambda)$ as $\lambda \to \infty$, where $C_1 = \sin\alpha / f(\alpha)$.

The following Theorem 1.2 is our main result in this paper.

Theorem 1.2. *Let $\pi < \alpha < 3\pi$ be fixed. Then*
(a) $\mu(\lambda) > 0$.
(b) $\mu(\lambda) \to 0$ as $\lambda \to \infty$. More precisely, as $\lambda \to \infty$,

$$\lambda \exp(-t_\alpha(1 + o(1))\sqrt{\lambda}) < \mu(\lambda) < \lambda \exp(-t_\alpha(1 - o(1))\sqrt{\lambda}),$$

where $t_\alpha := (F(\alpha) - F(\pi))T/(F(3\pi) - F(\pi))$, which is positive by the condition $\pi < \alpha < 3\pi$ and (A.1).
(c) Assume

$$3F(\alpha) < F(3\pi) + 2F(\pi). \tag{1.10}$$

Then as $\lambda \to \infty$

$$u_\lambda \to 3\pi \quad \textit{locally uniformly on } (-t_\alpha, t_\alpha),$$
$$u_\lambda \to \pi \quad \textit{locally uniformly on } (-T, -t_\alpha) \bigcup (t_\alpha, T).$$

Remark 1.3. (i) Theorem 1.1 (b) implies that for $0 < \alpha < \pi$, u_λ is almost flat inside I and develops boundary layer as $\lambda \to \infty$. On the other hand, Theorem 1.2 (c) implies that for $\pi < \alpha < 3\pi$ satisfying (1.10), u_λ has both boundary layers and interior layers. Moreover u_λ is almost flat in $(-t_\alpha, t_\alpha)$ and $(-T, -t_\alpha) \bigcup (t_\alpha, T)$. In other words, u_λ is almost a step function with two steps in this case. Therefore, the structure of u_λ for $0 < \alpha < \pi$ and $\pi < \alpha < 3\pi$ is totally different.
(ii) The raugh idea of the proof of Theorem 1.2 (c) is as follows. We first show that u_λ has bounbdary layers at $t = \pm T$. Secondly, we show that u_λ has a interior layer in $(0, T)$ and is

almost equal to π and 3π. Inequality (1.10) is a technical condition to obtain the estimate of u_λ from above. Then the position of the interior layer is established automatically. If $f(u) = |u|^{p-1}u$ $(p > 1)$, then (1.10) implies $\pi < \alpha < ((3^{p+1} + 2)/3)^{1/(p+1)}\pi$. For instance, if $p = 7$, then (1.10) is equivalent to

$$\pi < \alpha < (6563/3)^{1/8}\pi = 2.615 \cdots \pi.$$

(iii) It is certainly important to consider the asymptotic shape of u_λ as $\lambda \to \infty$ for the case $\alpha = \pi$. Clearly, u_λ is almost equal to π in $(-T, 0) \bigcup (0, T)$. By Theorem 1.1 (b), we see that if $\alpha < \pi$ and α is very close to π, then u_λ is almost flat and equal to π inside I when $\lambda \gg 1$. On the other hand, by Theorem 1.2 (c), if $\alpha > \pi$ and α is very close to π, then u_λ is almost flat and equal to π in $(-T, -t_\alpha) \bigcup (t_\alpha, T)$, and u_λ is almost equal to 3π in $(-t_\alpha, t_\alpha)$. Since $\alpha > \pi$ and α is nearly equal to π, we see that t_α is very small by definition of t_α and $t_\alpha \to 0$ as $\alpha \to \pi$. Therefore, if $\alpha = \pi$, then the asymptotic shape of u_λ when $\lambda \gg 1$ is expected to be a box with *spike* at $t = 0$. Therefore, it is quite interesting to determine whether the asymptotic shape of u_λ is like *a box with spike at* $t = 0$ as $\lambda \to \infty$ when $\alpha = \pi$. However, it is difficult to treat this problem by our methods here. The reason why is as follows. We regard α as a parameter and denote the minimizer by $u_\lambda = u_{\lambda,\alpha}$ if $u_\lambda \in M_\alpha$. Then it is quite natural to consider a sequence of minimizer $\{u_{\lambda,\alpha}\}$ for *a fixed* λ, and observe a limit function $u_{\lambda,\pi} = \lim_{\alpha\to\pi} u_{\lambda,\alpha}$. Then it is not so difficult to show that $u_{\lambda,\pi}$ is also a minimizer of (1.9) for $\alpha = \pi$, and satisfies (1.1)–(1.3). Therefore, it is expected that $\|u_{\lambda,\pi}\|_\infty \to 3\pi$ as $\lambda \to \infty$. However, to show this, the uniform estimate $\|u_{\lambda,\alpha}\|_\infty \geq 3\pi - \delta$ for all $\lambda > \lambda_0$ and $\pi < \alpha < \pi + \delta$ for some $0 < \delta \ll 1$ is necessary. Since this estimate is quite difficult to show, it is so hard to show whether $\|u_\lambda\|_\infty \to 3\pi$ or π as $\lambda \to \infty$.

