An LMI Approach to Guaranteed Cost Control for Uncertain Delay Systems

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***Abstract*—The guaranteed cost-control problem for uncertain linear systems which have delay in both state and control input is considered. Sufficient conditions for the existence of guaranteed cost controllers are given in terms of linear matrix inequality (LMI). It is shown that the state feedback controllers can be obtained by solving the LMI.**

***Index Terms*—Delay, guaranteed cost control, linear matrix inequality (LMI), uncertain linear systems.**

I. Introduction

Delay systems generally occur in modern society in the form of communication systems, transmission systems, chemical processing systems, power systems, and so on. If the presence of delays, is not considered in the controller design, it may cause instability or serious deterioration in the performance of the resulting control systems [1]. The study of time-delay systems has received ever greater attention in the past few decades.

In recent years, the problem of robust control of delay systems with parameter uncertainties has been widely studied in the literature [2]–[4]. Although there have been numerous results on robust control of uncertain delay systems, much effort has been made toward finding a controller which guarantees robust stability. However, when controlling such systems, it is also desirable to design the control systems which is not only robustly stable, but also guarantees an adequate level of performance. One approach to this problem is the so-called guaranteed cost-control approach [5]–[11]. This approach has the advantage of providing an upperbound on a given performance index. Recent advances in the theory of linear matrix inequality (LMI) have allowed a revisiting of the guaranteed cost-control approach [6], [7]. The LMI design method is a very well-known and powerful tool. Not only can it efficiently find feasible and global solutions, but also easily handle various kinds of additional linear constraints. The guaranteed cost-control problem for a class of linear uncertain delay systems which is based on the LMI design approach was solved [8]–[11]. However, in [8]–[11], delay of the control input has not been considered. Furthermore, it should be pointed out that although the robust-control design methods for parameter uncertain systems that have delay in both state and control input have been considered (see for example [3], [4]), the guaranteed cost control for such delay systems has not been discussed so far.

In this brief, the guaranteed cost-control problem of the robust control for uncertain system that has delay in both state and control input is considered. A sufficient condition for the existence of the robust feedback controllers is derived in terms of the LMI. The main result of this brief shows that guaranteed cost controllers can be constructed by solving the LMI. The crucial difference between the existing results [8]–[11] and our new one is that the controller which guarantees the stability and adequate level of performance for the time delay in both state and control input is given. Thus, the resulting controllers can be easily implemented for more practical delay system.

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The notations used in this brief are fairly standard. The superscript T denotes matrix transpose, $I_n \in \mathbf{R}^{n\times n}$ denote the identity matrices, block-diag denotes the block diagonal matrix, $\|\cdot\|$ denotes the Euclidean norm, and $\|\cdot\|_2$ denotes the largest singular value.

II. Preliminary

We consider the autonomous uncertain delay system of the form

$$\begin{aligned}\dot{x}(t) &= [\overline{A}+\Delta\overline{A}(t)]x(t)+\sum_{i=1}^{N}[A_i^{\tau}+\Delta A_i^{\tau}(t)]x(t-\tau_i)\\ &\quad+[H^h+\Delta H^h(t)]x(t-h)\\ x(t) &= \phi(t), \qquad t\in[-d,\,0], \qquad d=\max\{\tau_1,\,\ldots,\,\tau_N,\ h\}\end{aligned} \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the state, $\tau_i>0$ are the delay constants, and $\phi(t)$ is the given continuous vector valued initial function. $\overline{A}$, A_i^{τ}, and H^h are the constant matrices of appropriate dimensions. The parameter uncertainties considered here are assumed to be of the following form:

$$[\,\Delta\overline{A}(t) \quad \Delta H^h(t)\,] = DF(t)\,[\,\overline{E}_1 \quad \overline{E}\,_2^h\,] \tag{2a}$$

$$\Delta A_i^{\tau}(t) = D_i^{\tau}F_i^{\tau}(t)E_i^{\tau}, \qquad i=1,\,\ldots,\,N \tag{2b}$$

where D, $\overline{E}_1$, $\overline{E}\,_2^h$, D_i^{τ}, and E_i^{τ} are known constant real matrices of appropriate dimensions. $F(t)\in\mathbf{R}^{p\times q}$ and $F_i^{\tau}(t)\in\mathbf{R}^{r_i\times s_i}$ are unknown matrix functions with Lebesgue measurable elements and satisfying

$$F^T(t)F(t)\le I_q \quad \text{and} \quad F_i^{\tau T}(t)F_i^{\tau}(t)\le I_{s_i}. \tag{3}$$

