A New Design Approach for Solving Linear Quadratic Nash Games of Multiparameter Singularly Perturbed Systems

Hiroaki Mukaidani, *Member, IEEE*

***Abstract*—In this paper, the linear quadratic Nash games for infinite horizon nonstandard multiparameter singularly perturbed systems (MSPS) without the nonsingularity assumption that is needed for the existing result are discussed. The new strategies are obtained by solving the generalized cross-coupled multiparameter algebraic Riccati equations (GCMARE). Firstly, the asymptotic expansions for the GCMARE are newly established. The main result in this paper is that the proposed algorithm which is based on the Newton's method for solving the GCMARE guarantees the quadratic convergence. In fact, the simulation results show that the proposed algorithm succeed in improving the convergence rate dramatically compared with the previous results. It is also shown that the resulting controller achieves $O(\|\mu\|^{2^n})$ approximation of the optimal cost.**

***Index Terms*—Multiparameter singularly perturbed systems (MSPS), linear quadratic Nash games, generalized cross-coupled multiparameter algebraic Riccati equations (GCMARE), Newton's method.**

I. Introduction

THE linear quadratic Nash games and their applications have been studied intensively in many papers (see e.g., [1], [2]). Starr and Ho [1] derived the closed-loop perfect-state linear Nash equilibrium strategies for a class of analytic differential games. In [2], a state feedback mixed H_2/H_∞ control problem has been formulated as a dynamic Nash game. It is well-known that in order to obtain the Nash equilibrium strategies, we must solve the cross-coupled algebraic Riccati equations (CARE). Li and Gajić [3] proposed an algorithm called the Lyapunov iterations for solving the CARE. However, there are no results for the convergence rate of the Lyapunov iterations. It is easy to verify that the convergence speed is very slow when the simulation is carried out. In order to improve the convergence rate of the Lyapunov iterations, Mukaidani *et al.* [17] proposed the Riccati iterations which is based on the algebraic Riccati equation (ARE) for solving the CARE. On the other hand, Freiling *et al.* [4] proposed the algorithm which is different from [17] for solving the CARE. However, the convergence of these Riccati iterations have not been proved exactly.

Manuscript received January 9, 2003; revised November 3, 2004. This work was supported in part by the Electric Technology Research Foundation of Chugoku of Japan and a Grant-in-Aid for Young Scientists Research (B)-15700016 from the Ministry of Education, Culture, Sports, Science, and Technology of Japan. This paper was recommended by Associate Editor A. B. Kahng.

The author is with the Graduate School of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan (e-mail: mukaida@hiroshima-u.ac.jp).

Digital Object Identifier 10.1109/TCSI.2005.846668

Multimodeling control problems have been investigated extensively (see e.g., [5], [6], [22]). The multimodeling problems arise in large scale dynamic systems. For example, these multimodel situations in practice are illustrated by the multiarea power system [6] and the passenger car model [9]. The linear quadratic Nash games for the multiparameter singularly perturbed systems (MSPS) and the singularly perturbed systems (SPS) have been studied by using composite controller design [5]–[8], [23]. When the parameters represent small unknown perturbations whose values are not known exactly, the composite design is very useful. However, the composite Nash equilibrium solution achieves only a performance which is $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $[\varepsilon_1 \ \varepsilon_2]$) close to the full-order performance. Moreover, in [5]–[8], the assumptions that the fast state matrices are nonsingular are needed, while in [23], the conservative condition for the existence of the reduced-order solution is assumed. Therefore, the composite design cannot be applied for the wider class of the MSPS. As another important drawback, since the closed-loop solution of the reduced Nash problem depends on the path along $\varepsilon_1/\varepsilon_2$ as $\|\mu\| \to 0$, we cannot conclude that the closed-loop solution of the full problem converges to the closed-loop solution of the reduced problem [8]. Therefore, as long as the small perturbation parameters ε_i are known, much effort should be made toward finding the exact strategies which guarantees the Nash equilibrium without the ill-conditioning.

The recursive algorithm for solving the singularly perturbed Nash games has been attempted in [12] for the first time. In recent years, the recursive algorithm for solving the CARE for the SPS has been investigated [15], [16]. It has been shown that the recursive algorithm is very effective to solve the CARE when the system matrices are functions of a small perturbation parameter ε. However, the recursive algorithm converges only to the approximation solution because the convergence solutions depend on the zeroth-order solutions. In addition, the recursive algorithm has the property of the linear convergence. Thus, the convergence speed is very slow. Very recently, the numerical algorithm which is based on the Newton's method for solving the CARE for the SPS [20] and the cross-coupled multiparameter algebraic Riccati equations (CMARE) [21] has been proposed. However, the conservative assumption guaranteeing that the proposed algorithm converge to the required solution is made. Furthermore, so far, the asymptotic structure for the CMARE has not been investigated exactly.

In this paper, we study the linear quadratic Nash games for infinite horizon nonstandard MSPS without the nonsingularity assumption of the fast state matrices from a viewpoint of solving the CMARE. It should be noted that the cost functions are independent of the other player compared with the previous results [21]. As a result, although the availability of the result derived here is limited, the presented result has a good feature as a more complete formulation for the CMARE. After defining the generalized cross-coupled multiparameter algebraic Riccati equations (GCMARE), we first investigate the uniqueness and boundedness of the solution to the GCMARE and newly establish its asymptotic structure without nonsingularity assumption that the fast state matrices are invertiable. The proof of the existence of the solution to the GCMARE with asymptotic expansion is obtained by an implicit function theorem [10]. The main result of this paper is to propose a new iterative algorithm for solving the GCMARE. Since the new algorithm is based on the Newton's method, it is shown that the new algorithm has a quadratic convergence property. The quadratic convergence of the resulting algorithm is proved by using the Newton-Kantorovich theorem [24]. In particular, it is worth pointing out that the convergence rate of the proposed algorithm and its exact proof are first given in this paper. As a result, using the new algorithm, we will improve the convergence speed compared with the previous results [3], [4], [15]–[17]. Finally, for the practical power systems [6] the simulation results show that the proposed algorithm succeed in improving the convergence rate dramatically.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of M. $\operatorname{vec} M$ denotes an ordered stack of the columns of M [14]. $\otimes$ denotes Kronecker product. U_{lm} denotes a permutation matrix in Kronecker matrix sense [14] such that $U_{lm}\operatorname{vec} M = \operatorname{vec} M^T,\ M \in \mathbf{R}^{l\times m}$. $\mathrm{Re}(\lambda)$ denotes a real part of $\lambda \in \mathbf{C}$. $E[\cdot]$ denotes the expection operator.

II. Problem Formulation

Consider a linear time-invariant MSPS [5]

$$\dot{x}_0(t) = \sum_{k=0}^{2} A_{0i}x_i(t) + \sum_{k=1}^{2} B_{0i}u_i(t)$$
$$x_0(0) = x_0^0 \tag{1a}$$
$$\varepsilon_i\dot{x}_i(t) = A_{i0}x_0(t) + A_{ii}x_i(t) + B_{ii}u_i(t)$$
$$x_i(0) = x_i^0, \qquad i = 1,2 \tag{1b}$$

with quadratic cost functions

$$J_i = \frac{1}{2}\int_0^{\infty} \left[y_i^T(t)y_i(t) + u_i^T(t)R_{ii}u_i(t)\right] dt \tag{2a}$$
$$y_i(t) = C_{i0}x_0(t) + C_{ii}x_i(t) = C_ix(t)$$
$$x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}, \qquad R_{ii} > 0, \qquad i = 1,2 \tag{2b}$$

where $x_i \in \mathbf{R}^{n_i}, i = 0,1,2$ are the state vectors and $u_i \in \mathbf{R}^{m_i}, i = 1,2$ are the control inputs. All the matrices are constant matrices of appropriate dimensions.

ε_1 and ε_2 are two small positive singular perturbation parameters of the same order of magnitude such that

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty. \tag{3}$$

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero (see, e.g., [5], [6]), that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1\to+0 \\ \varepsilon_2\to+0}} \alpha. \tag{4}$$

It is worth pointing out that the matrices $A_{ii},\ i = 1,2$ may be singular. In fact such systems arise in some real physical applications like a flexible space structure [11]. In this case, it should be noted that the composite design [5]–[8] cannot be applied.

Let us introduce the partitioned matrices

$$A_e := \Lambda_e^{-1}A, \quad B_{ie} := \Lambda_e^{-1}B_i$$
$$S_{ie} := B_{ie}R_{ii}^{-1}B_{ie}^T = \Lambda_e^{-1}S_i\Lambda_e^{-1}, \qquad i = 1,2$$
$$\Lambda_e := \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix}$$
$$A := \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix}$$
$$B_1 := \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix}$$
$$B_2 := \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix}$$
$$S_1 := B_1R_{11}^{-1}B_1^T = \begin{bmatrix} S_{001} & S_{011} & 0 \\ S_{011}^T & S_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$S_2 := B_2R_{22}^{-1}B_2^T = \begin{bmatrix} S_{002} & 0 & S_{022} \\ 0 & 0 & 0 \\ S_{022}^T & 0 & S_{222} \end{bmatrix}$$
$$Q_1 := C_1^TC_1 = \begin{bmatrix} Q_{001} & Q_{011} & 0 \\ Q_{011}^T & Q_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$Q_2 := C_2^TC_2 = \begin{bmatrix} Q_{002} & 0 & Q_{022} \\ 0 & 0 & 0 \\ Q_{022}^T & 0 & Q_{222} \end{bmatrix}.$$

We now consider the linear quadratic Nash games for infinite horizon nonstandard MSPS (1) under the following basic assumptions (see, e.g., [3], [5]).

Assumption 1: There exists a $\mu^* > 0$ such that the triplet $(A_e, B_{ie}, C_i), i = 1,2$ are stabilizable and detectable for all $\mu \in (0, \mu^*]$, where $\mu := \sqrt{\varepsilon_1\varepsilon_2}$.

Assumption 2: The triplet $(A_{ii}, B_{ii}, C_{ii}), i = 1,2$ are stabilizable and detectable.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback controller (u_1^*, u_2^*) such that

$$J_i(u_i^*, u_j^*) \le J_i(u_i, u_j^*), \qquad i,j = 1,2, \quad i \ne j. \tag{5}$$

The Nash inequality shows that u_i^* regulates the state to zero with minimum output energy. The following lemma is already known [1].

Lemma 1: Under Assumption 1, there exists an admissible controller such that the inequality (5) holds iff the following full-order CMARE:

$$\begin{aligned} &A_e^T X_e + X_e A_e + Q_1 \\ &\quad - X_e S_{1e} X_e - X_e S_{2e} Y_e - Y_e S_{2e} X_e = 0 \quad (6a)\\ &A_e^T Y_e + Y_e A_e + Q_2 \\ &\quad - Y_e S_{2e} Y_e - Y_e S_{1e} X_e - X_e S_{1e} Y_e = 0 \quad (6b)\end{aligned}$$

have stabilizing solutions $X_e \geq 0$ and $Y_e \geq 0$, where

$$X_e = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ \varepsilon_1 X_{10} & \varepsilon_1 X_{11} & \sqrt{\varepsilon_1\varepsilon_2} X_{21}^T \\ \varepsilon_2 X_{20} & \sqrt{\varepsilon_1\varepsilon_2} X_{21} & \varepsilon_2 X_{22} \end{bmatrix}$$
$$Y_e = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ \varepsilon_1 Y_{10} & \varepsilon_1 Y_{11} & \sqrt{\varepsilon_1\varepsilon_2} Y_{21}^T \\ \varepsilon_2 Y_{20} & \sqrt{\varepsilon_1\varepsilon_2} Y_{21} & \varepsilon_2 Y_{22} \end{bmatrix}.$$

Then, the closed-loop linear Nash equilibrium solutions to the full-order problem are given by

$$u_1^*(t) = -R_{11}^{-1} B_{1e}^T X_e x(t) \quad (7a)$$
$$u_2^*(t) = -R_{22}^{-1} B_{2e}^T Y_e x(t). \quad (7b)$$

It should be noted that it is difficult to solve the CMARE (6) because of the different magnitude of their coefficient caused by the small perturbed parameter ε_i and high dimensions.

III. Asymptotic Structure

In order to obtain the solutions of the CMARE (6), we introduce the following useful lemma [20], [21].

Lemma 2: The CMARE (6) is equivalent to the following GCMARE (8), respectively:

$$A^T X + X^T A + Q_1 - X^T S_1 X - X^T S_2 Y - Y^T S_2 X = 0 \quad (8a)$$
$$A^T Y + Y^T A + Q_2 - Y^T S_2 Y - Y^T S_1 X - X^T S_1 Y = 0 \quad (8b)$$

where

$$X = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1} X_{21}^T \\ X_{20} & \sqrt{\alpha} X_{21} & X_{22} \end{bmatrix}$$
$$X_e = \Lambda_e X = X^T \Lambda_e, \quad X_{ii} = X_{ii}^T$$
$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix}$$
$$Y_e = \Lambda_e Y = Y^T \Lambda_e, \qquad Y_{ii} = Y_{ii}^T, \qquad i = 0, 1, 2.$$

Moreover, we can change the form of the strategies (7) as follows:

$$u_1^*(t) = -R_{11}^{-1} B_1^T X x(t) \quad (9a)$$
$$u_2^*(t) = -R_{22}^{-1} B_2^T Y x(t). \quad (9b)$$

