Nash Games for Multiparameter Singularly Perturbed Systems With Uncertain Small Singular Perturbation Parameters

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***Abstract*—In this paper, the linear quadratic Nash games for infinite horizon multiparameter singularly perturbed systems with uncertain singular perturbation parameters are discussed. The main contribution is that a construction of high-order approximations to a strategy that guarantees a desired performance level on the basis of the successive approximation is proposed. It is newly shown that the proposed high-order approximate strategy improves the cost performance.**

***Index Terms*—Generalized cross-coupled multiparameter algebraic Riccati equation (GCMARE), multiparameter singularly perturbed systems (MSPS), successive approximation, uncertain small singular perturbation parameters.**

I. Introduction

THE control problems for the multiparameter singularly perturbed systems (MSPS) have been investigated extensively (see, e.g., [1] and reference therein). In these various studies of the MSPS, the linear quadratic Nash games for MSPS have been studied [2], [3]. Recent advance in theory of the numerical computation algorithm for the SPS and MSPS has allowed a revisiting of Nash games [4], [6]–[8]. The numerical method is a very powerful tool, it can not only efficiently find feasible solutions, but also easily handle reduced-order calculation. However, a limitation of these approaches is that the small parameters are assumed to be known. Thus, they are not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly. In fact, these parameters are often not known [1].

It is well-known that one of the approaches for constructing Nash equilibrium strategies of the MSPS is the composite design method [2], [3]. When the parameters represent the small unknown perturbations, the composite strategies are very useful. However, the composite Nash equilibrium solution achieves only a performance that is $O(\mu)$ (where $\mu := \sqrt{\varepsilon_1 \varepsilon_2}$) close to the full-order performance for small enough μ. Furthermore, since the closed-loop solution of the reduced Nash problem depends on the path along $\varepsilon_1/\varepsilon_2$ as $\mu \to 0$, we cannot conclude that the closed-loop solution of the full problem converges to the closed-loop solution of the reduced problem [3].

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In this paper, the linear quadratic Nash games for infinite horizon MSPS with the uncertain singular perturbation parameters are studied. The main contribution of this paper is to derive the high-order approximate strategy via the successive approximation method. As a result, the desired approximation strategies are obtained by solving the linear equation directly without any iterative computations. It is worth pointing out that the new high-order approximate strategy can be constructed even though there exist the uncertain small singular perturbation parameters. It should be noted that so far there is not known work addressing the problems of control of the MSPS and the SPS that the uncertain singular perturbation parameters are included. Finally, we claim that our results include the existing strategies [2] because the high-order approximate strategy without the exact information of the small perturbation parameters can be constructed.

II. Problem Formulation

Consider a linear time-invariant MSPS [1]

$$\dot{x}_0 = \sum_{i=0}^{2} A_{0i} x_i + \sum_{i=1}^{2} B_{0i} u_i, \qquad x_0(0) = x_0^0 \tag{1a}$$

$$\varepsilon_i \dot{x}_i = A_{i0} x_0 + A_{ii} x_i + B_{ii} u_i, \qquad x_i(0) = x_i^0 \tag{1b}$$

with quadratic cost functions

$$J_i(u_i, u_j) = \frac{1}{2} \int_0^{\infty} \left[y_i^T y_i + u_i^T R_{ii} u_i + \mu u_j^T R_{ij} u_j \right] dt,$$

$$R_{ii} > 0; \quad R_{ij} \geq 0; \quad \mu := \sqrt{\varepsilon_1 \varepsilon_2} \tag{2a}$$

$$y_i = C_{i0} x_0 + C_{ii} x_i = C_i x \tag{2b}$$

$$x = [x_0^T \quad x_1^T \quad x_2^T]^T, \qquad i, j = 1, 2, \quad i \neq j \tag{2c}$$

where $x_i \in \mathbf{R}^{n_i}, i = 0, 1, 2$ are the state vectors and $u_i \in \mathbf{R}^{m_i}, i = 1, 2$ are the control inputs. All the matrices are constant matrices of appropriate dimensions. It should be noted that the special case of $R_{12} = 0$ and $R_{21} = 0$ has been studied in the existing results [2], [8]. Therefore, our result is extension of the existing one.