Associated with (1) is the cost function

$$J=\int_0^{\infty} x^T(t)\overline{Q}x(t)\,dt \tag{4}$$

where $\overline{Q}$ is the given positive definite symmetric matrices.

Definition 1: The matrix $P>0$ is said to be the quadratic cost matrix for the uncertain delay systems (1) if the following inequality holds:

$$\frac{d}{dt}\,x^T(t)Px(t)+x^T(t)\overline{Q}x(t)<0 \tag{5}$$

for all nonzero $x(t)\in\mathbf{R}^n$ and all uncertainties (2).

Theorem 1: Suppose there exist the symmetric positive definite matrices $P>0,\ S_i>0,\ i=1,\,\ldots,\,N,\ U>0\ \in\mathbf{R}^{n\times n}$ such that for all uncertain matrices (2) the following matrix inequality (6) holds:

$$\Lambda=\begin{bmatrix}\Xi & P\tilde{A}_1^{\tau} & \cdots & P\tilde{A}_N^{\tau} & P\tilde{H}^h\\ \tilde{A}_1^{\tau T}P & -S_1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \tilde{A}_N^{\tau T}P & 0 & \cdots & -S_N & 0\\ \tilde{H}^{hT}P & 0 & \cdots & 0 & -U\end{bmatrix}<0 \tag{6}$$

where $\Lambda\in\mathbf{R}^{\overline{N}\times\overline{N}}$, $\overline{N}=n\cdot(N+2)$, and

$$\begin{aligned}\Xi &:= \tilde{A}^TP+P\tilde{A}+\overline{Q}+\sum_{i=1}^{N}S_i+U\\ \tilde{A} &:= \overline{A}+\Delta\overline{A}(t)\\ \tilde{A}_i^{\tau} &:= A_i^{\tau}+\Delta A_i^{\tau}(t)\\ \tilde{H}^h &:= H^h+\Delta H^h(t).\end{aligned}$$

Then, the autonomous uncertain delay system (1) is quadratically stable and the corresponding value of the cost function (4) satisfies the following inequality (7):

$$J < \phi^T(0)P\phi(0) + \sum_{i=1}^{N}\int_{-\tau_i}^{0} \phi^T(s)S_i\phi(s)\,ds + \int_{-h}^{0} \phi^T(s)U\phi(s)\,ds. \tag{7}$$

The proof is given in Appendix I.

III. Problem Formulation

In this section, we consider the problem of optimal guaranteed cost control via the state feedback for a class of the delay system. The uncertain delay system under consideration are described by the following state equation:

$$\begin{aligned} \dot{x}(t) &= [A+\Delta A(t)]x(t) + [B+\Delta B(t)]u(t) \\ &\quad + \sum_{i=1}^{N}[A_i^\tau + \Delta A_i^\tau(t)]x(t-\tau_i) + [B^h + \Delta B^h(t)]u(t-h), \\ &\quad x(t) = \phi(t), \qquad t \in [-d,\, 0], \qquad d = \max\{\tau_1,\, \ldots,\, \tau_N,\, h\} \end{aligned} \tag{8}$$

where $u(t) \in \mathbf{R}^m$ is the control. The parameter uncertainties satisfy

$$[\,\Delta B(t) \quad \Delta B^h(t)\,] = DF(t)\,[\,E_2 \quad E_2^h\,]. \tag{9}$$

B, B^h, E_2, and E_2^h are the constant matrices of appropriate dimensions. The remainder constant real matrices and parameter uncertainties are the same as the delay systems (1). Associated with (8) is the cost function

$$\mathcal{J} = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]\,dt \tag{10}$$

where Q and R are the given positive definite symmetric matrices.