From these points of view, it is not easy to study the case where $\alpha = \pi$ by the simple calculation. The future direction of this study is certainly to extend our investigation to the case where $\alpha = \pi, 3\pi, \cdots$.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.1. For the sake of completeness, we show the existence of $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}^2 \times M_\alpha$ in Appendix.

2 Proof of Theorem 1.2

In what follows, we denote by C the various constants which are independent of $\lambda \gg 1$. In particular, the several characters C, which appear in an equality or an inequality repeatedly, may imply the different constants each other.

Proof of Theorem 1.2 (a). Assume that $\mu(\lambda) \leq 0$. Then $u_\lambda > 0$ satisfies

$$\begin{aligned} -u''(t)|\mu(\lambda)|f(u(t)) &= \lambda \sin u(t), \quad t \in I, \\ u(\pm T) &= 0. \end{aligned}$$

Then it follows from [5] that $\|u_\lambda\|_\infty < \pi$. Indeed, let $0 < m_\lambda < \pi$ satisfy $|\mu(\lambda)|f(m_\lambda) = \lambda \sin m_\lambda$. Then we know from [5] that $\|u_\lambda\|_\infty < m_\lambda$. This is impossible, since $u_\lambda \in M_\alpha$ and $\alpha > \pi$. ∎

We next prove Theorem 1.2 (c). To do this, we need some lemmas. For a given $\gamma > 0$, let $t_{\gamma,\lambda} \in [0, T]$ satisfy $u_\lambda(t_{\gamma,\lambda}) = \gamma$, which is unique if it exists, since

$$u'_\lambda(t) < 0, \quad 0 < t \le T. \tag{2.1}$$

Lemma 2.1. *Let $0 < \delta \ll 1$ be fixed. Then $t_{\pi-\delta,\lambda} \to T$ as $\lambda \to \infty$.*

Proof. Since $u_\lambda \in M_\alpha$ ($\pi < \alpha < 3\pi$), we see that $\|u_\lambda\|_\infty > \pi$. Therefore, there exists unique $t_{\pi,\lambda}$. By (1.1), we have

$$\{u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) + \lambda \sin u_\lambda(t)\}u'_\lambda(t) = 0.$$

This implies that for $t \in [0, T]$

$$\begin{aligned} &\frac{1}{2}u'_\lambda(t)^2 + \mu(\lambda)F(u_\lambda(t)) + \lambda(1 - \cos u_\lambda(t)) \equiv \text{constant} \\ &= \mu(\lambda)F(\|u_\lambda\|_\infty) + \lambda(1 - \cos \|u_\lambda\|_\infty) \qquad (\text{put } t = 0) \\ &= \frac{1}{2}u'_\lambda(t_{\pi,\lambda})^2 + \mu(\lambda)F(\pi) + 2\lambda \qquad (\text{put } t = t_{\pi,\lambda}). \end{aligned} \tag{2.2}$$

By this and (2.1), we see that for $t \in [0, T]$

$$-u'_\lambda(t) = \sqrt{2\lambda(1 + \cos u_\lambda(t)) + 2\mu(\lambda)(F(\pi) - F(u_\lambda(t)) + u'_\lambda(t_{\pi,\lambda})^2}. \tag{2.3}$$

By this and Theorem 1.2 (a), for $t_{\pi-\delta,\lambda} \le t \le T$,

$$-u'_\lambda(t) \ge \sqrt{2\lambda(1 - \cos \delta)}. \tag{2.4}$$

By this, we obtain

$$\pi - \delta = \int_{t_{\pi-\delta,\lambda}}^T -u'_\lambda(t)dt \ge \sqrt{2\lambda(1 - \cos \delta)}(T - t_{\pi-\delta,\lambda}).$$

This implies our conclusion. ∎

Lemma 2.2. *Assume that $\|u_\lambda\|_\infty \ge 3\pi$ for $\lambda \gg 1$. Let an arbitrary $0 < \delta \ll 1$ be fixed. Then $t_{\pi+\delta,\lambda} - t_{3\pi-\delta,\lambda} \to 0$ as $\lambda \to \infty$.*

Lemma 2.2 can be proved by the same argument as that in Lemma 2.1. So we omit the proof.

Lemma 2.3. *Let an arbitrary $0 < \delta \ll 1$ be fixed. Then for $\lambda \gg 1$*

$$t_{\pi,\lambda} - t_{\pi+\delta,\lambda} > t_{\pi-\delta,\lambda} - t_{\pi,\lambda}. \tag{2.5}$$

Proof. By (2.3) and putting $\theta := \pi - u_\lambda(t)$, we obtain

$$\begin{aligned} t_{\pi-\delta,\lambda} - t_{\pi,\lambda} &= \int_{t_{\pi,\lambda}}^{t_{\pi-\delta,\lambda}} \frac{-u'_\lambda(t)dt}{\sqrt{2\lambda(1 + \cos u_\lambda(t)) + 2\mu(\lambda)(F(\pi) - F(u_\lambda(t)) + u'_\lambda(t_{\pi,\lambda})^2}} \\ &= \int_0^\delta \frac{d\theta}{\sqrt{2\lambda(1 - \cos\theta) + 2\mu(\lambda)(F(\pi) - F(\pi - \theta)) + u'_\lambda(t_{\pi,\lambda})^2}}. \end{aligned} \tag{2.6}$$