The GCARE equation (8) can be partitioned as

$$\begin{aligned} f_{x1} = {} & A_{00}^T X_{00} + X_{00} A_{00} + A_{10}^T X_{10} + X_{10}^T A_{10} \\ & + A_{20}^T X_{20} + X_{20}^T A_{20} - X_{00} S_{001} X_{00} - X_{00} S_{002} Y_{00} \\ & - X_{10}^T S_{011}^T X_{00} - X_{00} S_{011} X_{10} - X_{10}^T S_{111} X_{10} \\ & - X_{20}^T S_{022}^T Y_{00} - X_{00} S_{022} Y_{20} - X_{20}^T S_{222} Y_{20} \\ & - Y_{00} S_{002} X_{00} - Y_{20}^T S_{022}^T X_{00} - Y_{00} S_{022} X_{20} \\ & - Y_{20}^T S_{222} X_{20} + Q_{001} = 0 \qquad (10a) \end{aligned}$$

$$\begin{aligned} f_{x2} = {} & X_{00} A_{01} + X_{10}^T A_{11} + \varepsilon_1 A_{00}^T X_{10}^T + A_{10}^T X_{11} \\ & + \sqrt{\alpha} A_{20}^T X_{21} - \varepsilon_1 \left(X_{00} S_{001} X_{10}^T + X_{10}^T S_{011}^T X_{10}^T \right) \\ & - \left(X_{00} S_{011} X_{11} + X_{10}^T S_{111} X_{11} \right) - \varepsilon_1 \left(X_{00} S_{002} Y_{10}^T \right. \\ & \left. + X_{20}^T S_{022}^T Y_{10}^T \right) - \sqrt{\alpha} \left(X_{00} S_{022} Y_{21} + X_{20}^T S_{222} Y_{21} \right) \\ & - \varepsilon_1 \left(Y_{00} S_{002} X_{10}^T + Y_{20}^T S_{022}^T X_{10}^T \right) \\ & - \sqrt{\alpha} \left(Y_{00} S_{022} X_{21} + Y_{20}^T S_{222} X_{21} \right) + Q_{011} = 0 \qquad (10b) \end{aligned}$$

$$\begin{aligned} f_{x3} = {} & X_{00} A_{02} + X_{20}^T A_{22} + \varepsilon_2 A_{00}^T X_{20}^T + A_{20}^T X_{22} \\ & + \frac{1}{\sqrt{\alpha}} A_{10}^T X_{21}^T - \varepsilon_2 \left(X_{00} S_{001} X_{20}^T + X_{10}^T S_{011}^T X_{20}^T \right) \\ & - \frac{1}{\sqrt{\alpha}} \left(X_{00} S_{011} X_{21}^T + X_{10}^T S_{111} X_{21}^T \right) - \varepsilon_2 \left(X_{00} S_{002} Y_{20}^T \right. \\ & \left. + X_{20}^T S_{022}^T Y_{20}^T \right) - \left(X_{00} S_{022} Y_{22} + X_{20}^T S_{222} Y_{22} \right) \\ & - \varepsilon_2 \left(Y_{00} S_{002} X_{20}^T + Y_{20}^T S_{022}^T X_{20}^T \right) \\ & - \left(Y_{00} S_{022} X_{22} + Y_{20}^T S_{222} X_{22} \right) = 0 \qquad (10c) \end{aligned}$$

$$\begin{aligned} f_{x4} = {} & A_{11}^T X_{11} + X_{11} A_{11} + \varepsilon_1 \left(A_{01}^T X_{10}^T + X_{10} A_{01} \right) \\ & - \varepsilon_1 \left(\varepsilon_1 X_{10} S_{001} X_{10}^T + X_{11} S_{011}^T X_{10}^T + X_{10} S_{011} X_{11} \right) \\ & - X_{11} S_{111} X_{11} - \varepsilon_1 \left(\varepsilon_1 X_{10} S_{002} Y_{10}^T \right. \\ & \left. + \sqrt{\alpha} X_{21}^T S_{022}^T Y_{10}^T + \sqrt{\alpha} X_{10} S_{022} Y_{21} \right) - \alpha X_{21}^T S_{222} Y_{21} \\ & - \varepsilon_1 \left(\varepsilon_1 Y_{10} S_{002} X_{10}^T + \sqrt{\alpha} Y_{21}^T S_{022}^T X_{10}^T \right. \\ & \left. + \sqrt{\alpha} Y_{10} S_{022} X_{21} \right) - \alpha Y_{21}^T S_{222} X_{21} + Q_{111} = 0 \qquad (10d) \end{aligned}$$

$$\begin{aligned} f_{x5} = {} & \varepsilon_1 X_{10} A_{02} + \varepsilon_2 A_{01}^T X_{20}^T + \sqrt{\alpha} X_{21}^T A_{22} + \frac{1}{\sqrt{\alpha}} A_{11}^T X_{21}^T \\ & - \varepsilon_1 \varepsilon_2 X_{10} S_{001} X_{20}^T - \varepsilon_2 X_{11} S_{011}^T X_{20}^T \\ & - \frac{\varepsilon_1}{\sqrt{\alpha}} X_{10} S_{011} X_{21}^T - \frac{1}{\sqrt{\alpha}} X_{11} S_{111} X_{21}^T \\ & - \varepsilon_1 \varepsilon_2 X_{10} S_{002} Y_{20}^T - \varepsilon_2 \sqrt{\alpha} X_{21}^T S_{022}^T Y_{20}^T \\ & - \varepsilon_1 X_{10} S_{022} Y_{22} - \sqrt{\alpha} X_{21}^T S_{222} Y_{22} \\ & - \varepsilon_1 \varepsilon_2 Y_{10} S_{002} X_{20}^T - \varepsilon_2 \sqrt{\alpha} Y_{21}^T S_{022}^T X_{20}^T \\ & - \varepsilon_1 Y_{10} S_{022} X_{22} - \sqrt{\alpha} Y_{21}^T S_{222} X_{22} = 0 \qquad (10e) \end{aligned}$$

$$\begin{aligned} f_{x6} = {} & A_{22}^T X_{22} + X_{22} A_{22} + \varepsilon_2 \left(A_{02}^T X_{20}^T + X_{20} A_{02} \right) \\ & - \varepsilon_2 \left(\varepsilon_2 X_{20} S_{001} X_{20}^T + \frac{1}{\sqrt{\alpha}} X_{21} S_{011}^T X_{20}^T \right. \\ & \left. + \frac{1}{\sqrt{\alpha}} X_{20} S_{011} X_{21}^T \right) - \frac{1}{\alpha} X_{21} S_{111} X_{21}^T \\ & - \varepsilon_2 \left(\varepsilon_2 X_{20} S_{002} Y_{20}^T + X_{22} S_{022}^T Y_{20}^T + X_{20} S_{022} Y_{22} \right) \\ & - X_{22} S_{222} Y_{22} - \varepsilon_2 \left(\varepsilon_2 Y_{20} S_{002} X_{20}^T + Y_{22} S_{022}^T X_{20}^T \right. \\ & \left. + Y_{20} S_{022} X_{22} \right) - Y_{22} S_{222} X_{22} = 0 \qquad (10f) \end{aligned}$$

$$\begin{aligned} f_{y1} = & A_{00}^T Y_{00} + Y_{00} A_{00} + A_{10}^T Y_{10} + Y_{10}^T A_{10} \\ & + A_{20}^T Y_{20} + Y_{20}^T A_{20} - Y_{00} S_{002} Y_{00} - Y_{00} S_{001} X_{00} \\ & - Y_{20}^T S_{022}^T Y_{00} - Y_{00} S_{022} Y_{20} - Y_{20}^T S_{222} Y_{20} \\ & - Y_{10}^T S_{011}^T X_{00} - Y_{00} S_{011} X_{10} - Y_{10}^T S_{111} X_{10} \\ & - X_{00} S_{001} Y_{00} - X_{10}^T S_{011}^T Y_{00} - X_{00} S_{011} Y_{10} \\ & - X_{10}^T S_{111} Y_{10} + Q_{002} = 0 \end{aligned} \quad (10g)$$

$$\begin{aligned} f_{y2} = & Y_{00} A_{01} + Y_{10}^T A_{11} + \varepsilon_1 A_{00}^T Y_{10}^T + A_{10}^T Y_{11} \\ & + \sqrt{\alpha} A_{20}^T Y_{21} - \varepsilon_1 \left(Y_{00} S_{002} Y_{10}^T + Y_{20}^T S_{022}^T Y_{10}^T \right) \\ & - \sqrt{\alpha} \left(Y_{00} S_{022} Y_{21} + Y_{20}^T S_{222} Y_{21} \right) - \varepsilon_1 \left(Y_{00} S_{001} X_{10}^T \right. \\ & \left. + Y_{10}^T S_{011}^T X_{10}^T \right) - \left(Y_{00} S_{011} X_{11} + Y_{10}^T S_{111} X_{11} \right) \\ & - \varepsilon_1 \left(X_{00} S_{001} Y_{10}^T + X_{10}^T S_{011}^T Y_{10}^T \right) \\ & - \left(X_{00} S_{011} Y_{11} + X_{10}^T S_{111} Y_{11} \right) = 0 \end{aligned} \quad (10h)$$

$$\begin{aligned} f_{y3} = & Y_{00} A_{02} + Y_{20}^T A_{22} + \varepsilon_2 A_{00}^T Y_{20}^T + A_{20}^T Y_{22} \\ & + \frac{1}{\sqrt{\alpha}} A_{10}^T Y_{21}^T - \varepsilon_2 \left(Y_{00} S_{002} Y_{20}^T + Y_{20}^T S_{022}^T Y_{20}^T \right) \\ & - \left(Y_{00} S_{022} Y_{22} + Y_{20}^T S_{222} Y_{22} \right) - \varepsilon_2 \left(Y_{00} S_{001} X_{20}^T \right. \\ & \left. + Y_{10}^T S_{011}^T X_{20}^T \right) - \frac{1}{\sqrt{\alpha}} \left(Y_{00} S_{011} X_{21}^T + Y_{10}^T S_{111} X_{21}^T \right) \\ & - \varepsilon_2 \left(X_{00} S_{001} Y_{20}^T + X_{10}^T S_{011}^T Y_{20}^T \right) \\ & - \frac{1}{\sqrt{\alpha}} \left(X_{00} S_{011} Y_{21}^T + X_{10}^T S_{111} Y_{21}^T \right) + Q_{022} = 0 \end{aligned} \quad (10i)$$

$$\begin{aligned} f_{y4} = & A_{11}^T Y_{11} + Y_{11} A_{11} + \varepsilon_1 \left(A_{01}^T Y_{10}^T + Y_{10} A_{01} \right) \\ & - \varepsilon_1 \left(\varepsilon_1 Y_{10} S_{002} Y_{10}^T + \sqrt{\alpha} Y_{21}^T S_{022}^T Y_{10}^T \right. \\ & \left. + \sqrt{\alpha} Y_{10} S_{022} Y_{21} \right) - \alpha Y_{21}^T S_{222} Y_{21} \\ & - \varepsilon_1 \left(\varepsilon_1 Y_{10} S_{001} X_{10}^T + Y_{11} S_{011}^T X_{10}^T + Y_{10} S_{011} X_{11} \right) \\ & - Y_{11} S_{111} X_{11} - \varepsilon_1 \left(\varepsilon_1 X_{10} S_{001} Y_{10}^T + X_{11} S_{011}^T Y_{10}^T \right. \\ & \left. + X_{10} S_{011} Y_{11} \right) - X_{11} S_{111} Y_{11} = 0 \end{aligned} \quad (10j)$$

$$\begin{aligned} f_{y5} = & \varepsilon_1 Y_{10} A_{02} + \varepsilon_2 A_{01}^T Y_{20}^T + \sqrt{\alpha} Y_{21}^T A_{22} + \frac{1}{\sqrt{\alpha}} A_{11}^T Y_{21}^T \\ & - \varepsilon_1 \varepsilon_2 Y_{10} S_{002} Y_{20}^T - \varepsilon_2 \sqrt{\alpha} Y_{21}^T S_{022}^T Y_{20}^T \\ & - \varepsilon_1 Y_{10} S_{022} Y_{22} - \sqrt{\alpha} Y_{21}^T S_{222} Y_{22} - \varepsilon_1 \varepsilon_2 Y_{10} S_{001} X_{20}^T \\ & - \varepsilon_2 Y_{11} S_{011}^T X_{20}^T - \frac{\varepsilon_1}{\sqrt{\alpha}} Y_{10} S_{011} X_{21}^T - \frac{1}{\sqrt{\alpha}} Y_{11} S_{111} X_{21}^T \\ & - \varepsilon_1 \varepsilon_2 X_{10} S_{001} Y_{20}^T - \varepsilon_2 X_{11} S_{011}^T Y_{20}^T \\ & - \frac{\varepsilon_1}{\sqrt{\alpha}} X_{10} S_{011} Y_{21}^T - \frac{1}{\sqrt{\alpha}} X_{11} S_{111} Y_{21}^T = 0 \end{aligned} \quad (10k)$$

$$\begin{aligned} f_{y6} = & A_{22}^T Y_{22} + Y_{22} A_{22} + \varepsilon_2 \left(A_{02}^T Y_{20}^T + Y_{20} A_{02} \right) \\ & - \varepsilon_2 \left(\varepsilon_2 Y_{20} S_{002} Y_{20}^T + Y_{22} S_{022}^T Y_{20}^T + Y_{20} S_{022} Y_{22} \right) \\ & - Y_{22} S_{222} Y_{22} - \varepsilon_2 \left(\varepsilon_2 Y_{20} S_{001} X_{20}^T \right. \\ & \left. + \frac{1}{\sqrt{\alpha}} Y_{21} S_{011}^T X_{20}^T + \frac{1}{\sqrt{\alpha}} Y_{20} S_{011} X_{21}^T \right) - \frac{1}{\alpha} Y_{21} S_{111} X_{21}^T \\ & - \varepsilon_2 \left(\varepsilon_2 X_{20} S_{001} Y_{20}^T + \frac{1}{\sqrt{\alpha}} X_{21} S_{011}^T Y_{20}^T \right. \\ & \left. + \frac{1}{\sqrt{\alpha}} X_{20} S_{011} Y_{21}^T \right) - \frac{1}{\alpha} X_{21} S_{111} Y_{21}^T + Q_{222} = 0. \end{aligned} \quad (10l)$$

Therefore, we obtain the (11) as $\varepsilon_i \to +0, i = 1, 2$, where $\bar{X}_{lm}, \bar{Y}_{lm}, lm = 00, 10, 20, 11, 21, 22$ are the zeroth-order solutions

$$\begin{aligned} & A_{00}^T \bar{X}_{00} + \bar{X}_{00} A_{00} + A_{10}^T \bar{X}_{10} + \bar{X}_{10}^T A_{10} \\ & + A_{20}^T \bar{X}_{20} + \bar{X}_{20}^T A_{20} - \bar{X}_{00} S_{001} \bar{X}_{00} - \bar{X}_{00} S_{002} \bar{Y}_{00} \\ & - \bar{X}_{10}^T S_{011}^T \bar{X}_{00} - \bar{X}_{00} S_{011} \bar{X}_{10} - \bar{X}_{10}^T S_{111} \bar{X}_{10} \\ & - \bar{X}_{20}^T S_{022}^T \bar{Y}_{00} - \bar{X}_{00} S_{022} \bar{Y}_{20} - \bar{X}_{20}^T S_{222} \bar{Y}_{20} \\ & - \bar{Y}_{00} S_{002} \bar{X}_{00} - \bar{Y}_{20}^T S_{022}^T \bar{X}_{00} - \bar{Y}_{00} S_{022} \bar{X}_{20} \\ & - \bar{Y}_{20}^T S_{222} \bar{X}_{20} + Q_{001} = 0 \end{aligned} \quad (11a)$$

$$\begin{aligned} & \bar{X}_{00} A_{01} + \bar{X}_{10}^T A_{11} + A_{10}^T \bar{X}_{11} + \sqrt{\bar{\alpha}} A_{20}^T \bar{X}_{21} \\ & - \left(\bar{X}_{00} S_{011} \bar{X}_{11} + \bar{X}_{10}^T S_{111} \bar{X}_{11} \right) \\ & - \sqrt{\bar{\alpha}} \left(\bar{X}_{00} S_{022} \bar{Y}_{21} + \bar{X}_{20}^T S_{222} \bar{Y}_{21} \right) \\ & - \sqrt{\bar{\alpha}} \left(\bar{Y}_{00} S_{022} \bar{X}_{21} + \bar{Y}_{20}^T S_{222} \bar{X}_{21} \right) + Q_{011} = 0 \end{aligned} \quad (11b)$$

$$\begin{aligned} & \bar{X}_{00} A_{02} + \bar{X}_{20}^T A_{22} + A_{20}^T \bar{X}_{22} + \frac{1}{\sqrt{\bar{\alpha}}} A_{10}^T \bar{X}_{21}^T \\ & - \frac{1}{\sqrt{\bar{\alpha}}} \left(\bar{X}_{00} S_{011} \bar{X}_{21}^T + \bar{X}_{10}^T S_{111} \bar{X}_{21}^T \right) \\ & - \left(\bar{X}_{00} S_{022} \bar{Y}_{22} + \bar{X}_{20}^T S_{222} \bar{Y}_{22} \right) \\ & - \left(\bar{Y}_{00} S_{022} \bar{X}_{22} + \bar{Y}_{20}^T S_{222} \bar{X}_{22} \right) = 0 \end{aligned} \quad (11c)$$

$$\begin{aligned} & A_{11}^T \bar{X}_{11} + \bar{X}_{11} A_{11} - \bar{X}_{11} S_{111} \bar{X}_{11} - \bar{\alpha} \bar{X}_{21}^T S_{222} \bar{Y}_{21} \\ & - \bar{\alpha} \bar{Y}_{21}^T S_{222} \bar{X}_{21} + Q_{111} = 0 \end{aligned} \quad (11d)$$