Definition 2: A control law $u(t) = Kx(t)$ is said to be a quadratic guaranteed cost control with associated cost matrix $P > 0$ for the delay system (8) and cost function (10) if the closed-loop system is quadratically stable and the closed-loop value of the cost function (10) satisfies the bound $\mathcal{J} \le \mathcal{J}^*$ for all admissible uncertainties, that is

$$\frac{d}{dt}\,x^T(t)Px(t) + x^T(t)[Q + K^TRK]x(t) < 0 \tag{11}$$

for all nonzero $x \in \mathbf{R}^n$.

The objective of this brief is to design a linear time-invariant feedback guaranteed cost-control law $u(t) = Kx(t)$ for the delay system (8) with uncertainties.

IV. Main Results

We now give the LMI design approach to the construction of a guaranteed cost controller.

Theorem 2: Suppose there exist the constant parameters $\mu > 0$ and $\varepsilon > 0$ such that the LMI shown in (12) at the bottom of the page, have the symmetric positive definite matrices $X > 0$, $\overline{S}_i > 0$, $i = 1, \ldots, N$, $Z > 0 \in \mathbf{R}^{n\times n}$ and a matrix $Y \in \mathbf{R}^{m\times n}$, where $\Phi := AX + BY + (AX+BY)^T + Z + \mu DD^T + H$, $H := \sum_{i=1}^{N} \varepsilon D_i^\tau D_i^{\tau T}$.

If such conditions are met, the linear-state feedback-control law

$$u(t) = Kx(t) = YX^{-1}x(t) \tag{13}$$

is the guaranteed cost controller and

$$\begin{aligned} \mathcal{J} &< \phi^T(0)X^{-1}\phi(0) + \sum_{i=1}^{N}\int_{-\tau_i}^{0}\phi^T(s)\overline{S}_i^{-1}\phi(s)\,ds \\ &\quad + \int_{-h}^{0}\phi^T(s)X^{-1}ZX^{-1}\phi(s)\,ds \end{aligned} \tag{14}$$

is the guaranteed cost for the closed-loop uncertain delay systems.

The proof is given in Appendix II.

Since the LMI (12) consists of a convex solution set of $(\mu,\, \varepsilon X,\, Y,\, \overline{S}_i,\, Z)$, various efficient convex-optimization algorithms can be applied. Moreover, its solutions represent the set of the guaranteed cost controllers. This parameterized representation can be exploited to design the guaranteed cost controllers which minimizes the value of the guaranteed cost for the closed-loop uncertain delay systems. Consequently, solving the following optimization problem allows us to determine the optimal bound:

$$\min_{\mathcal{X}} \overline{\mathcal{J}} = \min_{\mathcal{X}} \left[\alpha + \sum_{i=1}^{N}\mathrm{Trace}[\mathcal{M}_i] + c^2\|NN^T\|_2 \cdot \mathrm{Trace}[Z]\right] = \mathcal{J}^*, \quad \mathcal{X} \in (\mu,\, \varepsilon X,\, Y,\, \overline{S}_i,\, Z,\, \alpha,\, \mathcal{M}_i) \tag{15}$$

$$\begin{bmatrix}
\Phi & B^hY & (E_1X+E_2Y)^T & X & Y^T & A_1^\tau\overline{S}_1 & 0 & \cdots & A_N^\tau\overline{S}_N & 0 & X & \cdots & X \\
Y^TB^{hT} & -Z & Y^TE_2^{hT} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
E_1X+E_2Y & E_2^hY & -\mu I_q & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
X & 0 & 0 & -Q^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
Y & 0 & 0 & 0 & -R^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\overline{S}_1A_1^{\tau T} & 0 & 0 & 0 & 0 & -\overline{S}_1 & \overline{S}_1E_1^{\tau T} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & E_1^\tau\overline{S}_1 & -\varepsilon I_{s_1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\overline{S}_NA_N^{\tau T} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\overline{S}_N & \overline{S}_NE_N^{\tau T} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & E_N^\tau\overline{S}_N & -\varepsilon I_{s_N} & 0 & \cdots & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\overline{S}_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
X & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\overline{S}_N
\end{bmatrix} < 0 \tag{12}$$