Similarly, by (2.3)

$$t_{\pi,\lambda} - t_{\pi+\delta,\lambda} = \int_0^\delta \frac{d\theta}{\sqrt{2\lambda(1-\cos\theta) - 2\mu(\lambda)(F(\theta+\pi) - F(\pi)) + u_\lambda'(t_{\pi,\lambda})^2}}. \tag{2.7}$$

By this and (2.6), we obtain (2.5). ▮

Lemma 2.4. *Assume that there exists a constant* $0 < \delta_0 \ll 1$ *satisfying* $\limsup_{\lambda\to\infty} \|u_\lambda\|_\infty > 3\pi + \delta_0$. *Then for* $0 < \delta \ll \delta_0$ *and* $\lambda \gg 1$

$$t_{3\pi,\lambda} - t_{3\pi+\delta,\lambda} > t_{\pi,\lambda} - t_{\pi+\delta,\lambda}. \tag{2.8}$$

Proof. Let $0 < \delta \ll \delta_0$ be fixed. Put $t = t_{3\pi,\lambda}$ in (2.2). Then we obtain

$$\frac{1}{2}u_\lambda'(t_{\pi,\lambda})^2 + \mu(\lambda)F(\pi) + 2\lambda = \frac{1}{2}u_\lambda'(t_{3\pi,\lambda})^2 + \mu(\lambda)F(3\pi) + 2\lambda. \tag{2.9}$$

By this and (A.1), we see that

$$u_\lambda'(t_{3\pi,\lambda})^2 < u_\lambda'(t_{\pi,\lambda})^2. \tag{2.10}$$

By this and (2.2), for $t \in [t_{3\pi+\delta,\lambda}, t_{3\pi,\lambda}]$, we have

$$\begin{aligned} \frac{1}{2}u_\lambda'(t)^2 &= \lambda(1+\cos u_\lambda(t)) - \mu(\lambda)(F(u_\lambda(t)) - F(3\pi)) + \frac{1}{2}u_\lambda'(t_{3\pi,\lambda})^2 \\ &< \lambda(1+\cos u_\lambda(t)) - \mu(\lambda)(F(u_\lambda(t)) - F(3\pi)) + \frac{1}{2}u_\lambda'(t_{\pi,\lambda})^2. \end{aligned} \tag{2.11}$$

This along with (2.1) and the same argument as that to obtain (2.6) implies that

$$t_{3\pi,\lambda} - t_{3\pi+\delta,\lambda} > \int_0^\delta \frac{d\theta}{\sqrt{2\lambda(1-\cos\theta) - 2\mu(\lambda)(F(\theta+3\pi)) - F(3\pi)) + u_\lambda'(t_{\pi,\lambda})^2}}. \tag{2.12}$$

By (A.3), it is easy to see that for $0 \le \theta \le \delta$,

$$F(\theta+3\pi) - F(3\pi) > F(\theta+\pi) - F(\pi). \tag{2.13}$$

Then (2.7), (2.12) and (2.13) imply (2.8). Thus the proof is complete. ▮

Proof of Theorem 1.2 (c). We first show that

$$\limsup_{\lambda\to\infty} \|u_\lambda\|_\infty \le 3\pi. \tag{2.14}$$

To do this, we assume that there exists a constant $0 < \delta_0 \ll 1$ and a subsequence of $\{\|u_\lambda\|_\infty\}$, which is denoted by $\{\|u_\lambda\|_\infty\}$ again, such that $\|u_\lambda\|_\infty > 3\pi + \delta_0$ and derive a contradiction. For $0 < \delta < \delta_0$ and $\lambda \gg 1$, we put

$$I_{\pi,\delta,\lambda} := t_{\pi-\delta,\lambda} - t_{\pi+\delta,\lambda}, \quad I_{3\pi,\delta,\lambda} := t_{3\pi,\lambda} - t_{3\pi+\delta,\lambda}.$$

Then we see from Lemmas 2.3 and 2.4 that

$$I_{\pi,\delta,\lambda} < 2I_{3\pi,\delta,\lambda}, \quad I_{\pi,\delta,\lambda} + I_{3\pi,\delta,\lambda} < T. \tag{2.15}$$

By (2.15), for $\lambda \gg 1$, we obtain $I_{\pi,\delta,\lambda} < 2T/3$. Since $u_\lambda \in M_\alpha$ and $u_\lambda(t) = u_\lambda(-t)$ for $t \in [0, T]$, by this and Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} TF(\alpha) &= \int_0^T F(u_\lambda(t))dt \\ &\geq F(\pi - \delta)I_{\pi,\delta,\lambda} + F(3\pi - \delta)(T - I_{\pi,\delta,\lambda}) + o(1) \\ &\geq F(3\pi)T - (F(3\pi) - F(\pi))I_{\pi,\delta,\lambda} + O(\delta) + o(1) \\ &\geq F(3\pi)T - \frac{2}{3}T(F(3\pi) - F(\pi)) + O(\delta) + o(1) \\ &= \frac{1}{3}TF(3\pi) + \frac{2}{3}TF(\pi) + O(\delta) + o(1). \end{aligned} \tag{2.16}$$