$$\begin{aligned} & \sqrt{\bar{\alpha}} \bar{X}_{21}^T A_{22} + \frac{1}{\sqrt{\bar{\alpha}}} A_{11}^T \bar{X}_{21}^T - \frac{1}{\sqrt{\bar{\alpha}}} \bar{X}_{11} S_{111} \bar{X}_{21}^T \\ & - \sqrt{\bar{\alpha}} \bar{X}_{21}^T S_{222} \bar{Y}_{22} - \sqrt{\bar{\alpha}} \bar{Y}_{21}^T S_{222} \bar{X}_{22} = 0 \end{aligned} \quad (11e)$$

$$\begin{aligned} & A_{22}^T \bar{X}_{22} + \bar{X}_{22} A_{22} - \frac{1}{\bar{\alpha}} \bar{X}_{21} S_{111} \bar{X}_{21}^T - \bar{X}_{22} S_{222} \bar{Y}_{22} \\ & - \bar{Y}_{22} S_{222} \bar{X}_{22} = 0 \end{aligned} \quad (11f)$$

$$\begin{aligned} & A_{00}^T \bar{Y}_{00} + \bar{Y}_{00} A_{00} + A_{10}^T \bar{Y}_{10} + \bar{Y}_{10}^T A_{10} + A_{20}^T \bar{Y}_{20} + \bar{Y}_{20}^T A_{20} \\ & - \bar{Y}_{00} S_{002} \bar{Y}_{00} - \bar{Y}_{00} S_{001} \bar{X}_{00} - \bar{Y}_{20}^T S_{022}^T \bar{Y}_{00} \\ & - \bar{Y}_{00} S_{022} \bar{Y}_{20} - \bar{Y}_{20}^T S_{222} \bar{Y}_{20} - \bar{Y}_{10}^T S_{011}^T \bar{X}_{00} \\ & - \bar{Y}_{00} S_{011} \bar{X}_{10} - \bar{Y}_{10}^T S_{111} \bar{X}_{10} - \bar{X}_{00} S_{001} \bar{Y}_{00} \\ & - \bar{X}_{10}^T S_{011}^T \bar{Y}_{00} - \bar{X}_{00} S_{011} \bar{Y}_{10} - \bar{X}_{10}^T S_{111} \bar{Y}_{10} + Q_{002} = 0 \end{aligned} \quad (11g)$$

$$\begin{aligned} & \bar{Y}_{00} A_{01} + \bar{Y}_{10}^T A_{11} + A_{10}^T \bar{Y}_{11} + \sqrt{\bar{\alpha}} A_{20}^T \bar{Y}_{21} \\ & - \sqrt{\bar{\alpha}} \left(\bar{Y}_{00} S_{022} \bar{Y}_{21} + \bar{Y}_{20}^T S_{222} \bar{Y}_{21} \right) \\ & - \left(\bar{Y}_{00} S_{011} \bar{X}_{11} + \bar{Y}_{10}^T S_{111} \bar{X}_{11} \right) \\ & - \left(\bar{X}_{00} S_{011} \bar{Y}_{11} + \bar{X}_{10}^T S_{111} \bar{Y}_{11} \right) = 0 \end{aligned} \quad (11h)$$

$$\begin{aligned} & \bar{Y}_{00} A_{02} + \bar{Y}_{20}^T A_{22} + A_{20}^T \bar{Y}_{22} + \frac{1}{\sqrt{\bar{\alpha}}} A_{10}^T \bar{Y}_{21}^T \\ & - \left(\bar{Y}_{00} S_{022} \bar{Y}_{22} + \bar{Y}_{20}^T S_{222} \bar{Y}_{22} \right) \\ & - \frac{1}{\sqrt{\bar{\alpha}}} \left(\bar{Y}_{00} S_{011} \bar{X}_{21}^T + \bar{Y}_{10}^T S_{111} \bar{X}_{21}^T \right) \\ & - \frac{1}{\sqrt{\bar{\alpha}}} \left(\bar{X}_{00} S_{011} \bar{Y}_{21}^T + \bar{X}_{10}^T S_{111} \bar{Y}_{21}^T \right) + Q_{022} = 0 \end{aligned} \quad (11i)$$

$$\begin{aligned} & A_{11}^T \bar{Y}_{11} + \bar{Y}_{11} A_{11} - \bar{\alpha} \bar{Y}_{21}^T S_{222} \bar{Y}_{21} - \bar{Y}_{11} S_{111} \bar{X}_{11} \\ & - \bar{X}_{11} S_{111} \bar{Y}_{11} = 0 \end{aligned} \quad (11j)$$

$$\sqrt{\bar{\alpha}}\bar{Y}_{21}^T A_{22} + \frac{1}{\sqrt{\bar{\alpha}}} A_{11}^T \bar{Y}_{21}^T - \sqrt{\bar{\alpha}}\bar{Y}_{21}^T S_{222}\bar{Y}_{22} - \frac{1}{\sqrt{\bar{\alpha}}}\bar{Y}_{11}S_{111}\bar{X}_{21}^T - \frac{1}{\sqrt{\bar{\alpha}}}\bar{X}_{11}S_{111}\bar{Y}_{21}^T = 0 \tag{11k}$$

$$A_{22}^T\bar{Y}_{22} + \bar{Y}_{22}A_{22} - \bar{Y}_{22}S_{222}\bar{Y}_{22} - \frac{1}{\bar{\alpha}}\bar{Y}_{21}S_{111}\bar{X}_{21}^T - \frac{1}{\bar{\alpha}}\bar{X}_{21}S_{111}\bar{Y}_{21}^T + Q_{222} = 0. \tag{11l}$$

If Assumption 2 holds, there exist the matrices $\tilde{X}_{11} \geq 0$ and $\tilde{Y}_{22} \geq 0$ such that the matrices $A_{11} - S_{111}\tilde{X}_{11}$ and $A_{22} - S_{222}\tilde{Y}_{22}$ are stable, where $A_{11}^T\tilde{X}_{11} + \tilde{X}_{11}A_{11} - \tilde{X}_{11}S_{111}\tilde{X}_{11} + Q_{11} = 0$ and $A_{22}^T\tilde{X}_{22} + \tilde{X}_{22}A_{22} - \tilde{X}_{22}S_{222}\tilde{X}_{22} + Q_{22} = 0$. Now we choose $\bar{X}_{11}$ and $\bar{Y}_{22}$ to be $\tilde{X}_{11}$ and $\tilde{Y}_{22}$, respectively. Then, there exist $\lambda_x \in \mathbf{C}$ and $\lambda_y \in \mathbf{C}$ such that

$$(A_{11} - S_{111}\tilde{X}_{11})v_x = \lambda_x v_x, \quad \mathrm{Re}(\lambda_x) < 0 \tag{12a}$$

$$(A_{22} - S_{222}\tilde{Y}_{22})v_y = \lambda_y v_y, \quad \mathrm{Re}(\lambda_y) < 0 \tag{12b}$$

where $v_x \in \mathbf{C}^{n_1}$ and $v_y \in \mathbf{C}^{n_2}$ are any vectors.

Using (12) we can change (11f) and (11j) as follows:

(11f)

$$\Leftrightarrow v_y^T(A_{22} - S_{222}\tilde{Y}_{22})^T\bar{X}_{22}v_y + v_y^T\bar{X}_{22}(A_{22} - S_{222}\tilde{Y}_{22})v_y - \frac{1}{\bar{\alpha}}v_y^T\bar{X}_{21}S_{111}\bar{X}_{21}^T v_y = 0$$

$$\Leftrightarrow 2\lambda_y v_y^T\bar{X}_{22}v_y - \frac{1}{\bar{\alpha}}v_y^T\bar{X}_{21}S_{111}\bar{X}_{21}^T v_y = 0 \tag{13a}$$

(11j)

$$\Leftrightarrow v_x^T(A_{11} - S_{111}\tilde{X}_{11})^T\bar{Y}_{11}v_x + v_x^T\bar{Y}_{11}(A_{11} - S_{111}\tilde{X}_{11})v_x - \bar{\alpha}v_x^T\bar{Y}_{21}S_{222}\bar{Y}_{21}^T v_x = 0$$

$$\Leftrightarrow 2\lambda_x v_x^T\bar{Y}_{11}v_x - \bar{\alpha}v_x^T\bar{Y}_{21}S_{222}\bar{Y}_{21}^T v_x = 0. \tag{13b}$$

Taking $\mathrm{Re}(\lambda_x) < 0$ and $\mathrm{Re}(\lambda_y) < 0$ into account, we have $\bar{X}_{22} = \bar{Y}_{11} = 0$. Then, from (11e) and (11k), we obtain

$$\sqrt{\bar{\alpha}}\bar{X}_{21}^T(A_{22} - S_{222}\bar{Y}_{22}) + \frac{1}{\sqrt{\bar{\alpha}}}(A_{11} - S_{111}\bar{X}_{11})^T\bar{X}_{21}^T = 0 \tag{14a}$$

$$\sqrt{\bar{\alpha}}\bar{Y}_{21}^T(A_{22} - S_{222}\bar{Y}_{22}) + \frac{1}{\sqrt{\bar{\alpha}}}(A_{11} - S_{111}\bar{X}_{11})^T\bar{Y}_{21}^T = 0. \tag{14b}$$

Hence, the unique solutions of (11f) and (11j) are given by $\bar{X}_{21} = \bar{Y}_{21} = 0$ because of the stability of $A_{11} - S_{111}\bar{X}_{11}$ and $A_{22} - S_{222}\bar{Y}_{22}$. Thus the parameter $\bar{\alpha}$ does not appear in (11), namely, it does not affect the (11) in the limit when ε_1 and ε_2 tend to zero. Therefore, we obtain the zeroth-order equations (15).

$$\begin{aligned} &A_{00}^T\bar{X}_{00} + \bar{X}_{00}A_{00} + A_{10}^T\bar{X}_{10} + \bar{X}_{10}^T A_{10} + A_{20}^T\bar{X}_{20} + \bar{X}_{20}^T A_{20} - \bar{X}_{00}S_{001}\bar{X}_{00} - \bar{X}_{00}S_{002}\bar{Y}_{00} \\ &- \bar{X}_{10}^T S_{011}^T\bar{X}_{00} - \bar{X}_{00}S_{011}\bar{X}_{10} - \bar{X}_{10}^T S_{111}\bar{X}_{10} - \bar{X}_{20}^T S_{022}^T\bar{Y}_{00} - \bar{X}_{00}S_{022}\bar{Y}_{20} - \bar{X}_{20}^T S_{222}\bar{Y}_{20} \\ &- \bar{Y}_{00}S_{002}\bar{X}_{00} - \bar{Y}_{20}^T S_{022}^T\bar{X}_{00} - \bar{Y}_{00}S_{022}\bar{X}_{20} - \bar{Y}_{20}^T S_{222}\bar{X}_{20} + Q_{001} = 0 \end{aligned} \tag{15a}$$

$$\bar{X}_{00}A_{01} + \bar{X}_{10}^T A_{11} + A_{10}^T\bar{X}_{11} - \left(\bar{X}_{00}S_{011}\bar{X}_{11} + \bar{X}_{10}^T S_{111}\bar{X}_{11}\right) + Q_{011} = 0 \tag{15b}$$

$$\bar{X}_{00}A_{02} + \bar{X}_{20}^T A_{22} - \left(\bar{X}_{00}S_{022}\bar{Y}_{22} + \bar{X}_{20}^T S_{222}\bar{Y}_{22}\right) = 0 \tag{15c}$$

$$A_{11}^T\bar{X}_{11} + \bar{X}_{11}A_{11} - \bar{X}_{11}S_{111}\bar{X}_{11} + Q_{111} = 0 \tag{15d}$$

$$\begin{aligned} &A_{00}^T\bar{Y}_{00} + \bar{Y}_{00}A_{00} + A_{10}^T\bar{Y}_{10} + \bar{Y}_{10}^T A_{10} + A_{20}^T\bar{Y}_{20} + \bar{Y}_{20}^T A_{20} - \bar{Y}_{00}S_{002}\bar{Y}_{00} - \bar{Y}_{00}S_{001}\bar{X}_{00} \\ &- \bar{Y}_{20}^T S_{022}^T\bar{Y}_{00} - \bar{Y}_{00}S_{022}\bar{Y}_{20} - \bar{Y}_{20}^T S_{222}\bar{Y}_{20} - \bar{Y}_{10}^T S_{011}^T\bar{X}_{00} - \bar{Y}_{00}S_{011}\bar{X}_{10} - \bar{Y}_{10}^T S_{111}\bar{X}_{10} \\ &- \bar{X}_{00}S_{001}\bar{Y}_{00} - \bar{X}_{10}^T S_{011}^T\bar{Y}_{00} - \bar{X}_{00}S_{011}\bar{Y}_{10} - \bar{X}_{10}^T S_{111}\bar{Y}_{10} + Q_{002} = 0 \end{aligned} \tag{15e}$$

$$\bar{Y}_{00}A_{01} + \bar{Y}_{10}^T A_{11} - (\bar{Y}_{00}S_{011}\bar{X}_{11} + \bar{Y}_{10}^T S_{111}\bar{X}_{11}) = 0 \tag{15f}$$

$$\bar{Y}_{00}A_{02} + \bar{Y}_{20}^T A_{22} + A_{20}^T\bar{Y}_{22} - \left(\bar{Y}_{00}S_{022}\bar{Y}_{22} + \bar{Y}_{20}^T S_{222}\bar{Y}_{22}^T\right) + Q_{022} = 0 \tag{15g}$$

$$A_{22}^T\bar{Y}_{22} + \bar{Y}_{22}A_{22} - \bar{Y}_{22}S_{222}\bar{Y}_{22} + Q_{222} = 0 \tag{15h}$$

The Nash equilibrium strategies for the MSPS will be studied under the following basic assumption, so that one can apply the proposed method to the nonstandard MSPS.