such that (12) and

$$\begin{bmatrix} -\alpha & \phi^T(0) \\ \phi(0) & -X \end{bmatrix} < 0 \tag{16a}$$

$$\begin{bmatrix} -\mathcal{M}_i & M_i^T \\ M_i & -\overline{S}_i \end{bmatrix} < 0, \qquad i = 1, \ldots, N \tag{16b}$$

$$\begin{bmatrix} -cI_n & I_n \\ I_n & -X \end{bmatrix} < 0 \tag{16c}$$

where c is a given positive constant, $M_i M_i^T := \int_{-\tau_i}^{0} \phi(s)\phi^T(s)\,ds$, $NN^T := \int_{-h}^{0} \phi(s)\phi^T(s)\,ds$.

That is, the problem addressed in this brief is as follows. "Find $K = YX^{-1}$ such that LMI (12) and (16) are satisfied and the cost $\overline{\mathcal{J}}$ becomes as small as possible."

Finally, we are in a position to establish the main result of this section.

Theorem 3: If the above optimization problem has the solution μ, ε, X, Y, $\overline{S}_i$, Z, α, and $\mathcal{M}_i$, then, the control law of the form (13) is the linear-state feedback-control law which ensures the minimization of the guaranteed cost (14) for the uncertain delay systems.

Proof: By Theorem 2, the control law (13) constructed from the feasible solutions μ, ε, X, Y, $\overline{S}_i$, Z, α, and $\mathcal{M}_i$ is the guaranteed cost controllers of the uncertain delay systems (8). Applying the Schur complement to the LMI (16) and using the following inequality [12]:

$$\mathrm{Trace}[\mathcal{X}\mathcal{Y}] \le \|\mathcal{X}\|_2 \mathrm{Trace}[\mathcal{Y}], \qquad \mathcal{Y} = \mathcal{Y}^T \ge 0; \quad \mathcal{X} = \mathcal{X}^T$$

we have

$$\begin{aligned}
&(16a) \Leftrightarrow \phi^T(0)X^{-1}\phi(0) < \alpha \\
&(16b) \Rightarrow \int_{-\tau_i}^{0} \phi^T(s)\overline{S}_i^{-1}\phi(s)\,ds = \int_{-\tau_i}^{0} \mathrm{Trace}[\phi^T(s)\overline{S}_i^{-1}\phi(s)]\,ds \\
&\qquad = \mathrm{Trace}[M_i^T \overline{S}_i^{-1} M_i] < \mathrm{Trace}[\mathcal{M}_i] \\
&(16c) \Rightarrow \int_{-h}^{0} \phi^T(s)X^{-1}ZX^{-1}\phi(s)\,ds \\
&\qquad = \int_{-h}^{0} \mathrm{Trace}[\phi^T(s)X^{-1}ZX^{-1}\phi(s)]\,ds \\
&\qquad = \mathrm{Trace}[N^T X^{-1} Z X^{-1} N] \le \|NN^T\|_2 \cdot \|X^{-1}\|_2^2 \cdot \mathrm{Trace}[Z] \\
&\qquad < c^2 \|NN^T\|_2 \cdot \mathrm{Trace}[Z].
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{J} &< \phi^T(0)X^{-1}\phi(0) + \sum_{i=1}^{N} \int_{-\tau_i}^{0} \phi^T(s)\overline{S}_i^{-1}\phi(s)\,ds \\
&\quad + \int_{-h}^{0} \phi^T(s)X^{-1}ZX^{-1}\phi(s)\,ds \\
&< \alpha + \sum_{i=1}^{N} \mathrm{Trace}[\mathcal{M}_i] + c^2\|NN^T\|_2 \cdot \mathrm{Trace}[Z] \le \min_{\mathcal{X}} \overline{\mathcal{J}} = \mathcal{J}^*.
\end{aligned} \tag{17}$$