Since $0 < \delta \ll 1$ is arbitrary, this contradicts (1.10). Therefore, we obtain (2.14). Then there are two possibilities: (i) $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \pi$ or (ii) $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = 3\pi$. However, (i) is impossible, since $u_\lambda \in M_\alpha$ $(\pi < \alpha < 3\pi)$. Hence, we obtain (ii). Then we see from Lemmas 2.1 and 2.2 that for any $t \in [0, T)$, we have only two possibilities: (i) $\lim_{\lambda\to\infty} u_\lambda(t) = \pi$ or (ii) $\lim_{\lambda\to\infty} u_\lambda(t) = 3\pi$. Then by (2.1) and $u_\lambda \in M_\alpha$, obviously, Theorem 1.2 (c) holds, since $u_\lambda(t) = u_\lambda(-t)$ for $t \in [0, T]$. Thus the proof is complete. ∎

Proof of Theorem 1.2 (b). The proof is divided into three steps.
Step 1. We first show that for $\lambda \gg 1$

$$\|u_\lambda\|_\infty \leq 3\pi. \tag{2.17}$$

Indeed, if there exists a subsequence of $\{\|u_\lambda\|_\infty\}$, which is denoted by $\{\|u_\lambda\|_\infty\}$ again, such that $\|u_\lambda\|_\infty > 3\pi$, then by putting $\theta = u_\lambda(t) - 3\pi$ and $\delta_\lambda := \|u_\lambda\|_\infty - 3\pi > 0$, we see from the same calculation as that to obtain (2.12) that

$$t_{3\pi,\lambda} > \int_0^{\delta_\lambda} \frac{d\theta}{\sqrt{2\lambda(1 - \cos\theta) - 2\mu(\lambda)(F(\theta + 3\pi)) - F(3\pi)) + u_\lambda'(t_{\pi,\lambda})^2}}. \tag{2.18}$$

Further, by putting $\delta = \delta_\lambda$ in (2.6) and (2.7), we obtain

$$t_{\pi-\delta_\lambda,\lambda} - t_{\pi,\lambda} = \int_0^{\delta_\lambda} \frac{d\theta}{\sqrt{2\lambda(1 - \cos\theta) + 2\mu(\lambda)(F(\pi) - F(\pi - \theta)) + u_\lambda'(t_{\pi,\lambda})^2}}, \tag{2.19}$$

$$t_{\pi,\lambda} - t_{\pi+\delta_\lambda,\lambda} = \int_0^{\delta_\lambda} \frac{d\theta}{\sqrt{2\lambda(1 - \cos\theta) - 2\mu(\lambda)(F(\theta + \pi) - F(\pi)) + u_\lambda'(t_{\pi,\lambda})^2}}. \tag{2.20}$$

By (2.13) and (2.18)–(2.20), we obtain

$$t_{3\pi,\lambda} > t_{\pi,\lambda} - t_{\pi+\delta_\lambda,\lambda} > t_{\pi-\delta_\lambda,\lambda} - t_{\pi,\lambda}. \tag{2.21}$$

By this and the same argument to obtain (2.16), we obtain a contradiction. Therefore, we obtain (2.17).

Step 2. Assume that there exists a subsequence of $\{\mu(\lambda)\}$, denoted by $\{\mu(\lambda)\}$ again, such that $\mu(\lambda) \ge \delta_0 > 0$. By (2.2), we have

$$\frac{1}{2}u_\lambda'(t)^2 = \mu(\lambda)(F(\|u_\lambda\|_\infty) - F(u_\lambda(t))) + \lambda(\cos u_\lambda(t) - \cos\|u_\lambda\|_\infty). \tag{2.22}$$

Let $0 < \delta \ll 1$ be fixed. By mean value theorem and (A.3), for $t \in [0, t_{3\pi-2\delta,\lambda}]$ and $\lambda \gg 1$, we obtain

$$\begin{aligned} F(\|u_\lambda\|_\infty) - F(u_\lambda(t)) &\ge f(3\pi - 2\delta)(\|u_\lambda\|_\infty - u_\lambda(t)) \\ &\ge (f(3\pi) - C\delta)(\|u_\lambda\|_\infty - u_\lambda(t)). \end{aligned} \tag{2.23}$$

By (2.17) and the fact that $\|u_\lambda\|_\infty \to 3\pi$ as $\lambda \to \infty$, for $t \in [0, t_{3\pi-2\delta,\lambda}]$ and $\lambda \gg 1$,

$$\begin{aligned} \cos u_\lambda(t) - \cos\|u_\lambda\|_\infty &= -\sin\|u_\lambda\|_\infty(u_\lambda(t) - \|u_\lambda\|_\infty) \\ &\quad - \frac{1}{2}\cos(\theta\|u_\lambda\|_\infty + (1-\theta)u_\lambda(t))(\|u_\lambda\|_\infty - u_\lambda(t))^2 \\ &\ge \frac{1}{2}(1 - C\delta - o(1))(\|u_\lambda\|_\infty - u_\lambda(t))^2, \end{aligned} \tag{2.24}$$

where $0 < \theta < 1$. By (2.22)–(2.24), for $\lambda \gg 1$

$$\begin{aligned} -u_\lambda'(t) &\ge \sqrt{2(f(3\pi) - C\delta)\mu(\lambda)(\|u_\lambda\|_\infty - u_\lambda(t)) + \lambda(1 - C\delta - o(1))(\|u_\lambda\|_\infty - u_\lambda(t))^2} \\ &:= \sqrt{A_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^2 + B_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))}, \end{aligned} \tag{2.25}$$

where

$$A_\lambda = \lambda(1 - C\delta - o(1)), \quad B_\lambda = 2(f(3\pi) - C\delta)\mu(\lambda).$$