Assumption 3: The Hamiltonian matrices $T_{iii}, i = 1, 2$ are nonsingular, where

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \tag{16}$$

Under Assumptions 2 and 3, we obtain the following zeroth-order equations:

$$A_s^T\bar{X}_{00} + \bar{X}_{00}A_s + Q_{s1} - \bar{X}_{00}S_{s1}\bar{X}_{00} - \bar{X}_{00}S_{s2}\bar{Y}_{00} - \bar{Y}_{00}S_{s2}\bar{X}_{00} = 0 \tag{17a}$$

$$A_s^T\bar{Y}_{00} + \bar{Y}_{00}A_s + Q_{s2} - \bar{Y}_{00}S_{s2}\bar{Y}_{00} - \bar{Y}_{00}S_{s1}\bar{X}_{00} - \bar{X}_{00}S_{s1}\bar{Y}_{00} = 0 \tag{17b}$$

$$A_{11}^T\bar{X}_{11} + \bar{X}_{11}A_{11} - \bar{X}_{11}S_{111}\bar{X}_{11} + Q_{111} = 0 \tag{17c}$$

$$A_{22}^T\bar{Y}_{22} + \bar{Y}_{22}A_{22} - \bar{Y}_{22}S_{222}\bar{Y}_{22} + Q_{222} = 0 \tag{17d}$$

$$\bar{X}_{10} = -D_{x11}^{-T}D_{x01}^T\bar{X}_{00} - D_{x11}^{-T}N_{x01}^T \tag{17e}$$

$$\bar{Y}_{10} = -D_{x11}^{-T}D_{x01}^T\bar{Y}_{00} \tag{17f}$$

$$\bar{X}_{20} = -D_{y22}^{-T}D_{y02}^T\bar{X}_{00} \tag{17g}$$

$$\bar{Y}_{20} = -D_{y22}^{-T}D_{y02}^T\bar{Y}_{00} - D_{y22}^{-T}N_{y02}^T \tag{17h}$$

where

$$\begin{aligned} A_s &= A_{00} - D_{x01}D_{x11}^{-1}A_{10} - D_{y02}D_{y22}^{-1}A_{20} + \left(S_{011} - D_{x01}D_{x11}^{-1}S_{111}\right)D_{x11}^{-T}N_{x01}^T + \left(S_{022} - D_{y02}D_{y22}^{-1}S_{222}\right)D_{y22}^{-T}N_{y02}^T \\ S_{s1} &= S_{001} - D_{x01}D_{x11}^{-1}S_{011}^T - S_{011}D_{x11}^{-T}D_{x01}^T + D_{x01}D_{x11}^{-1}S_{111}D_{x11}^{-T}D_{x01}^T \\ S_{s2} &= S_{002} - D_{y02}D_{y22}^{-1}S_{022}^T - S_{022}D_{y22}^{-T}D_{y02}^T + D_{y02}D_{y22}^{-1}S_{222}D_{y22}^{-T}D_{y02}^T \\ Q_{s1} &= Q_{001} - A_{10}^T D_{x11}^{-T}N_{x01}^T - N_{x01}D_{x11}^{-1}A_{10} - N_{x01}D_{x11}^{-1}S_{111}D_{x11}^{-T}N_{x01}^T \\ Q_{s2} &= Q_{002} - A_{20}^T D_{y22}^{-T}N_{y02}^T - N_{y02}D_{y22}^{-1}A_{20} - N_{y02}D_{y22}^{-1}S_{222}D_{y22}^{-T}N_{y02}^T \end{aligned}$$

$$D_{x01} = A_{01} - S_{011}\bar{X}_{11}$$
$$D_{x11} = A_{11} - S_{111}\bar{X}_{11}$$
$$D_{y02} = A_{02} - S_{022}\bar{Y}_{22}$$
$$D_{y22} = A_{22} - S_{222}\bar{Y}_{22}$$
$$N_{x01} = A_{10}^T\bar{X}_{11} + Q_{011}$$
$$N_{y02} = A_{20}^T\bar{Y}_{22} + Q_{022}.$$

Lemma 3: The matrices $A_s, S_{s1}, S_{s2}, Q_{s1}$, and Q_{s2} do not depend on the matrices $\bar{X}_{11}, \bar{X}_{21}, \bar{Y}_{21}$, and $\bar{Y}_{22}$. Therefore, their matrices can be calculated by using the following Hamiltonian matrices:

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix} = \begin{bmatrix} A_{00} & * \\ * & -A_{00}^T \end{bmatrix} - T_{011}T_{111}^{-1}T_{101} - T_{022}T_{222}^{-1}T_{202} \tag{18a}$$

$$\begin{bmatrix} * & -S_{s1} \\ -Q_{s1} & * \end{bmatrix} = T_{001} - T_{011}T_{111}^{-1}T_{101} \tag{18b}$$

$$\begin{bmatrix} * & -S_{s2} \\ -Q_{s2} & * \end{bmatrix} = T_{002} - T_{022}T_{222}^{-1}T_{202} \tag{18c}$$

$$T_{001} = \begin{bmatrix} A_{00} & -S_{001} \\ -Q_{001} & -A_{00}^T \end{bmatrix}$$
$$T_{011} = \begin{bmatrix} A_{01} & -S_{011} \\ -Q_{011} & -A_{10}^T \end{bmatrix}$$
$$T_{101} = \begin{bmatrix} A_{10} & -S_{011}^T \\ -Q_{011}^T & -A_{01}^T \end{bmatrix}$$
$$T_{111} = \begin{bmatrix} A_{11} & -S_{111} \\ -Q_{111} & -A_{11}^T \end{bmatrix}$$
$$T_{002} = \begin{bmatrix} A_{00} & -S_{002} \\ -Q_{002} & -A_{00}^T \end{bmatrix}$$
$$T_{022} = \begin{bmatrix} A_{02} & -S_{022} \\ -Q_{022} & -A_{20}^T \end{bmatrix}$$
$$T_{202} = \begin{bmatrix} A_{20} & -S_{022}^T \\ -Q_{022}^T & -A_{02}^T \end{bmatrix}$$
$$T_{222} = \begin{bmatrix} A_{22} & -S_{222} \\ -Q_{222} & -A_{22}^T \end{bmatrix}$$

where $*$ stands for an appropriate matrix. Moreover, we can change the form of the solutions $\bar{X}_{10}, \bar{X}_{20}, \bar{Y}_{10}$, and $\bar{Y}_{20}$.

$$[\bar{X}_{10} \quad \bar{Y}_{10}] = [\bar{X}_{11} \quad -I_{n_1}]T_{111}^{-1}T_{101}\begin{bmatrix} I_{n_0} & 0 \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix} \tag{19a}$$

$$[\bar{X}_{20} \quad \bar{Y}_{20}] = [\bar{Y}_{22} \quad -I_{n_2}]T_{222}^{-1}T_{202}\begin{bmatrix} 0 & I_{n_0} \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix}. \tag{19b}$$

Proof: Note the relation

$$T_{111} = \begin{bmatrix} I_{n_1} & 0 \\ \bar{X}_{11} & I_{n_1} \end{bmatrix}\begin{bmatrix} D_{x11} & -S_{111} \\ 0 & -D_{x11}^T \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 \\ -\bar{X}_{11} & I_{n_1} \end{bmatrix}.$$

Since T_{111} is nonsingular under Assumption 3 and the ARE (17c) has a stabilizing solution under Assumption 2, D_{x11} is also nonsingular. This means that T_{111}^{-1} can be expressed explicitly in terms of D_{x11}^{-1}. Using the similar manner, we have the following relations:

$$T_{111}^{-1} = \begin{bmatrix} I_{n_1} & 0 \\ \bar{X}_{11} & I_{n_1} \end{bmatrix}\begin{bmatrix} D_{x11}^{-1} & -D_{x11}^{-1}S_{111}D_{x11}^{-T} \\ 0 & -D_{x11}^{-T} \end{bmatrix} \cdot \begin{bmatrix} I_{n_1} & 0 \\ -\bar{X}_{11} & I_{n_1} \end{bmatrix} \tag{20a}$$

$$T_{222}^{-1} = \begin{bmatrix} I_{n_2} & 0 \\ \bar{Y}_{22} & I_{n_2} \end{bmatrix}\begin{bmatrix} D_{y22}^{-1} & -D_{y22}^{-1}S_{222}D_{y22}^{-T} \\ 0 & -D_{y22}^{-T} \end{bmatrix} \cdot \begin{bmatrix} I_{n_2} & 0 \\ -\bar{Y}_{22} & I_{n_2} \end{bmatrix}. \tag{20b}$$

Therefore, it suffices the Proof of Lemma 3 to show that the relations (18) hold. These formulations can be proved after direct algebraic manipulations, which are omitted here for brevity. ■

The following theorem will establish the relation between the solutions X and Y and the zeroth-order solutions $\bar{X}_{lm}$ and $\bar{Y}_{lm}, lm = 00, 10, 20, 11, 21, 22$ for the reduced-order equations (17).

Theorem 1: Suppose that

$$\det\begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s & -[(S_{s1}\bar{Y}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s1}\bar{Y}_{00})^T] \\ -[(S_{s2}\bar{X}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s2}\bar{X}_{00})^T] & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s \end{bmatrix} \neq 0 \tag{21}$$

where $\hat{A}_s := A_s - S_{s1}\bar{X}_{00} - S_{s2}\bar{Y}_{00}$ and the matrix $\hat{A}_s$ is stable.

Under Assumptions 1–3, the GCMARE (8) admits the stabilizing solutions X and Y such that these matrices possess a power series expansion at $||\mu|| = 0$. That is

$$X = \begin{bmatrix} \bar{X}_{00} & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & 0 \end{bmatrix} + O(||\mu||) = \bar{X} + O(||\mu||) \tag{22a}$$

$$Y = \begin{bmatrix} \bar{Y}_{00} & 0 & 0 \\ \bar{Y}_{10} & 0 & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix} + O(||\mu||) = \bar{Y} + O(||\mu||). \tag{22b}$$

Proof: We apply the implicit function theorem [10] to the partitioned GCMARE (10). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $||\mu|| = 0$. It can be shown, after some algebra, that the Jacobian of (10) in the limit as $\mu \to \mu_0$ is given by (23), shown at the bottom of the next page, where

$$J_{00} = I_{n_0} \otimes D_{00}^T + D_{00}^T \otimes I_{n_0}$$
$$J_{01} = I_{n_0} \otimes D_{x10}^T + \left(D_{x10}^T \otimes I_{n_0}\right)U_{n_1n_0}$$
$$J_{02} = I_{n_0} \otimes D_{y20}^T + \left(D_{y20}^T \otimes I_{n_0}\right)U_{n_2n_0}$$
$$J_{10} = D_{x01}^T \otimes I_{n_0}, \quad J_{20} = D_{y02}^T \otimes I_{n_0}$$
$$J_{11} = \left(D_{x11}^T \otimes I_{n_0}\right)U_{n_1n_0}$$
$$J_{22} = \left(D_{y22}^T \otimes I_{n_0}\right)U_{n_2n_0}$$
$$J_{13} = I_{n_1} \otimes D_{x10}^T$$
$$J_{14} = \sqrt{\bar{\alpha}}\left(I_{n_1} \otimes D_{y20}^T\right)$$

$$
\begin{aligned}
J_{24} &= \sqrt{\bar{\alpha}}^{-1}\left(I_{n_2}\otimes D_{x10}^T\right)U_{n_2n_1}\\
J_{25} &= I_{n_2}\otimes D_{y20}^T\\
J_{33} &= I_{n_1}\otimes D_{x11}^T + D_{x11}^T\otimes I_{n_1}\\
J_{44} &= \sqrt{\bar{\alpha}}\left(D_{y22}^T\otimes I_{n_1}\right)U_{n_2n_1} + \sqrt{\bar{\alpha}}^{-1}\left(I_{n_2}\otimes D_{x11}^T\right)U_{n_2n_1}\\
J_{55} &= I_{n_2}\otimes D_{y22}^T + D_{y22}^T\otimes I_{n_2}\\
J_{06} &= -I_{n_0}\otimes E_{x00}^T - E_{x00}^T\otimes I_{n_0}\\
J_{08} &= -I_{n_0}\otimes E_{x20}^T - \left(E_{x20}^T\otimes I_{n_0}\right)U_{n_2n_0}\\
J_{110} &= -\sqrt{\bar{\alpha}}(I_{n_1}\otimes E_{x20})\\
J_{211} &= -I_{n_2}\otimes E_{x20}\\
J_{60} &= -I_{n_0}\otimes E_{y00}^T - E_{y00}^T\otimes I_{n_0}\\
J_{61} &= -I_{n_0}\otimes E_{y10}^T - \left(E_{y10}^T\otimes I_{n_0}\right)U_{n_1n_0}\\
J_{73} &= -I_{n_1}\otimes E_{y10}^T\\
J_{84} &= -\sqrt{\bar{\alpha}}^{-1}\left(I_{n_2}\otimes E_{y10}^T\right)U_{n_2n_1}\\
D_{00} &= A_{00} - S_{001}\bar{X}_{00} - S_{002}\bar{Y}_{00} - S_{011}\bar{X}_{10} - S_{022}\bar{Y}_{20}\\
D_{x10} &= A_{10} - S_{011}^T\bar{X}_{00} - S_{111}\bar{X}_{10}\\
D_{y20} &= A_{20} - S_{022}^T\bar{Y}_{00} - S_{222}\bar{Y}_{20}\\
E_{x00} &= S_{002}\bar{X}_{00} + S_{022}\bar{X}_{20}\\
E_{x20} &= S_{022}^T\bar{X}_{00} + S_{222}\bar{X}_{20}\\
E_{y00} &= S_{001}\bar{Y}_{00} + S_{011}\bar{Y}_{10}\\
E_{y10} &= S_{011}^T\bar{Y}_{00} + S_{111}\bar{Y}_{10}.
\end{aligned}
$$

The Jacobian (23) can be expressed as

$$
\begin{aligned}
\det J &= \prod_{j=3}^{5}(\det J_{jj})^2\\
&\quad\cdot\det\begin{bmatrix} J_{00} & J_{01} & J_{02} & J_{06} & 0 & J_{08}\\ J_{10} & J_{11} & 0 & 0 & 0 & 0\\ J_{20} & 0 & J_{22} & 0 & 0 & 0\\ J_{60} & J_{61} & 0 & J_{00} & J_{01} & J_{02}\\ 0 & 0 & 0 & J_{10} & J_{11} & 0\\ 0 & 0 & 0 & J_{20} & 0 & J_{22}\end{bmatrix}\\
&= \left[\prod_{j=1}^{5}(\det J_{jj})^2\right]\\
&\quad\cdot\det\begin{bmatrix} J_0 & J_{06}-J_{08}J_{22}^{-1}J_{20}\\ J_{60}-J_{61}J_{11}^{-1}J_{10} & J_0\end{bmatrix} \qquad (24)
\end{aligned}
$$

where $J_s = J_{00} - J_{01}J_{11}^{-1}J_{10} - J_{02}J_{22}^{-1}J_{20}$.

After some straightforward but tedious algebra, it is easy to show that the following relations hold:

$$J_s = \hat{A}_s^T\otimes I_{n_0} + I_{n_0}\otimes\hat{A}_s^T \qquad (25a)$$

$$J_{06} - J_{08}J_{22}^{-1}J_{20} = -[(S_{s2}\bar{X}_{00})^T\otimes I_{n_0} + I_{n_0}\otimes(S_{s2}\bar{X}_{00})^T] \qquad (25b)$$

$$J_{60} - J_{61}J_{11}^{-1}J_{10} = -[(S_{s1}\bar{Y}_{00})^T\otimes I_{n_0} + I_{n_0}\otimes(S_{s1}\bar{Y}_{00})^T] \qquad (25c)$$

where

$$
\begin{aligned}
\hat{A}_s &= D_{00} - D_{x01}D_{x11}^{-1}D_{x10} - D_{y02}D_{y22}^{-1}D_{y20}\\
S_{s1}\bar{Y}_{00} &= E_{y00} - D_{x01}D_{x11}^{-1}E_{y10}\\
S_{s2}\bar{X}_{00} &= E_{x00} - D_{y02}D_{y22}^{-1}E_{x20}.
\end{aligned}
$$

Hence, we have

$$\det J = \left[\prod_{j=1}^{5}(\det J_{jj})^2\right]\cdot\det\Gamma. \qquad (26)$$

Obviously, $J_{jj}, j = 1,2,\ldots,5$ are nonsingular because the matrices $D_{x11} = A_{11} - S_{111}\bar{X}_{11}$ and $D_{y22} = A_{22} - S_{222}\bar{Y}_{22}$ are stable under Assumption 2. By the nonsingularity assumption of the matrix $\Gamma, \det J \neq 0$, i.e., J is nonsingular at $(\mu,\mathbf{P}) = (\mu_0,\mathbf{P}_0)$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ■

It is worth pointing out that the asymptotic structures of the solutions X and Y are established without the assumption that the matrices $A_{ii}, i = 1,2$ are nonsingular compared with the existing result in [5]. Moreover, for the existence of these solutions the conservative condition which is made in [21] is not

$$
\begin{aligned}
J &= \left.\frac{\partial\mathrm{vec}(f_{x1},f_{x2},f_{x3},f_{x4},f_{x5},f_{x6},f_{y1},f_{y2},f_{y3},f_{y4},f_{y5},f_{y6})}{\partial\mathrm{vec}(X_{00},X_{10},X_{20},X_{11},X_{21},X_{22},Y_{00},Y_{10},Y_{20},Y_{11},Y_{21},Y_{22})^T}\right|_{\mu=\mu_0,\mathbf{P}=\mathbf{P}_0}\\
&= \begin{bmatrix}
J_{00} & J_{01} & J_{02} & 0 & 0 & 0 & J_{06} & 0 & J_{08} & 0 & 0 & 0\\
J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 & 0 & 0 & 0 & 0 & J_{110} & 0\\
J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} & 0 & 0 & 0 & 0 & 0 & J_{211}\\
0 & 0 & 0 & J_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & J_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & J_{55} & 0 & 0 & 0 & 0 & 0 & 0\\
J_{60} & J_{61} & 0 & 0 & 0 & 0 & J_{00} & J_{01} & J_{02} & 0 & 0 & 0\\
0 & 0 & 0 & J_{73} & 0 & 0 & J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0\\
0 & 0 & 0 & 0 & J_{84} & 0 & J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{33} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{44} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{55}
\end{bmatrix}\\
\mu &= [\varepsilon_1 \quad \varepsilon_2]^T\\
\mu_0 &= [0 \quad 0]^T\\
\mathbf{P} &= (X_{00},X_{10},X_{20},X_{11},X_{21},X_{22},Y_{00},Y_{10},Y_{20},Y_{11},Y_{21},Y_{22})\\
\mathbf{P}_0 &= (\bar{X}_{00},\bar{X}_{10},\bar{X}_{20},\bar{X}_{11},0,0,\bar{Y}_{00},\bar{Y}_{10},\bar{Y}_{20},0,0,\bar{Y}_{22}) \qquad (23)
\end{aligned}
$$

needed. That is, the existence condition is newly derived in detail compared with [5], [21]. As a result, we succeed in finding the asymptotic structures of the solutions X and Y in same dimension of the reduced-order subsystems even if the matrices $A_{ii}, i = 1, 2$ are singular.