Thus, the minimization of $\overline{\mathcal{J}}$ implies the minimum value $\mathcal{J}^*$ of the guaranteed cost for the uncertain delay systems (8). The optimality of the solution of the optimization problem follows from the convexity of the objective function under the LMI constraints. This is the required result. ∎

Remark 1: The constant parameter c which is included in the inequality (16c) need to be optimized as the LMI constraint. In that case, it is hard to obtain the optimum guaranteed cost because the resulting problem is a nonconvex optimization problem. Hence, we propose the above suboptimal guaranteed cost control instead of solving the nonconvex optimization problem. As a result, the robust suboptimal guaranteed cost controller which minimizes the value of the guaranteed cost for the closed-loop uncertain delay system can be easily solved by using the LMI.

The constant parameter c needs to be chosen as small as possible. However, if there exists no solution of the considered optimization problem, then we need to take the large parameter c. On the other hand, it should be noted that the parameter c cannot become large because the matrix X is constrained by the inequality (16a).

V. Numerical Example

In order to demonstrate the efficiency of our proposed control, we have run a simple numerical example. The system matrices with the uncertainties are given as follows:

$$\begin{aligned}
A &= \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \\
A_1^\tau &= \begin{bmatrix} -0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix} \\
A_2^\tau &= \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix} \\
A_3^\tau &= \begin{bmatrix} -0.15 & 0.01 \\ 0 & 0.2 \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
B^h &= \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \\
D &= D_1^\tau = D_2^\tau = D_3^\tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
E_1 &= E_1^\tau = E_2^\tau = E_3^\tau = [0 \quad 0.1] \\
E_2 &= E_2^h = [0.2] \\
\tau_1 &= 1 \\
\tau_2 &= 2 \\
\tau_3 &= 3 \\
h &= 1 \\
\phi(t) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
F(t) &= \cos(60\pi t) \\
F_i^\tau(t) &= 1 - \exp(-2t) \\
i &= 1, 2, 3 \\
x &= \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}.
\end{aligned}$$

Note that we cannot apply the technique proposed in [8]–[11] to the above delay system, since the considered system has delay in control input. Moreover, compared with the multiple delay systems presented in [9], the uncertain matrices related to state delays are more general forms. Now, we choose as $R = 0.1$ and $Q = \mathrm{diag}\,[0.2 \ 0.1]$. Moreover, we take $c = 2$. By applying Theorem 3 and solving the corresponding optimization problem (15), we obtain the linear optimal state feedback-control law

$$K = [\,-1.0931 \times 10^{-2} \quad -4.6304\,].$$

Consequently, the optimal guaranteed cost of the uncertain closed-loop delay system is $\mathcal{J}^* = 16.364\,413$.

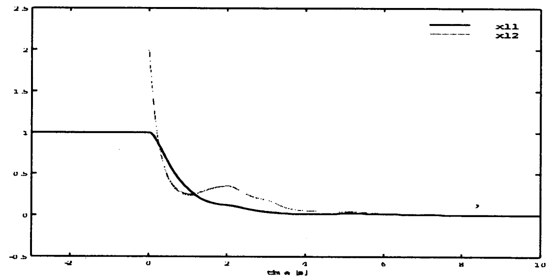

Fig. 1. Response of the closed-loop system with the proposed control method.

Finally, the result of the simulation of this example is depicted in Fig. 1. It is shown from Fig. 1 that the closed-loop systems are asymptotically stable.

VI. Conclusion

In this brief, a solution of the guaranteed cost-control problem for uncertain system that have delay in both state and control input has been presented. The robust optimal guaranteed cost controller which minimizes the value of the guaranteed cost for the closed-loop uncertain delay system can be solved by using software such as MATLAB's LMI control Toolbox [13]. Thus, the resulting linear feedback controller can guarantee the quadratic stability and the optimal cost bound for the uncertain delay systems. On the other hand, there is one drawback that we must not ignore for the proposed controller design. It should be noted that the uncertain system with the kind of uncertainty structure as in(2), usually results in very conservative controller design. However, since it is well known that there are many physical systems in which the uncertainty can be modeled by this manner [14], the assumption that the uncertain system has the structure given by the uncertainty structure (2) is reliable.