By this, for $\lambda \gg 1$,

$$\begin{aligned} t_{3\pi-2\delta,\lambda} &= \int_0^{t_{3\pi-2\delta,\lambda}} dt \\ &\le \int_0^{t_{3\pi-2\delta,\lambda}} \frac{-u_\lambda'(t)}{\sqrt{A_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^2 + B_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))}} dt \\ &= \int_0^{\|u_\lambda\|_\infty - 3\pi + 2\delta} \frac{1}{\sqrt{A_\lambda\theta^2 + B_\lambda\theta}} d\theta \\ &< \frac{1}{\sqrt{A_\lambda}} \int_0^{3\delta} \frac{1}{\sqrt{(\theta + B_\lambda/(2A_\lambda))^2 - B_\lambda^2/(4A_\lambda^2)}} d\theta \\ &= \frac{1}{\sqrt{A_\lambda}} \left[\log\left| 3\delta + \frac{B_\lambda}{2A_\lambda} + \sqrt{\left(3\delta + \frac{B_\lambda}{2A_\lambda}\right)^2 - \frac{B_\lambda^2}{4A_\lambda^2}} \right| - \log\frac{B_\lambda}{2A_\lambda} \right]. \end{aligned} \tag{2.26}$$

Step 3. There are three cases to consider.
Case (i). Assume that there exists a subsequence of $\{\mu(\lambda)\}$ satisfying $\mu(\lambda)/\lambda \to 0$ as $\lambda \to \infty$. Then since $\mu(\lambda) \ge \delta_0$, as $\lambda \to \infty$

$$t_{3\pi-2\delta,\lambda} \le \frac{C}{\sqrt{\lambda}}\left(C + C\log\frac{\lambda}{\mu(\lambda)}\right) \le \frac{C}{\sqrt{\lambda}}\left(C + C\log\frac{\lambda}{\delta_0}\right) \to 0.$$

This along with Lemmas 2.1 and 2.2 implies that $u_\lambda(t) \to \pi$ ($t \in I \setminus \{0\}$) as $\lambda \to \infty$. This is a contradiction, since $u_\lambda \in M_\alpha$ and $\pi < \alpha$.
Case (ii). Assume that there exists a subsequence of $\{\mu(\lambda)\}$ satisfying $C \le \mu(\lambda)/\lambda \le C^{-1}$. Then by this and (2.26), as $\lambda \to \infty$

$$t_{3\pi-2\delta,\lambda} \ \le \ \frac{C}{\sqrt{\lambda}} \to 0. \tag{2.27}$$

By the same reason as that of Case (i), this is a contradiction.
Case (iii). Assume that there exists a subsequence of $\{\mu(\lambda)\}$ satisfying $\mu(\lambda)/\lambda \to \infty$. Then by this and (2.26), as $\lambda \to \infty$

$$t_{3\pi-2\delta,\lambda} \ \le \ o\left(\frac{1}{\sqrt{\lambda}}\right) \to 0. \tag{2.28}$$

By the same reason as that of Case (i), this is a contradiction. Therefore, we see that $\mu(\lambda) \to 0$ as $\lambda \to \infty$.

Finally, we show the decay rate of $\mu(\lambda)$ as $\lambda \to \infty$. Since $t_{3\pi-2\delta,\lambda} \to t_\alpha$ as $\lambda \to \infty$, by (2.26),

$$\begin{aligned} L_1 \ &:= \ \frac{1}{\sqrt{A_\lambda}} \int_0^{2\delta} \frac{1}{\sqrt{A_\lambda \theta^2 + B_\lambda \theta}} d\theta < t_{3\pi-2\delta,\lambda} \\ &< \ \frac{1}{\sqrt{A_\lambda}} \int_0^{3\delta} \frac{1}{\sqrt{A_\lambda \theta^2 + B_\lambda \theta}} d\theta := L_2. \end{aligned} \tag{2.29}$$

Since $\mu(\lambda) \to 0$ as $\lambda \to \infty$,

$$L_1 = \frac{1}{\sqrt{\lambda}}(1 + o(1)) \left(\log(3\delta + o(1)) + \log \frac{\lambda}{C\mu(\lambda)} \right) < t_\alpha(1 + o(1)). \tag{2.30}$$

This implies that

$$\log \frac{\lambda}{\mu(\lambda)} < t_\alpha(1 + o(1))\sqrt{\lambda}. \tag{2.31}$$

By this, we obtain

$$\mu(\lambda) > \lambda \exp(-t_\alpha(1 + o(1))\sqrt{\lambda}). \tag{2.32}$$

By the same argument as above, we also obtain

$$\mu(\lambda) < \lambda \exp(-t_\alpha(1 - o(1))\sqrt{\lambda}). \tag{2.33}$$

Thus the proof is complete. ∎

3 Proof of Theorem 1.1

To prove Theorem 1.1, we follows the idea of the proof of [13, Theorem 2].