IV. Newton's Method

In order to improve the convergence rate of the Lyapunov iterations [3], we propose the following new algorithm which is based on the Newton's method:

$$\Phi^{(n)T}\mathcal{P}^{(n+1)} + \mathcal{P}^{(n+1)T}\Phi^{(n)} - \Theta^{(n)T}\mathcal{P}^{(n+1)}\mathcal{J} - \mathcal{J}\mathcal{P}^{(n+1)T}\Theta^{(n)} + \Xi^{(n)} = 0, \qquad n = 0, 1, \ldots$$
$$\Leftrightarrow \begin{cases} \Phi_1^{(n)T}X^{(n+1)} + X^{(n+1)T}\Phi_1^{(n)} \\ -\Theta_2^{(n)T}Y^{(n+1)} - Y^{(n+1)T}\Theta_2^{(n)} + \Xi_1^{(n)} = 0 \\ \Phi_2^{(n)T}Y^{(n+1)} + Y^{(n+1)T}\Phi_2^{(n)} \\ -\Theta_1^{(n)T}X^{(n+1)} - X^{(n+1)T}\Theta_1^{(n)} + \Xi_2^{(n)} = 0 \end{cases} \tag{27}$$

where

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\mathcal{P}^{(n)} - \mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix}$$
$$\Theta^{(n)} := \tilde{S}\mathcal{J}\mathcal{P}^{(n)} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix}$$
$$\Xi^{(n)} := \tilde{Q} + \mathcal{P}^{(n)T}\tilde{S}\mathcal{P}^{(n)} + \mathcal{J}\mathcal{P}^{(n)T}\tilde{S}\mathcal{J}\mathcal{P}^{(n)} + \mathcal{P}^{(n)T}\mathcal{J}\tilde{S}\mathcal{P}^{(n)}\mathcal{J} = \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix}$$
$$\Phi_i^{(n)} := \begin{bmatrix} \Phi_{00i}^{(n)} & \Phi_{01i}^{(n)} & \Phi_{02i}^{(n)} \\ \Phi_{10i}^{(n)} & \Phi_{11i}^{(n)} & \Phi_{12i}^{(n)} \\ \Phi_{20i}^{(n)} & \Phi_{21i}^{(n)} & \Phi_{22i}^{(n)} \end{bmatrix}$$
$$\Theta_i^{(n)} := \begin{bmatrix} \Theta_{00i}^{(n)} & \Theta_{01i}^{(n)} & \Theta_{02i}^{(n)} \\ \Theta_{10i}^{(n)} & \Theta_{11i}^{(n)} & \Theta_{12i}^{(n)} \\ \Theta_{20i}^{(n)} & \Theta_{21i}^{(n)} & \Theta_{22i}^{(n)} \end{bmatrix}$$
$$\Xi_i^{(n)} := \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} & \Xi_{02i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} & \Xi_{12i}^{(n)} \\ \Xi_{02i}^{(n)T} & \Xi_{12i}^{(n)T} & \Xi_{22i}^{(n)} \end{bmatrix}, \qquad i = 1, 2$$
$$\mathcal{P}^{(n)} := \begin{bmatrix} X^{(n)} & 0 \\ 0 & Y^{(n)} \end{bmatrix}$$
$$X^{(n)} := \begin{bmatrix} X_{00}^{(n)} & \varepsilon_1 X_{10}^{(n)T} & \varepsilon_2 X_{20}^{(n)T} \\ X_{10}^{(n)} & X_{11}^{(n)} & \sqrt{\alpha}^{-1}X_{21}^{(n)T} \\ X_{20}^{(n)} & \sqrt{\alpha}X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}$$
$$Y^{(n)} := \begin{bmatrix} Y_{00}^{(n)} & \varepsilon_1 Y_{10}^{(n)T} & \varepsilon_2 Y_{20}^{(n)T} \\ Y_{10}^{(n)} & Y_{11}^{(n)} & \sqrt{\alpha}^{-1}Y_{21}^{(n)T} \\ Y_{20}^{(n)} & \sqrt{\alpha}Y_{21}^{(n)} & Y_{22}^{(n)} \end{bmatrix}$$
$$\tilde{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$
$$\tilde{Q} := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$
$$\tilde{S} := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$
$$\mathcal{J} := \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$$

and the initial condition $\mathcal{P}^{(0)}$ has the following form:

$$\mathcal{P}^{(0)} = \begin{bmatrix} X^{(0)} & 0 \\ 0 & Y^{(0)} \end{bmatrix} = \begin{bmatrix} \bar{X}_{00} & \varepsilon_1\bar{X}_{10}^T & \varepsilon_2\bar{X}_{20}^T & 0 & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 & 0 & 0 & 0 \\ \bar{X}_{20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{Y}_{00} & \varepsilon_1\bar{Y}_{10}^T & \varepsilon_2\bar{Y}_{20}^T \\ 0 & 0 & 0 & \bar{Y}_{10} & 0 & 0 \\ 0 & 0 & 0 & \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix} \tag{28}$$

Note that the considered algorithm (27) is original. The new algorithm (27) can be constructed by setting $\mathcal{P}^{(n+1)} = \mathcal{P}^{(n)} + \Delta\mathcal{P}^{(n)}$ and neglecting $O(\Delta\mathcal{P}^{(n)T}\Delta\mathcal{P}^{(n)})$ term. Newton's method is well known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood.

First we show that the algorithm (27) is equivalent to the Newton's method. Now, let us define the following matrix function:

$$\mathcal{F}(\mathcal{P}) := \tilde{A}^T\mathcal{P} + \mathcal{P}^T\tilde{A} + \tilde{Q} - \mathcal{P}^T\tilde{S}\mathcal{P} - \mathcal{J}\mathcal{P}^T\tilde{S}\mathcal{J}\mathcal{P} - \mathcal{P}^T\mathcal{J}\tilde{S}\mathcal{P}\mathcal{J} \tag{29}$$

where

$$\mathcal{P} := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

Taking the partial derivative of the function $\mathcal{F}(\mathcal{P})$ with respect to $\mathcal{P}$ yields

$$\begin{aligned} \nabla\mathcal{F}(\mathcal{P}) &:= \left.\frac{\partial \mathrm{vec}\mathcal{F}(\mathcal{P})}{\partial(\mathrm{vec}\mathcal{P})^T}\right|_{\mathcal{P}=\mathcal{P}} \\ &= (\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T \otimes I_{2N} \\ &\quad + [I_{2N} \otimes (\tilde{A} - \tilde{S}\mathcal{P} - \mathcal{J}\tilde{S}\mathcal{P}\mathcal{J})^T]U_{2N2N} \\ &\quad - (\tilde{S}\mathcal{J}\mathcal{P})^T \otimes \mathcal{J} - [\mathcal{J} \otimes (\tilde{S}\mathcal{J}\mathcal{P})]^T U_{2N2N}. \end{aligned} \tag{30}$$

Taking the vec-operator transformation on both sides of (27) and (29), we obtain

$$\begin{aligned} &\left[\left(\Phi^{(n)T} \otimes I_{2N}\right)U_{2N2N} + I_{2N} \otimes \Phi^{(n)T}\right]\mathrm{vec}\,\mathcal{P}^{(n+1)} \\ &- \left[\left(\Theta^{(n)T} \otimes \mathcal{J}\right)U_{2N2N} + \mathcal{J} \otimes \Theta^{(n)T}\right]\mathrm{vec}\,\mathcal{P}^{(n+1)} \\ &+ \mathrm{vec}\Xi^{(n)} = 0 \end{aligned} \tag{31a}$$
$$\begin{aligned} &\mathrm{vec}\,\mathcal{F}(\mathcal{P}^{(n)}) \\ &= \left[\left(\Phi^{(n)T} \otimes I_{2N}\right)U_{2N2N} + I_{2N} \otimes \Phi^{(n)T}\right]\mathrm{vec}\,\mathcal{P}^{(n)} \\ &\quad - \left[\left(\Theta^{(n)T} \otimes \mathcal{J}\right)U_{2N2N} + \mathcal{J} \otimes \Theta^{(n)T}\right]\mathrm{vec}\mathcal{P}^{(n)} \\ &\quad + \mathrm{vec}\,\Xi^{(n)}. \end{aligned} \tag{31b}$$

Subtracting (31b) from (31a) and noting that

$$\nabla\mathcal{F}(\mathcal{P}^{(n)}) := \left[\left(\Phi^{(n)T} \otimes I_{2N}\right)U_{2N2N} + I_{2N} \otimes \Phi^{(n)T}\right] - \left[\left(\Theta^{(n)T} \otimes \mathcal{J}\right)U_{2N2N} + \mathcal{J} \otimes \Theta^{(n)T}\right]$$

we have

$$\mathrm{vec}\,\mathcal{P}^{(n+1)} = \mathrm{vec}\mathcal{P}^{(n)} - \left[\nabla\mathcal{F}(\mathcal{P}^{(n)})\right]^{-1}\mathrm{vec}\mathcal{F}(\mathcal{P}^{(n)})$$

which is the desired result.

We are concerned with good choices of the starting points which guarantee to find a required solution of a given GCMARE (8). Our new idea is to set the initial conditions to the matrix (28). The fundamental idea is based on $\|\mathcal{P} - \mathcal{P}^{(0)}\| = O(\|\mu\|)$ which is derived from (22). Consequently, we will get the required solution with rate of the quadratic convergence via the Newton's method. Moreover, using the Newton-Kantorovich theorem [24], we will also prove the existence of the unique solution for the GCMARE (8). The main result of this section is as follows.

Theorem 2: Under Assumptions 1–3, the new iterative algorithm (27) converges to the exact solution $\mathcal{P}^*$ of the GCMARE (8) with the rate of quadratic convergence. Furthermore, the unique bounded solution $\mathcal{P}^{(n)}$ of the GCMARE (8) is in the neighborhood of the exact solution $\mathcal{P}^*$. That is, the following conditions are satisfied:

$$\left\|\mathcal{P}^{(n)} - \mathcal{P}^*\right\| \le O\left(\|\mu\|^{2^n}\right), \qquad n = 0, 1, \dots \tag{32a}$$

$$\left\|\mathcal{P}^{(n)} - \mathcal{P}^*\right\| \le \frac{1}{\beta\mathcal{L}}[1 - \sqrt{1-2\theta}], \qquad n = 0, 1, \dots \tag{32b}$$

where

$$\mathcal{P} = \mathcal{P}^* = \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}$$

$$\mathcal{L} := 6\|\tilde{S}\|, \quad \beta := \left\|\left[\nabla\mathbf{F}\left(\mathbf{P}^{(0)}\right)\right]^{-1}\right\|$$

$$\eta := \left\|\left[\nabla\mathbf{F}\left(\mathbf{P}^{(0)}\right)\right]^{-1}\right\| \cdot \left\|\mathbf{F}\left(\mathbf{P}^{(0)}\right)\right\|, \qquad \theta := \beta\eta\mathcal{L}$$

$$\mathbf{F}(\mathbf{P}) := (f_{x1}, \dots, f_{x6}, f_{y1}, \dots, f_{y6}).$$

$$\mathbf{P}^{(0)} := \mathbf{P}_0.$$

Proof: The proof is given directly by applying the Newton-Kantorovich theorem [24] for the GCMARE (8). It is immediately obtained from (30) that

$$\|\nabla\mathcal{F}(\mathcal{P}_1) - \nabla\mathcal{F}(\mathcal{P}_2)\| \le \mathcal{L}\|\mathcal{P}_1 - \mathcal{P}_2\|$$
$$\Rightarrow \|\nabla\mathbf{F}(\mathbf{P}_1) - \nabla\mathbf{F}(\mathbf{P}_2)\| \le \mathcal{L}\|\mathbf{P}_1 - \mathbf{P}_2\|. \tag{33}$$

Moreover, using (22), we get

$$\nabla\mathbf{F}(\mathbf{P}) = J + O(\|\mu\|). \tag{34}$$

Hence, it follows that $\nabla\mathbf{F}(\mathbf{P}^{(0)})$ is nonsingular under the condition (21) for sufficiently small $\|\mu\|$. Therefore, there exists β such that $\beta = \|[\nabla\mathbf{F}(\mathbf{P}^{(0)})]^{-1}\|$. On the other hand, since $\mathbf{F}(\mathbf{P}^{(0)}) = O(\|\mu\|)$, there exists η such that $\eta = \|[\nabla\mathbf{F}(\mathbf{P}^{(0)})]^{-1}\| \cdot \|\mathbf{F}(\mathbf{P}^{(0)})\| = O(\|\mu\|)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because $\eta = O(\|\mu\|)$. Now, let us define

$$t^* := \frac{1}{\beta\mathcal{L}}[1 - \sqrt{1-2\theta}]$$
$$= \frac{1}{6\|\tilde{S}\| \cdot \|\left[\nabla\mathbf{F}\left(\mathbf{P}^{(0)}\right)\right]^{-1}\|}[1 - \sqrt{1-2\theta}]. \tag{35}$$

Using the Newton-Kantorovich theorem, we can show that $\mathcal{P}^*$ is the unique solution in the subset $\mathcal{S} \equiv \{\mathcal{P} : \|\mathcal{P}^{(0)} - \mathcal{P}\| \le t^*\}$. Moreover, using the Newton-Kantorovich theorem, the error estimate is given by

$$\left\|\mathbf{P}^{(n)} - \mathbf{P}^*\right\| \le \frac{(2\theta)^{2^n}}{2^n\beta\mathcal{L}}$$
$$\Rightarrow \left\|\mathcal{P}^{(n)} - \mathcal{P}^*\right\| \le \frac{(2\theta)^{2^n}}{2^n\beta\mathcal{L}}, \qquad n = 1, 2, \dots. \tag{36}$$

Substituting $2\theta = O(\|\mu\|)$ into (36), we have (32a). Furthermore, substituting $\mathcal{P}^*$ into $\mathcal{P}$ of the subset $\mathcal{S}$, we can also get (32b). Therefore, (32) holds for the small $\|\mu\|$. ■

Remark 1: Using the Newton–Kantorovich theorem instead of the implicit function theorem, we can also prove (22). That is, the structure of the solutions X and Y are established by setting $n = 0$ in (32a). It should be noted that the asymptotic structure for the GCMARE (8) is established by using the Newton–Kantorovich theorem which is different from implicit function theorem. As a result, the condition of the small parameter $\|\mu\|$ for existence of the solutions of the GCMARE (8) would be clear.