After submitting this brief, we have noticed that there exist the useful results for the guaranteed cost-control problem for uncertain systems that have delay in both state and control [18]. Although the proposed design approach is also based on the LMI technique, the controllers are not unique. Therefore, one needs to construct a guaranteed cost controller through trial and error. On the other hand, our controllers are unique. Moreover, since the considered uncertain systems contain the multiple delay in the states, it is applicable to wider class.

Finally, it is expected that the LMI approach is also applied to the output feedback case [15]. Such a problem is more realistic than the state feedback case. This problem will be addressed in future investigations.

Appendix I
Proof of Theorem 1

Using the definitions $\tilde{A}$, $\tilde{A}_i^{\tau}$, and $\tilde{H}^h$, we can change the form (1) as

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=1}^{N} \tilde{A}_i^{\tau} x(t-\tau_i) + \tilde{H}^h x(t-h). \tag{18}$$

Suppose now there exist the symmetric positive definite matrices P, S_i and U such that the matrix inequality (6) holds for all admissible uncertainties (2). In order to prove the asymptotic stability of the delay systems (18), let us define the following Lyapunov function candidate:

$$V(x(t)) = x^T(t)Px(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} x^T(s)S_i x(s)\,ds + \int_{t-h}^{t} x^T(s)Ux(s)\,ds. \tag{19}$$

Note that $V(x(t)) > 0$ whenever $x(t) \neq 0$. Then, the time derivative of $V(x(t))$ along any trajectory of the delay system (18) is given by

$$\begin{aligned}
\frac{d}{dt}V(x(t)) &= x^T(t)(\tilde{A}^T P + P\tilde{A})x(t) + \sum_{i=1}^{N} 2x^T(t)P\tilde{A}_i^{\tau} x(t-\tau_i) \\
&\quad + 2x^T(t)P\tilde{H}^h x(t-h) \\
&\quad + \sum_{i=1}^{N} [x^T(t)S_i x(t) - x^T(t-\tau_i)S_i x(t-\tau_i)] \\
&\quad + x^T(t)Ux(t) - x^T(t-h)Ux(t-h) \\
&= z^T(t) \begin{bmatrix} \Xi - \overline{Q} & P\tilde{A}_1^{\tau} & \cdots & P\tilde{A}_N^{\tau} & P\tilde{H}^h \\ \tilde{A}_1^{\tau T}P & -S_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{A}_N^{\tau T}P & 0 & \cdots & -S_N & 0 \\ \tilde{H}^{hT}P & 0 & \cdots & 0 & -U \end{bmatrix} z(t) \\
&= z^T(t)\Lambda z(t) - x^T(t)\overline{Q}x(t)
\end{aligned}$$

where $z(t) = \left[x^T(t)\; x^T(t-\tau_1)\; \cdots\; x^T(t-\tau_N)\; x^T(t-h)\right]^T \in \mathbf{R}^{\overline{N}}$ and Ξ and Λ are given in (6). Taking the fact that the inequality (6) holds into account, it follows immediately that

$$\frac{d}{dt}V(x(t)) < -x^T(t)\overline{Q}x(t) < 0. \tag{20}$$

Hence, $V(x(t))$ is a Lyapunov function for the delay system (18). Therefore, (18) is asymptotically stable. Furthermore, by integrating both sides of the inequality (20) from 0 to T and using the initial conditions, we have

$$V(x(T)) - V(x(0)) < -\int_0^T x^T(t)\overline{Q}x(t)\,dt. \tag{21}$$

Since (18) is asymptotically stable, that is, $x(T) \to 0$, when $T \to \infty$, we obtain $V(x(T)) \to 0$. Thus, we get

$$\begin{aligned}
J &= \int_0^T x^T(t)\overline{Q}x(t)\,dt < V(x(0)) \\
&= \phi^T(0)P\phi(0) + \sum_{i=1}^{N} \int_{-\tau_i}^{0} \phi^T(s)S_i\phi(s)\,ds + \int_{-h}^{0} \phi^T(s)U\phi(s)\,ds.
\end{aligned}$$