Proof of Theorem 1.1 (a). We assume that $\mu(\lambda) \ge 0$ and derive a contradiction. There are three cases to consider.
Case 1. Assume that there exists a subsequence of $\{\|u_\lambda\|_\infty\}$, denoted by $\{\|u_\lambda\|_\infty\}$ again, such that $\|u_\lambda\|_\infty \ge \pi$. Let $0 < \delta \ll 1$ be fixed. Then by (2.2), for $t \in [t_{\pi-\delta,\lambda}, T]$

$$\frac{1}{2}u_\lambda'(t)^2 \ge \lambda(1 + \cos u_\lambda(t)) \ge \lambda(1 - \cos\delta).$$

By this and (2.1),

$$\pi - \delta \;=\; \int_{\pi-\delta,\lambda}^{T} -u_\lambda'(t)dt \ge \sqrt{2\lambda(1-\cos\delta)}(T - t_{\pi-\delta,\lambda}).$$

This implies that $t_{\pi-\delta,\lambda} \to T$ as $\lambda \to \infty$. This is a contradiction, since $u_\lambda \in M_\alpha$ and $0 < \alpha < \pi$.
Case 2. Assume that there exists a subsequence of $\{\|u_\lambda\|_\infty\}$, denoted by $\{\|u_\lambda\|_\infty\}$ again, such that $\|u_\lambda\|_\infty \to \pi$ as $\lambda \to \infty$. Then we see that $\alpha + \delta < \|u_\lambda\|_\infty < \pi$ for $0 < \delta \ll 1$. Then it is clear that $t_{\alpha+\delta,\lambda} \not\to T$ as $\lambda \to \infty$. Indeed, if $t_{\alpha+\delta,\lambda} \to T$ as $\lambda \to \infty$, then

$$2TF(\alpha) = \limsup_{\lambda\to\infty} Q(u_\lambda) \ge 2TF(\alpha + \delta).$$

This is a contradiction. Therefore, there exists a constant $0 < \epsilon_0 \ll 1$ such that $0 < t_{\alpha+\delta,\lambda} \le T - \epsilon_0$ for $\lambda \gg 1$. We choose $\phi \in C_0^\infty(I)$ satisfying $\mathrm{supp}\phi \subset (T - \epsilon_0, T)$. Since $0 \le u_\lambda(t) \le \alpha + \delta$ for $t \in (T - \epsilon_0, T)$ by (2.1), we see that for $t \in (T - \epsilon_0, T)$ and $\lambda \gg 1$

$$\frac{\sin u_\lambda(t)}{u_\lambda(t)} \ge \delta_0 > 0. \tag{3.1}$$

Then for $\lambda \gg 1$

$$\begin{aligned} \mu(\lambda) &= \inf_{v\in H_0^1(I), v\not\equiv 0} \frac{\|v'\|_2^2 - \lambda\int_I \frac{\sin u_\lambda(t)}{u_\lambda(t)} v^2 dt}{\int_I \frac{f(u_\lambda(t))}{u_\lambda(t)} v^2 dt} \le \frac{\|\phi'\|_2^2 - \lambda\delta_0 \int_I \phi^2 dt}{\int_I \frac{f(u_\lambda(t))}{u_\lambda(t)} \phi^2 dt} \\ &< 0. \end{aligned} \tag{3.2}$$

Case 3. Assume that there exists a subsequence of $\{\|u_\lambda\|_\infty\}$, denoted by $\{\|u_\lambda\|_\infty\}$ again, such that $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty < \pi$. Since (3.1) holds for any $t \in [0, T]$ in this case, we choose $\phi \in C_0^\infty(I)$ and obtain (3.2). Thus the proof is complete. ∎

For simplicity, we put $\tilde{\mu}(\lambda) := |\mu(\lambda)| > 0$. Then (1.1) is equivalent to

$$-u''(t) + \tilde{\mu}(\lambda)f(u(t)) = \lambda \sin u(t), \quad t \in I. \tag{3.3}$$

Therefore, in what follows, we consider (3.3) with the conditions (1.2) and (1.3).

Lemma 3.1. $\tilde{\mu}(\lambda) \le C\lambda$ *for* $\lambda \gg 1$

Proof. Let $\theta_\lambda = \{t \in (0, \pi) : \tilde{\mu}(\lambda) f(\theta) = \lambda \sin\theta\}$. Then by [5], we see that

$$\|u_\lambda\|_\infty < \theta_\lambda < \pi. \tag{3.4}$$

Furthermore, since $u_\lambda \in M_\alpha$, we see that for $\lambda \gg 1$

$$\|u_\lambda\|_\infty > \delta_1 > 0. \tag{3.5}$$

By the same calculation as that to obtain (2.2), we have

$$\begin{aligned} &\frac{1}{2}u_\lambda'(t)^2 - \lambda\cos u_\lambda(t) - \tilde{\mu}(\lambda)F(u_\lambda(t)) \\ &= -\lambda\cos\|u_\lambda\|_\infty - \tilde{\mu}(\lambda)F(\|u_\lambda\|_\infty) \\ &= \frac{1}{2}u_\lambda'(T)^2 - \lambda. \end{aligned} \tag{3.6}$$