ε_1 and ε_2 are two small positive singular bounded perturbation parameters of the same order of magnitude that are constrained by the known positive parameters $\bar{\varepsilon}_i$ and σ_i such that

$$\bar{\varepsilon}_i - \sigma_i \bar{\mu}^{\eta} \leq \varepsilon_i \leq \bar{\varepsilon}_i + \sigma_i \bar{\mu}^{\eta}, \qquad i = 1, 2 \tag{3a}$$

$$0 < k_1 \leq \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \leq k_2 < \infty \tag{3b}$$

where $\bar{\mu} := \sqrt{\bar{\varepsilon}_1 \bar{\varepsilon}_2}, \bar{\varepsilon}_i$, and $\sigma_i > 0, i = 1, 2$ are known constants. η is known constant that has an appropriate accuracy for the parameter ε_i. This hypothesis that is affected by the fixed uncertainties of the small parameters particularly will be allowed. It should be noted that the parameters ε_i are unknown but their bounds are known. Moreover, so far, there is no known work addressing the problem of control of such systems.

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero (see, e.g., [1]), that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha. \tag{4}$$

Let us introduce the partitioned matrices

$$\begin{aligned}
A_e &:= \Phi_e^{-1} A, \quad B_{1e} := \Phi_e^{-1} B_1 \\
B_{2e} &:= \Phi_e^{-1} B_2 \\
\Phi_e &:= \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix} \\
A &:= \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix} \\
B_1 &:= \begin{bmatrix} B_{01} \\ B_{11} \\ 0 \end{bmatrix} \\
B_2 &:= \begin{bmatrix} B_{02} \\ 0 \\ B_{22} \end{bmatrix} \\
S_1 &:= \begin{bmatrix} S_{001} & S_{011} & 0 \\ S_{011}^T & S_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
S_2 &:= \begin{bmatrix} S_{002} & 0 & S_{022} \\ 0 & 0 & 0 \\ S_{022}^T & 0 & S_{222} \end{bmatrix} \\
G_1^{\mu} &:= \begin{bmatrix} G_{001}^{\mu} & G_{011}^{\mu} & 0 \\ G_{011}^{\mu T} & G_{111}^{\mu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
G_2^{\mu} &:= \begin{bmatrix} G_{002}^{\mu} & 0 & G_{022}^{\mu} \\ 0 & 0 & 0 \\ G_{022}^{\mu T} & 0 & G_{222}^{\mu} \end{bmatrix} \\
Q_1 &:= \begin{bmatrix} Q_{001} & Q_{011} & 0 \\ Q_{011}^T & Q_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
Q_2 &:= \begin{bmatrix} Q_{002} & 0 & Q_{022} \\ 0 & 0 & 0 \\ Q_{022}^T & 0 & Q_{222} \end{bmatrix} \\
S_i &:= B_i R_{ii}^{-1} B_i^T \\
G_i^{\mu} &= \mu B_i R_{ii}^{-1} R_{ji} R_{ii}^{-1} B_i^T \\
Q_i &:= C_i^T C_i.
\end{aligned}$$

We now consider the linear quadratic Nash games for infinite horizon MSPS (1) under the following basic assumptions (see e.g., [2], [5]).

Assumption 1: There exists a $\mu^* > 0$ such that the triples $(A_e, B_{ie}, C_i), i = 1, 2$ are stabilizable and detectable for all $\mu \in (0, \mu^*]$, where $\mu = \sqrt{\varepsilon_1 \varepsilon_2}$.

Assumption 2: The triples $(A_{ii}, B_{ii}, C_{ii}), i = 1, 2$ are stabilizable and detectable.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Particularly, Assumption 1 has to be needed to guarantee the existence of the strategy for the full-order MSPS (1). It should be noted that Assumption 1 is less conservative under the study of the MSPS because such condition is based on the control-oriented assumption that are made in the existing results [2], [3].

The decision makers are required to select the closed loop control laws u_i^*, if they exist, such that

$$J_i(u_i^*, u_j^*) \le J_i(u_i, u_j^*), \qquad i, j = 1, 2, \qquad i \ne j. \tag{5}$$

The pair (u_1^*, u_2^*) are called Nash optimal strategy. Nash inequality shows that u_i^* regulates the state to zero with minimum output energy. The following lemma is already known [2], [3].