The proof of Theorem 1 is completed. ∎

Appendix II
Proof of Theorem 2

Let us introduce the matrices $X := P^{-1}, Y := KP^{-1}, \overline{S}_i := S_i^{-1}$, and $Z := P^{-1}UP^{-1}$. Premultiplying and postmultiplying both sides of the inequality (12) by

$$\text{block diag}\,[P\;\; P\;\; I_q\;\; I_n\;\; I_m\;\; S_1\;\; I_{s_1}\;\; \cdots\;\; S_N\;\; I_{s_N}\;\; I_n\;\; \cdots\;\; I_n]$$

yields (22), shown at the top of the next page, where $\Psi := \overline{A}^T P + P\overline{A} + U + \mu PDD^T P + PHP, \overline{A} := A + BK, \overline{E} := E_1 + E_2K$:

Using the Schur complement [16], (22) holds if, and only if (23), shown at the top of the next page, holds,where $\Gamma := \overline{A}^T P + P\overline{A} + \overline{R} + \sum_{i=1}^{N} S_i + U + \mu PDD^T P + PHP + \mu^{-1}\overline{E}^T\overline{E}, \overline{R} := Q + K^T RK$.

$$\begin{bmatrix} \Psi & PB^hK & \overline{E}^T & I_n & K^T & PA_1^\tau & 0 & \cdots & PA_N^\tau & 0 & I_n & \cdots & I_n \\ K^TB^{hT}P & -U & K^TE_2^{hT} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \overline{E} & E_2^hK & -\mu I_q & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ I_n & 0 & 0 & -Q^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ K & 0 & 0 & 0 & -R^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A_1^{\tau T}P & 0 & 0 & 0 & 0 & -S_1 & E_1^{\tau T} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1^\tau & -\varepsilon I_{s_1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N^{\tau T}P & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -S_N & E_N^{\tau T} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & E_N^\tau & -\varepsilon I_{s_N} & 0 & \cdots & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\overline{S}_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\overline{S}_N \end{bmatrix} < 0 \quad (22)$$

$$\mathcal{F} := \begin{bmatrix} \Gamma & PA_1^\tau & \cdots & PA_N^\tau & PB^hK + \mu^{-1}\overline{E}^TE_2^hK \\ A_1^{\tau T}P & \varepsilon^{-1}E_1^{\tau T}E_1^\tau - S_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_N^{\tau T}P & 0 & \cdots & \varepsilon^{-1}E_N^{\tau T}E_N^\tau - S_N & 0 \\ K^TB^{hT}P + \mu^{-1}K^TE_2^{hT}\overline{E} & 0 & \cdots & 0 & \mu^{-1}K^TE_2^{hT}E_2^hK - U \end{bmatrix} < 0 \quad (23)$$

Using a standard matrix inequality [17], for all admissible uncertainties (2) and (9), the following matrix inequality holds:

$$\begin{aligned} 0 > \mathcal{F} \\ \geq & \begin{bmatrix} \overline{A}^TP + P\overline{A} + \overline{R} + \sum_{i=1}^{N} S_i + U & PA_1^\tau & \cdots & PA_N^\tau & PB^hK \\ A_1^{\tau T}P & -S_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_N^{\tau T}P & 0 & \cdots & -S_N & 0 \\ K^TB^{hT}P & 0 & \cdots & 0 & -U \end{bmatrix} \\ & + \begin{bmatrix} PD \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} \overline{E}^T \\ 0 \\ \vdots \\ 0 \\ K^TE_2^{hT} \end{bmatrix}^T + \begin{bmatrix} \overline{E}^T \\ 0 \\ \vdots \\ 0 \\ K^TE_2^{hT} \end{bmatrix} F^T(t) \begin{bmatrix} PD \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T \\ & + \begin{bmatrix} 0 & PD_1^\tau & \cdots & PD_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & F_1^\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ & \cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & E_1^\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & E_1^\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T \\ & \cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & F_1^\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & PD_1^\tau & \cdots & PD_N^\tau & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T \\ = & \mathcal{L}. \end{aligned} \quad (24)$$