By (A.3), (3.5) and (3.6), we obtain

$$\begin{aligned} \tilde{\mu}(\lambda)F(\delta_1) &\le \tilde{\mu}(\lambda)F(\|u_\lambda\|_\infty) = -\frac{1}{2}u_\lambda'(T)^2 + \lambda(1 - \cos\|u_\lambda\|_\infty) \\ &\le 2\lambda. \end{aligned}$$

Thus the proof is complete. ∎

We put

$$g_\lambda(u) := \frac{\sin u - \tilde{\mu}(\lambda)f(u)/\lambda}{u}. \tag{3.7}$$

Then by (3.3),

$$-u_\lambda''(t) = \lambda g(u_\lambda(t))u_\lambda(t), \quad t \in I. \tag{3.8}$$

Lemma 3.2. *$g_\lambda(u_\lambda(t)) \to 0$ locally uniformly on I as $\lambda \to \infty$.*

Proof. We assume that there exists a constant $\delta > 0$, $t_0 \in [0, T)$ and a subsequence of $\{\lambda\}$, denoted by $\{\lambda\}$ again, such that $g_\lambda(u_\lambda(t_0)) \ge \delta$ for $\lambda \gg 1$. Since $g_\lambda(u)$ is decreasing for $0 < u < \pi$ by (A.3), we see from (2.1) that for any $t \in [t_0, T)$ and $\lambda \gg 1$

$$g_\lambda(u_\lambda(t)) \ge g_\lambda(u_\lambda(t_0)) \ge \delta. \tag{3.9}$$

We choose $\phi \in C_0^\infty(I)$ with supp$\phi \subset (t_0, T)$. Then by (3.8), we obtain

$$\lambda = \inf_{v \in H_0^1(I), v \not\equiv 0} \frac{\|v'\|_2^2}{\int_I g_\lambda(u_\lambda(t))v^2 dt} \le \frac{\|\phi'\|_2^2}{\delta\|\phi\|_2^2}.$$

This is a contradiction. Thus the proof is complete. ∎

Lemma 3.3. *$\lambda \le C\tilde{\mu}(\lambda)$ for $\lambda \gg 1$.*

Proof. We assume that there exists a subsequence of $\{\lambda/\tilde{\mu}(\lambda)\}$, denoted by $\{\lambda/\tilde{\mu}(\lambda)\}$ again, such that $\lambda/\tilde{\mu}(\lambda) \to \infty$ as $\lambda \to \infty$, and derive a contradiction. Multiply (3.3) by u_λ. Then integration by parts along with (3.4), (3.8) and Lemma 3.2 implies that as $\lambda \to \infty$

$$\begin{aligned} \|u_\lambda'\|_2^2 &= \lambda \int_I u_\lambda(t) \sin u_\lambda(t) dt - \tilde{\mu}(\lambda) \int_I f(u_\lambda(t)) u_\lambda(t) dt \\ &= \lambda \int_I g_\lambda(u_\lambda(t)) u_\lambda(t)^2 dt = o(\lambda). \end{aligned} \tag{3.10}$$

By (3.4) and the assumption,

$$\tilde{\mu}(\lambda) \int_I f(u_\lambda(t)) u_\lambda(t) dt = o(\lambda).$$

By this and (3.10), as $\lambda \to \infty$

$$\int_I u_\lambda(t) \sin u_\lambda(t) dt \to 0. \tag{3.11}$$

Since $u_\lambda \in M_\alpha$, by (3.4) and (3.11), we see that

$$u_\lambda(t) \to \pi \quad t \in [0, t_1), \tag{3.12}$$

$$u_\lambda(t) \to 0 \quad t \in (t_1, T], \tag{3.13}$$

where $t_1 := F(\alpha)T/F(\pi)$. Then by (3.4), (3.6), (3.12) and (3.13), for $t \in (t_1, T]$ and $\lambda \gg 1$

$$\begin{aligned} \frac{1}{2} u_\lambda'(t)^2 &= \lambda(\cos u_\lambda(t) - \cos \|u_\lambda\|_\infty) + \tilde{\mu}(\lambda)(F(u_\lambda(t) - F(\|u_\lambda\|_\infty)) \\ &= 2(1 + o(1))\lambda. \end{aligned} \tag{3.14}$$

Let $0 < \delta \ll 1$ be fixed. Then by (2.1) and (3.14), for $\lambda \gg 1$

$$\pi > u_\lambda(t_1 + \delta) - u_\lambda(t_1 + 2\delta) = \int_{t_1+\delta}^{t_1+2\delta} -u_\lambda'(t) dt \ge 2\sqrt{\lambda}\delta(1 + o(1)).$$

This is a contradiction. Thus the proof is complete. ∎

Proof of Theorem 1.1 (b). Let an arbitrary $0 < t_2 < T$ be fixed. We first prove that $u_\lambda(t_2) \ge \delta$ for $\lambda \gg 1$. Indeed, if there exists a subsequence of $\{u_\lambda(t_2)\}_\lambda$, denoted by $\{u_\lambda(t_2)\}_\lambda$ again, such that $u_\lambda(t_2) \to 0$ as $\lambda \to \infty$, then by (A.2) and Lemmas 3.1–3.3, as $\lambda \to \infty$

$$\frac{\sin u_\lambda(t_2)}{u_\lambda(t_2)} = g_\lambda(u_\lambda(t_2)) + \frac{\tilde{\mu}(\lambda)}{\lambda} \frac{f(u_\lambda(t_2))}{u_\lambda(t_2)} \to 0. \tag{3.15}$$

This is a contradiction, since the left hand side of (3.15) tends to 1 as $\lambda \to \infty$. This implies that $u_\lambda(t) \ge \delta$ for any $0 \le t \le t_2$ and $\lambda \gg 1$.