Remark 2: In view of [13], we know that the solution of the GCMARE (8) is not unique and several nonnegative solutions exist. In this paper, it is very important to note that if the sufficient condition (21) holds, the new algorithm (27) converge to the desired positive semidefinite solution in the same way as the Lyapunov iterations [3].

Remark 3: In order to obtain the initial condition (28), we have to solve the CARE (17a) and (17b) which are independent of the perturbation parameters ε_i. In this situation, we can also apply the Newton's method to these equations (17a) and (17b). The resulting algorithm is given by

$$\begin{aligned}&\left(A_s - S_{s1}\bar{X}_{00}^{(k)} - S_{s2}\bar{Y}_{00}^{(k)}\right)^T \bar{X}_{00}^{(k+1)}\\ &+ \bar{X}_{00}^{(k+1)}\left(A_s - S_{s1}\bar{X}_{00}^{(k)} - S_{s2}\bar{Y}_{00}^{(k)}\right)\\ &- \bar{X}_{00}^{(k)}S_{s2}\bar{Y}_{00}^{(k+1)} - \bar{Y}_{00}^{(k+1)}S_{s2}\bar{X}_{00}^{(k)} + Q_{s1}\\ &+ \bar{X}_{00}^{(k)}S_{s1}\bar{X}_{00}^{(k)} + \bar{X}_{00}^{(k)}S_{s2}\bar{Y}_{00}^{(k)}\\ &+ \bar{Y}_{00}^{(k)}S_{s2}\bar{X}_{00}^{(k)} = 0\end{aligned} \tag{37a}$$

$$\begin{aligned}&\left(A_s - S_{s1}\bar{X}_{00}^{(k)} - S_{s2}\bar{Y}_{00}^{(k)}\right)^T \bar{Y}_{00}^{(k+1)}\\ &+ \bar{Y}_{00}^{(k+1)}(A_s - S_{s1}\bar{X}_{00}^{(k)} - S_{s2}\bar{Y}_{00}^{(k)})\\ &- \bar{Y}_{00}^{(k)}S_{s1}\bar{X}_{00}^{(k+1)} - \bar{X}_{00}^{(k+1)}S_{s1}\bar{Y}_{00}^{(k)} + Q_{s2}\\ &+ \bar{Y}_{00}^{(k)}S_{s2}\bar{Y}_{00}^{(k)} + \bar{Y}_{00}^{(k)}S_{s1}\bar{X}_{00}^{(k)}\\ &+ \bar{X}_{00}^{(k)}S_{s1}\bar{Y}_{00}^{(k)} = 0\end{aligned} \tag{37b}$$

with

$$A_s^T\bar{X}_{00}^{(0)} + \bar{X}_{00}^{(0)}A_s - \bar{X}_{00}^{(0)}S_{s1}\bar{X}_{00}^{(0)} + Q_{s1} = 0 \tag{38a}$$

$$A_s^T\bar{Y}_{00}^{(0)} + \bar{Y}_{00}^{(0)}A_s - \bar{Y}_{00}^{(0)}S_{s2}\bar{Y}_{00}^{(0)} + Q_{s2} = 0. \tag{38b}$$

In the rest of this section, we explain the method for solving the generalized cross-coupled multiparameter algebraic Lyapunov equation (GCMALE) (27) with the dimension $2N = 2(n_0 + n_1 + n_2)$. So far, there is no argument as to the numerical method for solving the considered GCMALE (27). In order to reduce the dimension of the workspace, a new algorithm for solving the GCMALE (27) which is based on the existing algorithm [19] is established. Firstly, we set

$$\bar{A} := \Psi \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix}$$

$$\bar{B}_1 := \Psi \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix}$$

$$\bar{B}_2 := \Psi \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix}$$

$$\bar{S}_i := \bar{B}_i R_{ii}^{-1} \bar{B}_i^T, \qquad i = 1, 2$$

$$\Psi := \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \frac{\mu}{\varepsilon_1} I_{n_1} & 0 \\ 0 & 0 & \frac{\mu}{\varepsilon_2} I_{n_2} \end{bmatrix}.$$

Using the above relations, we can get the following Newton's method:

$$\begin{aligned} &\bar{\Phi}^{(n)T} \bar{\mathcal{P}}^{(n+1)} + \bar{\mathcal{P}}^{(n+1)T} \bar{\Phi}^{(n)} - \bar{\Theta}^{(n)T} \bar{\mathcal{P}}^{(n+1)} \mathcal{J} \\ &\quad - \mathcal{J} \bar{\mathcal{P}}^{(n+1)T} \bar{\Theta}^{(n)} + \bar{\Xi}^{(n)} = 0, \quad n = 0, 1, \dots \\ &\Leftrightarrow \begin{cases} \bar{\Phi}_1^{(n)T} \mathcal{X}^{(n+1)} + \mathcal{X}^{(n+1)T} \bar{\Phi}_1^{(n)} \\ \quad -\bar{\Theta}_2^{(n)T} \mathcal{Y}^{(n+1)} - \mathcal{Y}^{(n+1)T} \bar{\Theta}_2^{(n)} + \bar{\Xi}_1^{(n)} = 0, \\ \bar{\Phi}_2^{(n)T} \mathcal{Y}^{(n+1)} + \mathcal{Y}^{(n+1)T} \bar{\Phi}_2^{(n)} \\ \quad -\bar{\Theta}_1^{(n)T} \mathcal{X}^{(n+1)} - \mathcal{X}^{(n+1)T} \bar{\Theta}_1^{(n)} + \bar{\Xi}_2^{(n)} = 0 \end{cases} \end{aligned} \tag{39}$$

where

$$\bar{\Phi}^{(n)} := \mathcal{A} - \mathcal{S}\bar{\mathcal{P}}^{(n)} - \mathcal{J}\mathcal{S}\bar{\mathcal{P}}^{(n)}$$

$$\mathcal{J} = \begin{bmatrix} \bar{\Phi}_1^{(n)} & 0 \\ 0 & \bar{\Phi}_2^{(n)} \end{bmatrix}$$

$$\bar{\Theta}^{(n)} := \mathcal{S}\mathcal{J}\bar{\mathcal{P}}^{(n)} = \begin{bmatrix} 0 & \bar{\Theta}_1^{(n)} \\ \bar{\Theta}_2^{(n)} & 0 \end{bmatrix}$$

$$\begin{aligned} \bar{\Xi}^{(n)} &:= \tilde{Q} + \bar{\mathcal{P}}^{(n)T} \mathcal{S} \bar{\mathcal{P}}^{(n)} + \mathcal{J} \bar{\mathcal{P}}^{(n)T} \mathcal{S} \mathcal{J} \bar{\mathcal{P}}^{(n)} \\ &\quad + \bar{\mathcal{P}}^{(n)T} \mathcal{J} \mathcal{S} \bar{\mathcal{P}}^{(n)} \mathcal{J} \\ &:= \begin{bmatrix} \bar{\Xi}_1^{(n)} & 0 \\ 0 & \bar{\Xi}_2^{(n)} \end{bmatrix} \end{aligned}$$

$$\bar{\Phi}_i^{(n)} := \begin{bmatrix} \bar{\Phi}_{00i}^{(n)} & \bar{\Phi}_{01i}^{(n)} \\ \bar{\Phi}_{10i}^{(n)} & \bar{\Phi}_{11i}^{(n)} \end{bmatrix}$$

$$\bar{\Theta}_i^{(n)} := \begin{bmatrix} \bar{\Theta}_{00i}^{(n)} & \bar{\Theta}_{01i}^{(n)} \\ \bar{\Theta}_{10i}^{(n)} & \bar{\Theta}_{11i}^{(n)} \end{bmatrix}$$

$$\bar{\Xi}_i^{(n)} := \begin{bmatrix} \bar{\Xi}_{00i}^{(n)} & \bar{\Xi}_{01i}^{(n)} \\ \bar{\Xi}_{01i}^{(n)T} & \bar{\Xi}_{11i}^{(n)} \end{bmatrix}, \qquad i = 1, 2$$

$$\bar{\mathcal{P}}^{(n)} := \begin{bmatrix} \mathcal{X}^{(n)} & 0 \\ 0 & \mathcal{Y}^{(n)} \end{bmatrix}$$

$$\mathcal{X}^{(n)} := \begin{bmatrix} \mathcal{X}_{00}^{(n)} & \mu \mathcal{X}_{10}^{(n)T} \\ \mathcal{X}_{10}^{(n)} & \mathcal{X}_{11}^{(n)} \end{bmatrix}$$

$$\mathcal{Y}^{(n)} := \begin{bmatrix} \mathcal{Y}_{00}^{(n)} & \mu \mathcal{Y}_{10}^{(n)T} \\ \mathcal{Y}_{10}^{(n)} & \mathcal{Y}_{11}^{(n)} \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix}$$

$$\mathcal{S} = \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix}$$

$$\bar{\Phi}_{00i}^{(n)},\ \bar{\Theta}_{00i}^{(n)},\ \bar{\Xi}_{00i}^{(n)},\ \mathcal{X}_{00}^{(n)}, \mathcal{Y}_{00}^{(n)} \in \mathbf{R}^{n_0 \times n_0}$$

$$\bar{\Phi}_{11i}^{(n)},\ \bar{\Theta}_{11i}^{(n)},\ \bar{\Xi}_{11i}^{(n)},\ \mathcal{X}_{11}^{(n)}, \mathcal{Y}_{11}^{(n)} \in \mathbf{R}^{(n_1+n_2) \times (n_1+n_2)}$$

and the initial condition $\bar{\mathcal{P}}^{(0)}$ has the following form:

$$\bar{\mathcal{P}}^{(0)} = \begin{bmatrix} \Psi^{-1} X^{(0)} & 0 \\ 0 & \Psi^{-1} Y^{(0)} \end{bmatrix}. \tag{40}$$

It should be noted that the algorithm (39) has not 3×3 blocks but 2×2 blocks because we will use the existing results in [19]. That is, the coefficient matrices of the algorithm (39) can be changed such that $\varepsilon_1 = \varepsilon_2 = \mu$.

Secondly, we convert the GCMALE (39) into the following form:

$$\mathcal{T}\mathbf{X} = -\mathbf{Q} \tag{41}$$

where

$$\mathcal{T} := \begin{bmatrix} \mathcal{A}_1 & -\mathcal{B}_2 \\ -\mathcal{B}_1 & \mathcal{A}_2 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}$$

and the set of equations shown at the bottom of the next page, hold.

$\mathcal{A}_i$ and $\mathcal{B}_i$ are the appropriate matrices. In order to show the form of these matrices, let us consider a simple fourth-order example with $n_0 = 2, n_1 = n_2 = 1$. The matrix $\mathcal{A}_i$ is given in (42) at the bottom of the next page, where

$$\Phi_i = \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & a_{14}^i \\ a_{21}^i & a_{22}^i & a_{23}^i & a_{24}^i \\ a_{31}^i & a_{32}^i & a_{33}^i & a_{34}^i \\ a_{41}^i & a_{42}^i & a_{43}^i & a_{44}^i \end{bmatrix}.$$

It should be noted that there exists an algorithm for constructing the matrices $\mathcal{A}_i$ and $\mathcal{B}_i$. See for example [19] in detail. Moreover, it is well-known that although the matrices $\mathcal{A}_i$ and $\mathcal{B}_i$ contain the small parameters, these matrices are well-conditioned [18], [19].

Since the Newton–Kantorovich theorem guarantees the invertibility of the matrix $\mathcal{T}$, there exists the matrix $\mathcal{T}^{-1}$ for all $n, n = 0, 1, 2, \dots$. Hence we have $\mathbf{X} = -\mathcal{T}^{-1}\mathbf{Q}$.

Finally, we can get the desired solution by multiplying the μ on the matrices $\mathcal{X}_{00}^{(n)}, \mathcal{X}_{10}^{(n)}, \mathcal{Y}_{00}^{(n)}$ and $\mathcal{Y}_{10}^{(n)}$ in the matrix $\bar{\mathcal{P}}^{(n)}$. As a result, the following relation holds:

$$\begin{aligned} &\begin{bmatrix} \Lambda_e X^{(n)} & 0 \\ 0 & \Lambda_e Y^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{X}_{00}^{(n)} & \mu \mathcal{X}_{10}^{(n)T} & 0 & 0 \\ \mu \mathcal{X}_{10}^{(n)} & \mu \mathcal{X}_{11}^{(n)} & 0 & 0 \\ 0 & 0 & \mathcal{Y}_{00}^{(n)} & \mu \mathcal{Y}_{10}^{(n)T} \\ 0 & 0 & \mu \mathcal{Y}_{10}^{(n)} & \mu \mathcal{Y}_{11}^{(n)} \end{bmatrix} \\ &\to \begin{bmatrix} X_e & 0 \\ 0 & Y_e \end{bmatrix}, \quad (n \to \infty). \end{aligned}$$

V. High-Oreder Approximate Nash Strategy

In this section, we give the high-order approximate Nash strategy which is obtained by using the iterative solution (27)

$$u_{1\mathrm{app}}^{(n)}(t) = -R_{11}^{-1} B_1^T X^{(n)} x(t), \quad n = 0, 1, \dots \tag{43a}$$

$$u_{2\mathrm{app}}^{(n)}(t) = -R_{22}^{-1} B_2^T Y^{(n)} x(t), \quad n = 0, 1, \dots . \tag{43b}$$

Theorem 3: Assume that $\mathbf{Re}\lambda[\Pi_e^{-1}(A - S_1 X^{(0)} - S_2 Y^{(0)})] < 0$. Under Assumptions 1–3, the use of the high-order approximate strategy (43) results in $J_i(u_{i\,\mathrm{app}}^{(n)}, u_{j\,\mathrm{app}}^{(n)})$ satisfying

$$J_i\left(u_{i\,\mathrm{app}}^{(n)}, u_{j\,\mathrm{app}}^{(n)}\right) = J_i(u_i^*, u_j^*) + O\left(\|\mu\|^{2^n}\right) \quad i = 1, 2, \quad n = 0, 1, \ldots \tag{44}$$

where $J_i(u_i^*, u_j^*), i = 1, 2$ are the optimal equilibrium values of the cost functions (2).