Let $\overline{A} + DF(t)\overline{E} \to \tilde{A} = \overline{A} + \Delta\overline{A}(t)$, $[B^h + \Delta B^h(t)]K \to \tilde{H}^h = H^h + \Delta H^h(t)$ and $Q + K^TRK = \overline{R} \to \overline{Q}$. Then, $\mathcal{L} = \Lambda$. Hence, the closed-loop systems are asymptotically stable under Theorem 1. On the other hand, since the results of the cost bound (14) can be proved by using the similar argument for the proof of Theorem 1, it is omitted. ■

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Pursley's Aperiodic Cross-Correlation Functions Revisited

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***Abstract*—Pursley's aperiodic cross-correlation function of one delay parameter, which plays an important role in the quasi-synchronous state, is revisited. Using sequences up-sampled by a factor of M, we generalize this function to the one with two discrete delay parameters which play an important role in asynchronous state. Furthermore, Markov spreading sequences are shown to be simply generated by a two-state Markov chain. Applying the central limit theorems, in particular, the Fortet–Kac theorem to the aperiodic cross-correlation function of spreading sequences with Markovity, we can get theoretical estimate of the variance of multiple-access interference.**

***Index Terms*—Asynchronous direct-sequence code-division-multiple-access (DS/CDMA) system, average interference parameter (AIP), multiple-access interference (MAI), Markov chains, up-sampled sequences.**

I. Introduction

Consider baseband direct-sequence spread-spectrum (DS/SS) communications of J users as shown in Fig. 1. We define the data signal of the jth user and its assigned SS code signal $(j = 1, 2, \ldots, J)$ respectively, by $d^{(j)}(t) = \sum_{p=-\infty}^{\infty} d_p^{(j)} u_T(t - pT)$, $d_p^{(j)} \in \{1, -1\}$, $X^{(j)}(t) = \sum_{q=-\infty}^{\infty} X_q^{(j)} u_{T_c}(t - qT_c)$, and $X_q^{(j)} \in \{1, -1\}$, where

$$u_D(t) = \begin{cases} 1, & \text{for } 0 \le t \le D \\ 0, & \text{otherwise.} \end{cases}$$

Since the jth user's code sequence has period $N = T/T_c$, it is denoted by $\boldsymbol{X}^{(j)} = \{X_q^{(j)}\}_{q=0}^{N-1}$. Without loss of generality, we assume $T_c = 1$. Let $s^{(j)}(t) = X^{(j)}(t)d^{(j)}(t)$ be the jth user's SS modulated signal, and t_j be its time delay.

In an asynchronous DS code-division-multiple-access (DS/CDMA) system, values of the *aperiodic cross-correlation function* $R_N^A(\ell)$, introduced by Pursley [1] determine the magnitude of multiple-access interference (MAI) from other channels. The average interference parameter (AIP), a quadratic form of $R_N^A(\ell)$, has often been discussed as a measure of bit-error probabilities in asynchronous DS/CDMA systems [1]–[8] or as a measure of correlational properties of spreading sequences [9], [10].

Recently, by discussing a variance of MAI with respect to *code* symbols, it has been confirmed that sequences generated by some Markov chains are superior/inferior [11]–[14] to sequences of independent and identically distributed (i.i.d.) random variables in an asynchronous/quasi-synchronous (or chip-synchronous) state [15], [16].

In this brief, we show that from the viewpoint of symmetry, it would be better to evaluate the MAI in terms of the even and odd cross-correlation functions rather than in terms of the aperiodic cross-correlation function. Generalization of the *aperiodic cross-correlation function*

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