Now let $\xi > 0$ be an arbitrary accumulation point of $\{\tilde{\mu}(\lambda)/\lambda\}$. We see that $\theta_\lambda \not\to \pi$ as $\lambda \to \infty$. Indeed, if $\theta_\lambda \to \pi$ as $\lambda \to \infty$, then

$$\frac{\tilde{\mu}(\lambda)}{\lambda} = \frac{\sin \theta_\lambda}{f(\theta_\lambda)} \to 0.$$

This contradicts Lemma 3.3. Therefore, we see from (3.4), (3.5) and the argument above that for $t \in [0, t_2]$ and $\lambda \gg 1$

$$\delta \le u_\lambda(t) \le \pi - \delta.$$

By this and Lemma 3.2, for $0 \le t \le t_2$, as $\lambda \to \infty$

$$\xi = \lim_{\lambda\to\infty} \frac{\tilde{\mu}(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \left(\frac{\sin u_\lambda(t)}{f(u_\lambda(t))} - \frac{g_\lambda(u_\lambda(t))u_\lambda(t)}{f(u_\lambda(t))} \right) = \lim_{\lambda\to\infty} \frac{\sin u_\lambda(t)}{f(u_\lambda(t))}. \tag{3.16}$$

Since $\sin u_\lambda(t)/f(u_\lambda(t))$ is increasing for $t \in [0, T]$, and $u_\lambda \in M_\alpha$, this implies that $u_\lambda \to \alpha$ locally uniformly as $\lambda \to \infty$ and $\xi = C_1$. Now our assertion follows from a standard compactness argument. Thus the proof is complete. ∎

4 Appendix

In this section, we show the existence of $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}^2 \times M_\alpha$, where u_λ is the minimizer of the problem (1.9). Let $\lambda > 0$ and $\alpha > 0$ be fixed. Since $K_\lambda(v) \ge -4T\lambda$ for any $v \in M_\alpha$, we can choose a minimizing sequence $\{u_n\}_{n=1}^\infty \subset M_\alpha$ such that as $n \to \infty$

$$K_\lambda(u_n) \to \beta(\lambda, \alpha) \ge -4T\lambda. \tag{4.1}$$

Since $K_\lambda(u_n) = K_\lambda(|u_n|)$ and $|u_n| \in M_\alpha$ by (A.1), without loss of generality, we may assume that $u_n \ge 0$ for $n \in \mathbf{N}$. By (4.1), for $n \in \mathbf{N}$,

$$\frac{1}{2}\|u_n'\|_2^2 \le K_\lambda(u_n) + 4T\lambda < C.$$

Therefore, we can choose a subsequence of $\{u_n\}_{n=1}^\infty$, denoted by $\{u_n\}_{n=1}^\infty$ again, such that as $n \to \infty$

$$u_n \to u_\lambda \quad \text{weakly in } H_0^1(I), \tag{4.2}$$

$$u_n \to u_\lambda \quad \text{in } C(\bar{I}). \tag{4.3}$$

By (4.3), we see that $u_\lambda \in M_\alpha$. In particular, $u_\lambda \not\equiv 0$ in I. Furthermore, by (4.2) and (4.3),

$$\begin{aligned} K_\lambda(u_\lambda) &= \frac{1}{2}\|u_\lambda'\|_2^2 - \lambda \int_I (1 - \cos u_\lambda(t))dt \\ &\le \liminf_{n\to\infty} \frac{1}{2}\|u_n'\|_2^2 - \lim_{n\to\infty} \lambda \int_I (1 - \cos u_n(t))dt \\ &\le \liminf_{n\to\infty} \left(\frac{1}{2}\|u_n'\|_2^2 - \lambda \int_I (1 - \cos u_n(t))dt \right) = \beta(\lambda, \alpha). \end{aligned}$$

This implies that $u_\lambda \ge 0$ is a minimizer of (1.9). Then

$$Q'(u_\lambda)u_\lambda = \int_I f(u_\lambda(t))u_\lambda(t)dt > 0,$$

where the prime denotes the Fréchet derivative of Q. Now we apply the Lagrange multiplier theorem to our situation and obtain $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}^2 \times M_\alpha$, which satisfies (1.1) and (1.3) in a weak sense. Here $\mu(\lambda)$ is the Lagrange multiplier. Then by a standard regularity theorem, we see that $u_\lambda \in C^2(\bar{I})$ and it follows from the strong maximum principle that $u_\lambda > 0$ in I. Thus the proof is complete. ∎

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