Proof: We prove only the case $i = 1$. The proof of the case $i = 2$ is similar. When $u_{1\mathrm{app}}^{(n)}$ is used, the value of the performance index is

$$J_1\left(u_{1\mathrm{app}}^{(n)}, u_{2\mathrm{app}}^{(n)}\right) = \frac{1}{2} x(0)^T W_{1e}^{(n)} x(0) \tag{45}$$

$$\begin{aligned}
\mathbf{x} := \big[& x_{11}^{00} \ \ x_{12}^{00} \ \cdots \ x_{1n_0}^{00} \ \ x_{11}^{10} \ \cdots \ x_{1(n_1+n_2)}^{10} \ \ x_{22}^{00} \ \cdots \ x_{2n_0}^{00} \ \ x_{21}^{10} \ \cdots \ \cdots \ x_{2(n_1+n_2)}^{10} \\
& \cdots \ x_{n_0 n_0}^{00} \ \ x_{n_0 1}^{10} \ \cdots \ x_{n_0(n_1+n_2)}^{10} \ \cdots \ x_{11}^{11} \ \ x_{12}^{11} \ \cdots \ x_{1(n_1+n_2)}^{11} \ \ x_{22}^{11} \ \cdots \ x_{2(n_1+n_2)}^{11} \\
& \cdots \ x_{(n_1+n_2-1)(n_1+n_2-1)}^{11} \ \ x_{(n_1+n_2-1)(n_1+n_2)}^{11} \ \ x_{(n_1+n_2)(n_1+n_2)}^{11} \big]^T \in \mathbf{R}^{N(N+1)} \\
\mathbf{y} := \big[& y_{11}^{00} \ \ y_{12}^{00} \ \cdots \ y_{1n_0}^{00} \ \ y_{11}^{10} \ \cdots \ y_{1(n_1+n_2)}^{10} \ \ y_{22}^{00} \ \cdots \ y_{2n_0}^{00} \ \ y_{21}^{10} \ \cdots \ \cdots \ y_{2(n_1+n_2)}^{10} \\
& \cdots \ y_{n_0 n_0}^{00} \ \ y_{n_0 1}^{10} \ \cdots \ y_{n_0(n_1+n_2)}^{10} \ \cdots \ y_{11}^{11} \ \ y_{12}^{11} \ \cdots \ y_{1(n_1+n_2)}^{11} \ \ y_{22}^{11} \ \cdots \ y_{2(n_1+n_2)}^{11} \\
& \cdots \ y_{(n_1+n_2-1)(n_1+n_2-1)}^{11} \ \ y_{(n_1+n_2-1)(n_1+n_2)}^{11} \ \ y_{(n_1+n_2)(n_1+n_2)}^{11} \big]^T \in \mathbf{R}^{N(N+1)} \\
\mathbf{q}_i := \big[& q_{11}^i \ \ q_{12}^i \ \cdots \ q_{1N}^i \ \ q_{22}^i \ \ q_{23}^i \ \cdots \ q_{2N}^i \ \cdots \ q_{(N-1)(N-1)}^i \ \ q_{(N-1)N}^i \ \cdots \ q_{NN}^i \big]^T \in \mathbf{R}^{N(N+1)}
\end{aligned}$$

$$\mathcal{X}^{(n)} = \begin{bmatrix} \mathcal{X}_{00}^{(n)} & \mu\mathcal{X}_{10}^{(n)T} \\ \mathcal{X}_{10}^{(n)} & \mathcal{X}_{11}^{(n)} \end{bmatrix} = \left[\begin{array}{cccc|cccc}
x_{11}^{00} & x_{12}^{00} & \cdots & x_{1n_0}^{00} & \mu x_{11}^{10} & \mu x_{12}^{10} & \cdots & \mu x_{1(n_1+n_2)}^{10} \\
\cdot & x_{22}^{00} & \cdots & x_{2n_0}^{00} & \mu x_{21}^{10} & \mu x_{22}^{10} & \cdots & \mu x_{2(n_1+n_2)}^{10} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdots & x_{n_0 n_0}^{00} & \mu x_{n_0 1}^{10} & \mu x_{n_0 2}^{10} & \cdots & \mu x_{n_0(n_1+n_2)}^{10} \\ \hline
\cdot & \cdot & \cdots & \cdot & x_{11}^{11} & x_{12}^{11} & \cdots & x_{1(n_1+n_2)}^{11} \\
\cdot & \cdot & \cdots & \cdot & \cdot & x_{22}^{11} & \cdots & x_{2(n_1+n_2)}^{11} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & x_{(n_1+n_2)(n_1+n_2)}^{11}
\end{array}\right]$$

$$\mathcal{Y}^{(n)} = \begin{bmatrix} \mathcal{Y}_{00}^{(n)} & \mu\mathcal{Y}_{10}^{(n)T} \\ \mathcal{Y}_{10}^{(n)} & \mathcal{Y}_{11}^{(n)} \end{bmatrix} = \left[\begin{array}{cccc|cccc}
y_{11}^{00} & y_{12}^{00} & \cdots & y_{1n_0}^{00} & \mu y_{11}^{10} & \mu y_{12}^{10} & \cdots & \mu y_{1(n_1+n_2)}^{10} \\
\cdot & y_{22}^{00} & \cdots & y_{2n_0}^{00} & \mu y_{21}^{10} & \mu y_{22}^{10} & \cdots & \mu y_{2(n_1+n_2)}^{10} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdots & y_{n_0 n_0}^{00} & \mu y_{n_0 1}^{10} & \mu y_{n_0 2}^{10} & \cdots & \mu y_{n_0(n_1+n_2)}^{10} \\ \hline
\cdot & \cdot & \cdots & \cdot & y_{11}^{11} & y_{12}^{11} & \cdots & y_{1(n_1+n_2)}^{11} \\
\cdot & \cdot & \cdots & \cdot & \cdot & y_{22}^{11} & \cdots & y_{2(n_1+n_2)}^{11} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & y_{(n_1+n_2)(n_1+n_2)}^{11}
\end{array}\right]$$

$$\Xi_i^{(n)} := \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} \end{bmatrix} = \begin{bmatrix}
q_{11}^i & q_{12}^i & q_{13}^i & \cdots & q_{1N}^i \\
\cdot & q_{22}^i & q_{23}^i & \cdots & q_{2N}^i \\
\cdot & \cdot & q_{33}^i & \cdots & q_{3N}^i \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdot & \cdots & q_{NN}^i
\end{bmatrix}$$

$$\mathcal{A}_i = \begin{bmatrix}
2a_{11}^i & 2a_{21}^i & 2a_{31}^i & 2a_{41}^i & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{12}^i & a_{11}^i + a_{22}^i & a_{32}^i & a_{42}^i & a_{21}^i & a_{31}^i & a_{41}^i & 0 & 0 & 0 \\
a_{13}^i & a_{23}^i & \mu a_{11}^i + a_{33}^i & a_{43}^i & 0 & \mu a_{21}^i & 0 & a_{31}^i & a_{41}^i & 0 \\
a_{14}^i & a_{24}^i & a_{34}^i & \mu a_{11}^i + a_{44}^i & 0 & 0 & \mu a_{21}^i & 0 & a_{31}^i & a_{41}^i \\
0 & 2a_{12}^i & 0 & 0 & 2a_{22}^i & 2a_{32}^i & 2a_{42}^i & 0 & 0 & 0 \\
0 & a_{13}^i & \mu a_{12}^i & 0 & a_{23}^i & \mu a_{22}^i + a_{33}^i & a_{43}^i & a_{32}^i & a_{42}^i & 0 \\
0 & a_{14}^i & 0 & \mu a_{12}^i & a_{24}^i & a_{34}^i & \mu a_{22}^i + a_{44}^i & 0 & a_{32}^i & a_{42}^i \\
0 & 0 & 2\mu a_{13}^i & 0 & 0 & 2\mu a_{23}^i & 0 & 2a_{33}^i & 2a_{43}^i & 0 \\
0 & 0 & \mu a_{14}^i & \mu a_{13}^i & 0 & \mu a_{24}^i & \mu a_{23}^i & a_{34}^i & a_{33}^i + a_{44}^i & a_{43}^i \\
0 & 0 & 0 & 2\mu a_{14}^i & 0 & 0 & 2\mu a_{24}^i & 0 & 2a_{34}^i & 2a_{44}^i
\end{bmatrix} \tag{42}$$

where $W_{1e}^{(n)}$ is the positive semidefinite solution of the following multiparameter algebraic Lyapunov equation (MALE):

$$\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right)^T W_{1e}^{(n)} + W_{1e}^{(n)}\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right) + Q_1 + X_e^{(n)}S_{1e}X_e^{(n)} = 0. \quad (46)$$

Subtracting (6a) from (46) we find that $V_{1e}^{(n)} = W_{1e}^{(n)} - X_e$ satisfies the following MALE:

$$\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right)^T V_{1e}^{(n)} + V_{1e}^{(n)}\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right) + \left(X_e^{(n)} - X_e\right)S_{1e}(X_e^{(n)} - X_e) + Y_eS_{2e}\left(X_e^{(n)} - X_e\right) + \left(X_e^{(n)} - X_e\right)S_{2e}Y_e = 0. \quad (47)$$

Using the relations $X_e^{(n)} - X_e = O(\|\mu\|^{2^n})$ and $Y_e^{(n)} - Y_e = O(\|\mu\|^{2^n})$ from (32a), we can change the form of (47) into (48)

$$\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right)^T V_{1e}^{(n)} + V_{1e}^{(n)}\left(A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)}\right) + O\left(\|\mu\|^{2^n}\right) = 0. \quad (48)$$

It is easy to verify that $V_{1e}^{(n)} = O(\|\mu\|^{2^n})$ because $A_e - S_{1e}X_e^{(n)} - S_{2e}Y_e^{(n)} = \Pi_e^{-1}[A - S_1X^{(0)} - S_2Y^{(0)} + O(\|\mu\|)]$ is stable by using the standard Lyapunov theorem [25] for sufficiently small $\|\mu\|$. Consequently, the equality (44) holds. ■

Consequently, when ε_j is known, we can design the high-order $O(\|\mu\|^{2^n})$ approximate strategy which achieves the $O(\|\mu\|^{2^n})$ approximation for the equilibrium value of the cost functional.

In addition, we will present an important implication. If the parameter ε_j are unknown, then the following corollary is easily seen in view of Theorem 3.

Corollary 1: Under Assumptions 1–3, the use of the parameter-independent strategies

$$\bar{u}_{1\mathrm{app}}(t) = -R_{11}^{-1}B_1^T\bar{X}x(t) = -R_{11}^{-1}B_1^T\begin{bmatrix}\bar{X}_{00} & 0 & 0\\ \bar{X}_{10} & \bar{X}_{11} & 0\\ \bar{X}_{20} & 0 & 0\end{bmatrix}x(t) \quad (49a)$$

$$\bar{u}_{2\mathrm{app}}(t) = -R_{22}^{-1}B_2^T\bar{Y}x(t) = -R_{22}^{-1}B_2^T\begin{bmatrix}\bar{Y}_{00} & 0 & 0\\ \bar{Y}_{10} & 0 & 0\\ \bar{Y}_{20} & 0 & \bar{Y}_{22}\end{bmatrix}x(t) \quad (49b)$$

results in $J_i(\bar{u}_{i\,\mathrm{app}}, \bar{u}_{j\,\mathrm{app}})$ satisfying

$$J_i(\bar{u}_{i\,\mathrm{app}}, \bar{u}_{j\,\mathrm{app}}) = J_i(u_i^*, u_j^*) + O(\|\mu\|). \quad (50)$$

Proof: Since the result of Corollary 1 can be proved by using the similar technique in Theorem 3 under the fact that

$$\|\bar{\mathcal{P}} - \mathcal{P}^*\| = O(\|\mu\|) \quad (51)$$

where

$$\bar{\mathcal{P}} = \begin{bmatrix}\bar{X} & 0\\ 0 & \bar{Y}\end{bmatrix}$$

the proof is omitted. ■

It is worth noting that the proposed approximate strategies (49) can be implemented with the feature of (50) even if the perturbation parameters ε_j are unknown.

Remark 4: In [5], the following theorem has been derived:

$$J_i(\bar{u}_{i\,\mathrm{app}}, \bar{u}_{j\,\mathrm{app}}) \le J_i(u_i, \bar{u}_{j\,\mathrm{app}}) + O(\|\mu\|). \quad (52)$$

By using the similar manner which has been established in [5], if the condition (21) holds under Assumptions 1–3, then the inequality (52) is also satisfied without the nonsingularity assumption of the fast state matrices. Since the proof can be done by using the similar step in [5], it is omitted.

VI. Numerical Example

In order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given as Appendix A in [6].

$$A_{00} = \begin{bmatrix}0 & 0 & 4.5 & 0 & 1\\ 0 & 0 & 0 & 4.5 & -1\\ 0 & 0 & -0.05 & 0 & -0.1\\ 0 & 0 & 0 & -0.05 & 0.1\\ 0 & 0 & 32.7 & -32.7 & 0\end{bmatrix}$$

$$A_{01} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0.1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

$$A_{02} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0.1 & 0\\ 0 & 0\end{bmatrix}$$

$$A_{10} = \begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & -0.4 & 0 & 0\end{bmatrix}$$

$$A_{20} = \begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.4 & 0\end{bmatrix}$$

$$A_{11} = A_{22} = \begin{bmatrix}-0.05 & 0.05\\ 0 & -0.1\end{bmatrix}$$

$$B_{01} = B_{02} = [0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

$$B_{11} = B_{22} = \begin{bmatrix}0\\ 0.1\end{bmatrix}$$

$$R_{11} = R_{22} = 20$$

$$Q_1 = \mathrm{diag}(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0)$$

$$Q_2 = \mathrm{diag}(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1).$$

The small parameters are chosen as $\varepsilon_1 = \varepsilon_2 = 0.01$. We give an initial condition and a solution of the GCMARE (8) as follows:

$$\bar{X} = \begin{bmatrix} \bar{X}_{00} & \varepsilon_1 \bar{X}_{10}^T & \varepsilon_2 \bar{X}_{20}^T \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & 0 \end{bmatrix}$$
$$\bar{Y} = \begin{bmatrix} \bar{Y}_{00} & \varepsilon_1 \bar{Y}_{10}^T & \varepsilon_2 \bar{Y}_{20}^T \\ \bar{Y}_{10} & 0 & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix}$$

and in the sets of equations shown at the bottom of this page and at the bottom of the next page. Table I shows the results of the error $\|\mathcal{F}(\mathcal{P}^{(n)})\|$ per iterations, where "$e-x$" stands for "$\times 10^{-x}$". We find that the solutions of the GCMARE (8) converge to the exact solution with accuracy of $\|\mathcal{F}(\mathcal{P}^{(n)})\| < 10^{-10}$ after 5 iterative iterations. Moreover, it is interested in pointing out that the result of Table I shows that the algorithms (27) are quadratic convergence. Table II shows the necessary number of iterations for the convergence of the Lyapunov iterations [3] versus the new algorithm under the same accuracy of $\|\mathcal{F}(\mathcal{P}^{(n)})\| < 10^{-10}$. It can be seen that the convergence rate of the resulting algorithm is stable for all ε_i since the initial conditions $\mathcal{P}^{(0)}$ is quite good. On the other hand, the Lyapunov iterations converge to the exact solutions very slowly.

TABLE I
REMAINDER PER ITERATION

i	$\|\mathcal{F}(\mathcal{P}^{(n)})\|$
0	$1.18679e+1$
1	$2.71255e+0$
2	$1.81162e+0$
3	$1.07481e-2$
4	$1.19273e-7$
5	$4.52670e-11$

TABLE II
ERROR $\|\mathcal{F}(\mathcal{P}^{(n)})\|$

$\varepsilon_1 = \varepsilon_2$	Newton's Method	Lyapunov Iteration
$1.0e-2$	5	19
$1.0e-3$	3	17
$1.0e-4$	3	15
$1.0e-5$	2	13
$1.0e-6$	2	12
$1.0e-7$	2	10
$1.0e-8$	1	8

Finally, we evaluate the costs using the near-optimal strategies (49). We assume that the initial conditions are zero mean independent random vector with covariance matrix

$$\bar{X}_{00} = \begin{bmatrix} 5.5643e+0 & 7.7043e-1 & 4.6904e+1 & -2.9538e-2 & 2.4629e-1 \\ 7.7043e-1 & 3.2860e+0 & 1.9692e-2 & 2.3452e+1 & -3.1520e-1 \\ 4.6904e+1 & 1.9692e-2 & 6.4215e+2 & -2.1432e+2 & 4.7125e+0 \\ -2.9538e-2 & 2.3452e+1 & -2.1432e+2 & 4.0104e+2 & -5.7292e+0 \\ 2.4629e-1 & -3.1520e-1 & 4.7125e+0 & -5.7292e+0 & 1.2843e+0 \end{bmatrix}$$
$$\bar{X}_{10} = \begin{bmatrix} 9.2357e+1 & 3.8775e-2 & 1.2397e+3 & -4.2201e+2 & 9.2791e+0 \\ 4.4721e+1 & 1.8776e-2 & 5.7503e+2 & -2.0435e+2 & 4.4932e+0 \end{bmatrix}$$
$$\bar{X}_{20} = \begin{bmatrix} -5.8162e-2 & 4.6178e+1 & -4.2201e+2 & 7.8967e+2 & -1.1281e+1 \\ -2.8163e-2 & 2.2361e+1 & -2.0435e+2 & 3.8237e+2 & -5.4626e+0 \end{bmatrix}$$
$$\bar{X}_{11} = \begin{bmatrix} 9.9473e+0 & 3.2453e+0 \\ 3.2453e+0 & 6.5165e+0 \end{bmatrix}$$
$$\bar{Y}_{00} = \begin{bmatrix} 3.2860e+0 & 7.7043e-1 & 2.3452e+1 & 1.9692e-2 & 3.1520e-1 \\ 7.7043e-1 & 5.5643e+0 & -2.9538e-2 & 4.6904e+1 & -2.4629e-1 \\ 2.3452e+1 & -2.9538e-2 & 4.0104e+2 & -2.1432e+2 & 5.7292e+0 \\ 1.9692e-2 & 4.6904e+1 & -2.1432e+2 & 6.4215e+2 & -4.7125e+0 \\ 3.1520e-1 & -2.4629e-1 & 5.7292e+0 & -4.7125e+0 & 1.2843e+0 \end{bmatrix}$$
$$\bar{Y}_{10} = \begin{bmatrix} 4.6178e+1 & -5.8162e-2 & 7.8967e+2 & -4.2201e+2 & 1.1281e+1 \\ 2.2361e+1 & -2.8163e-2 & 3.8237e+2 & -2.0435e+2 & 5.4626e+0 \end{bmatrix}$$
$$\bar{Y}_{20} = \begin{bmatrix} 3.8775e-2 & 9.2357e+1 & -4.2201e+2 & 1.2397e+3 & -9.2791e+0 \\ 1.8776e-2 & 4.4721e+1 & -2.0435e+2 & 5.7503e+2 & -4.4932e+0 \end{bmatrix}$$
$$\bar{Y}_{22} = \begin{bmatrix} 9.9473e+0 & 3.2453e+0 \\ 3.2453e+0 & 6.5165e+0 \end{bmatrix}$$
$$X = \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1} X_{21}^T \\ X_{20} & \sqrt{\alpha} X_{21} & X_{22} \end{bmatrix}$$
$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix}$$

TABLE III
THE NASH EQUILIBRIUM VALUES

$\varepsilon_1 = \varepsilon_2$	$E[J_i(\bar{u}_{i\text{app}},\ \bar{u}_{j\text{app}})]$	$E[J_i(u_i^*,\ u_j^*)]$	ϕ_i
$1.0e-2$	$6.8876e+2$	$6.6354e+2$	$2.5221e+3$
$1.0e-3$	$5.3203e+2$	$5.3452e+2$	$2.4864e+3$
$1.0e-4$	$5.2712e+2$	$5.2739e+2$	$2.6915e+3$
$1.0e-5$	$5.2671e+2$	$5.2673e+2$	$2.7105e+3$
$1.0e-6$	$5.2666e+2$	$5.2667e+2$	$2.7124e+3$
$1.0e-7$	$5.2666e+2$	$5.2666e+2$	$2.7126e+3$
$1.0e-8$	$5.2666e+2$	$5.2666e+2$	$2.7126e+3$

$E[x(0)x(0)^T] = I_9$. The values of the cost functionals for various ε_1 and ε_2 are given in Table III, where

$$\phi_i := \frac{|E[J_i(\bar{u}_{i\,\text{app}}, \bar{u}_{j\,\text{app}})] - E[J_i(u_i^*, u_j^*)]|}{\|\mu\|} = \frac{|E[J_i(\bar{u}_{i\,\text{app}}, \bar{u}_{j\,\text{app}})] - E[J_i(u_i^*, u_j^*)]|}{\sqrt{\varepsilon_1\varepsilon_2}}$$

and $J_1(\bar{u}_{1\text{app}}, \bar{u}_{2\text{app}}) = J_2(\bar{u}_{1\text{app}}, \bar{u}_{2\text{app}}),\ J_1(u_1^*, u_2^*) = J_2(u_1^*, u_2^*)$.

It is easy to verify that $E[J_i(\bar{u}_{i\,\text{app}}, \bar{u}_{j\,\text{app}})] = E[J_i(u_i^*, u_j^*)] + O(\|\mu\|)$ because $\phi_i < \infty$ is the same order.

VII. CONCLUSION

The linear quadratic Nash games for infinite horizon nonstandard MSPS have been studied. Firstly, the uniqueness, the boundedness and the asymptotic structure of the solution to the GCMARE have been newly proved without the nonsingularity assumption. Secondly, in order to solve the GCMARE, we have proposed the new iterations method which is based on the

$$X_{00} = \begin{bmatrix} 5.7625e+0 & 6.9645e-1 & 5.0815e+1 & -1.2583e+0 & 2.2975e-1 \\ 6.9645e-1 & 3.4933e+0 & -2.0412e+0 & 2.7957e+1 & -3.0351e-1 \\ 5.0815e+1 & -2.0412e+0 & 7.7279e+2 & -3.2044e+2 & 4.4814e+0 \\ -1.2583e+0 & 2.7957e+1 & -3.2044e+2 & 5.4282e+2 & -5.5643e+0 \\ 2.2975e-1 & -3.0351e-1 & 4.4814e+0 & -5.5643e+0 & 1.5315e+0 \end{bmatrix}$$

$$X_{10} = \begin{bmatrix} 9.4880e+1 & -1.3549e+0 & 1.3467e+3 & -4.8954e+2 & -7.0647e+0 \\ 4.4721e+1 & -4.0168e-2 & 5.8211e+2 & -1.9452e+2 & -6.3223e+0 \end{bmatrix}$$

$$X_{20} = \begin{bmatrix} -7.2011e-1 & 4.9071e+1 & -4.9963e+2 & 9.1055e+2 & 5.4793e+0 \\ 6.0252e-2 & 2.2361e+1 & -2.0063e+2 & 3.9311e+2 & 5.9125e+0 \end{bmatrix}$$

$$X_{11} = \begin{bmatrix} 3.5630e+1 & 1.5095e+1 \\ 1.5095e+1 & 1.2160e+1 \end{bmatrix}$$

$$X_{22} = \begin{bmatrix} 1.6960e+1 & 7.7510e+0 \\ 7.7510e+0 & 3.6449e+0 \end{bmatrix}$$

$$X_{21} = \begin{bmatrix} -9.1241e+0 & -4.0306e+0 \\ -4.0697e+0 & -1.8928e+0 \end{bmatrix}$$

$$Y_{00} = \begin{bmatrix} 3.4933e+0 & 6.9645e-1 & 2.7957e+1 & -2.0412e+0 & 3.0351e-1 \\ 6.9645e-1 & 5.7625e+0 & -1.2583e+0 & 5.0815e+1 & -2.2975e-1 \\ 2.7957e+1 & -1.2583e+0 & 5.4282e+2 & -3.2044e+2 & 5.5643e+0 \\ -2.0412e+0 & 5.0815e+1 & -3.2044e+2 & 7.7279e+2 & -4.4814e+0 \\ 3.0351e-1 & -2.2975e-1 & 5.5643e+0 & -4.4814e+0 & 1.5315e+0 \end{bmatrix}$$

$$Y_{10} = \begin{bmatrix} 4.9071e+1 & -7.2011e-1 & 9.1055e+2 & -4.9963e+2 & -5.4793e+0 \\ 2.2361e+1 & 6.0252e-2 & 3.9311e+2 & -2.0063e+2 & -5.9125e+0 \end{bmatrix}$$

$$Y_{20} = \begin{bmatrix} -1.3549e+0 & 9.4880e+1 & -4.8954e+2 & 1.3467e+3 & 7.0647e+0 \\ -4.0168e-2 & 4.4721e+1 & -1.9452e+2 & 5.8211e+2 & 6.3223e+0 \end{bmatrix}$$

$$Y_{11} = \begin{bmatrix} 1.6960e+1 & 7.7510e+0 \\ 7.7510e+0 & 3.6449e+0 \end{bmatrix}$$

$$Y_{22} = \begin{bmatrix} 3.5630e+1 & 1.5095e+1 \\ 1.5095e+1 & 1.2160e+1 \end{bmatrix}$$

$$Y_{21} = \begin{bmatrix} -9.1241e+0 & -4.0697e+0 \\ -4.0306e+0 & -1.8928e+0 \end{bmatrix}$$

Newton's method. The proposed algorithm has the property of the quadratic convergence. It has been shown that the Newton's method can be used well to solve the GCMARE under the appropriate initial condition. It may be noted that the convergence rate of the proposed algorithm and its exact proof have been first given in this paper.

When the dimension of the MSPS is quite large, the algorithm appearing in Theorem 2 seems to be formidable. However, this is, in fact, quite numerically tractable for small dimension of the MSPS. Comparing with Lyapunov iterations [3], even if the singular perturbation parameter is extremely small, we have succeeded in improving the convergence rate dramatically. It is expected that the Newton's method for solving the GCMARE is applied to the wider class of the control law synthesis involving the solution of the CMARE with indefinite sign quadratic term such as the mixed H_2/H_∞ control problem [2]. This problem will be addressed in the near future.

REFERENCES

[1] A. W. Starr and Y. C. Ho, "Nonzero-sum differential games," *J. Optim. Theory Appl.*, vol. 3, no. 3, pp. 184–206, 1969.

[2] D. J. N. Limebeer, B. D. O. Anderson, and B. Hendel, "A Nash game approach to mixed H_2/H_∞ control," *IEEE Trans. Autom. Contr.*, vol. 39, no. 1, pp. 69–82, Jan. 1994.

[3] T.-Y. Li and Z. Gajić, "Lyapunov iterations for solving coupled algebraic lyapunov equations of Nash differential games and algebraic riccati equations of zero-sum games," in *New Trends in Dynamic Games and Applications*. Boston, MA: Birkhauser, 1994, pp. 333–351.

[4] G. Freiling, G. Jank, and H. Abou-Kandil, "On global existence of solutions to coupled matrix Riccati equations in closed-loop Nash games," *IEEE Trans. Autom. Contr.*, vol. 41, no. 2, pp. 264–269, Feb. 1996.

[5] H. K. Khalil, "Multimodel design of a Nash strategy," *J. Optim. Theory Appl.*, vol. 31, no. 4, pp. 553–564, 1980.

[6] H. K. Khalil and P. V. Kokotoviæ, "Control strategies for decision makers using different models of the same system," *IEEE Trans. Autom. Contr.*, vol. 23, no. 2, pp. 289–298, Apr. 1978.

[7] ——, "Feedback and well-posedness of singularly perturbed Nash games," *IEEE Trans. Autom. Contr.*, vol. 24, no. 5, pp. 699–708, Oct. 1979.

[8] ——, "Control of linear systems with multiparameter singular perturbations," *Automatica*, vol. 15, no. 2, pp. 197–207, 1979.

[9] C. Coumarbatch and Z. Gajic, "Parallel optimal Kalman filtering for stochastic systems in multimodeling form," *Trans. ASME*, vol. 122, pp. 542–550, Sept. 2000.

[10] Z. Gajić, D. Petkovski, and X. Shen, "Singularly perturbed and weakly coupled linear system—A recursive approach," in *Lecture Notes in Control and Information Sciences*. Berlin, Germany: Springer-Verlag, 1990, vol. 140.

[11] M. T. Lim, C.-G. Kang, and B.-S. Kim, "Optimal control of linear nonstandard singularly perturbed discrete systems," *Dyn. Cont.Discrete, and Impuls. Syst.*, vol. 9, no. 2, pp. 163–174, 2002.

[12] S. Koskie, D. Skataric, and B. Petrovic, "Convergence proof for recursive solution of linear-quadratic Nash games for quasi-singularly perturbed systems," *Dyn. Cont.Discrete, and Impuls. Syst.*, vol. 9b, no. 2, pp. 317–333, 2002.

[13] G. Jank and G. Kun *et al.*, "Solutions of generalized Riccati differential equations and their approximation," in *Computational Methods and Function, Theory* , N. Papamichael *et al.*, Eds. Singapore: World Scientific, 1998, pp. 1–18.

[14] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*. New York: Wiley, 1999.

[15] H. Mukaidani, H. Xu, and K. Mizukami, "A new algorithm for solving cross-coupled algebraic Riccati equations of singularly perturbed Nash games," in *Proc. IEEE Conf. Decision and Control*, Sydney, NSW, Australia, 2000, pp. 3648–3653.

[16] ——, "Recursive algorithm for mixed H_2/H_∞ control problem of singularly perturbed systems," *Int. J. Syst. Sci.*, vol. 31, no. 10, pp. 1299–1312, 2000.

[17] ——, "A new algorithm for solving cross-coupled algebraic Riccati equations of singularly perturbed systems for mixed H_2/H_∞ control problem," in *Proc. Int. Symp. Dynamic Games and Applications*, Adelaide, Australia, 2000, pp. 365–374.

[18] ——, "New iterative algorithm for algebraic Riccati equation related to H_∞ control problem of singularly perturbed systems," *IEEE Trans. Autom. Contr.*, vol. 46, no. 10, pp. 1659–1666, Oct. 2001.

[19] ——, "A revised Kleinman algorithm to solve algebraic Riccati equation of singularly perturbed systems," *Automatica*, vol. 38, no. 3, pp. 553–558, 2002.

[20] ——, "Numerical algorithm for solving cross-coupled algebraic Riccati equations of singularly perturbed systems," in *Advances in Dynamic Games: Applications to Economics, Finance, Optimization and Stochastic Control*, A. S. Nowak and K. Szajowski, Eds. Boston: Birkhauser, 2005, pp. 545–570.

[21] H. Mukaidani, T. Shimomura, and H. Xu, "Numerical algorithm for solving cross-coupled algebraic Riccati equations related to Nash games of multimodeling systems," in *Proc. IEEE Conf. Decision and Control*, 2002, pp. 4167–4172.

[22] H. Mukaidani, "Near-optimal Kalman filters for multiparameter singularly perturbed linear systems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 5, pp. 717–721, May 2003.

[23] H. Xu, H. Mukaidani, and K. Mizukami, "An order reduction procedure to "Composite Nash solution of singularly perturbed systems"," in *Proc. IFAC World Congr.*, vol. F, 1999, pp. 355–360.

[24] T. Yamamoto, "A method for finding sharp error bounds for Newton's method under the Kantorvich assumptions," *Num. Math.*, vol. 49, pp. 203–220, 1986.

[25] K. Zhou, *Essentials of Robust Control*. Englewood Cliffs, NJ: Prentice-Hall, 1998.

Hiroaki Mukaidani (M'01) received thes B.S. degree in integrated arts and sciences and the M.Eng. and Dr.Eng. degrees in information engineering from Hiroshima University, Hiroshima, Japan, in 1992, 1994 and 1997, respectively.

He worked with Hiroshima City University as a Research Associate from 1998 to 2002. Since 2002, he has been with the Graduate School of Education, Hiroshima University as an Assistant Professor, and currently an Associate Professor. His current research interests include robust control and its application of singularly perturbed systems.