Lemma 1: Under Assumption 1, there exists an admissible controller such that the inequality (5) holds iff the following full-order generalized cross-coupled multiparameter algebraic Riccati equation (GCMARE):

$$\begin{aligned}
&A^T X + X^T A + Q_1 - X^T S_1 X \\
&\quad - X^T S_2 Y - Y^T S_2 X + Y^T G_2^{\mu} Y = 0 && \text{(6a)} \\
&A^T Y + Y^T A + Q_2 - Y^T S_2 Y \\
&\quad - Y^T S_1 X - X^T S_1 Y + X^T G_1^{\mu} X = 0 && \text{(6b)}
\end{aligned}$$

have solutions $\Phi_e X \ge 0$ and $\Phi_e Y \ge 0$, where

$$X := \begin{bmatrix} X_{00} & \varepsilon_1 X_{10}^T & \varepsilon_2 X_{20}^T \\ X_{10} & X_{11} & \sqrt{\alpha}^{-1} X_{21}^T \\ X_{20} & \sqrt{\alpha} X_{21} & X_{22} \end{bmatrix}$$

$$Y := \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \sqrt{\alpha}^{-1} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix}.$$

Then, the closed-loop linear Nash equilibrium solutions to the full-order problem are given by

$$u_1^* = -R_{11}^{-1} B_1^T X x \quad \text{and} \quad u_2^* = -R_{22}^{-1} B_2^T Y x. \tag{7}$$

It should be noted that it is impossible to solve the GCMARE (6) because the small perturbed parameters ε_i are partially unknown. Thus, our purpose is to find the parameter-independent high-order approximate strategies. Moreover, it should also be noted that G_i^{μ} of the GCMARE (6) has not been considered in [2], [8] because the special case of $R_{12} = 0$ and $R_{21} = 0$ has only been studied.

III. Asymptotic Structure

Nash equilibrium strategies for the MSPS will be studied under the following basic assumption, so that we can apply the proposed method to the MSPS.

Assumption 3: The Hamiltonian matrices $T_{iii}, i = 1, 2$ are nonsingular, where

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \tag{8}$$

In order to guarantee the existence of the solutions for the reduced-order AREs that will appear later as (9), Assumption 3 has to be needed. It should be noted that the assumption that T_{iii} is nonsingular is weak than the assumption that A_{ii} is nonsingular that are made in [2] and [3].

Under Assumptions 2 and 3, we can obtain the following zeroth-order equations of the GCMARE (6) as $\mu \to +0$ by using the similar technique in [8]:

$$A_s^T\bar{X}_{00} + \bar{X}_{00}A_s + Q_{s1} - \bar{X}_{00}S_{s1}\bar{X}_{00} - \bar{X}_{00}S_{s2}\bar{Y}_{00} - \bar{Y}_{00}S_{s2}\bar{X}_{00} = 0 \quad (9a)$$

$$A_s^T\bar{Y}_{00} + \bar{Y}_{00}A_s + Q_{s2} - \bar{Y}_{00}S_{s2}\bar{Y}_{00} - \bar{Y}_{00}S_{s1}\bar{X}_{00} - \bar{X}_{00}S_{s1}\bar{Y}_{00} = 0 \quad (9b)$$

$$A_{11}^T\bar{X}_{11} + \bar{X}_{11}A_{11} - \bar{X}_{11}S_{111}\bar{X}_{11} + Q_{111} = 0 \quad (9c)$$

$$A_{22}^T\bar{Y}_{22} + \bar{Y}_{22}A_{22} - \bar{Y}_{22}S_{222}\bar{Y}_{22} + Q_{222} = 0 \quad (9d)$$

$$\bar{X}_{22} = 0 \quad \text{and} \quad \bar{Y}_{11} = 0 \quad (9e)$$

$$\begin{bmatrix} \bar{X}_{10}^T \\ \bar{Y}_{10}^T \end{bmatrix}^T = [\,\bar{X}_{11} \quad -I_{n_1}\,]\, T_{111}^{-1}T_{101} \begin{bmatrix} I_{n_0} & 0 \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix} \quad (9f)$$

$$\begin{bmatrix} \bar{X}_{20}^T \\ \bar{Y}_{20}^T \end{bmatrix}^T = [\,\bar{Y}_{22} \quad -I_{n_2}\,]T_{222}^{-1}T_{202} \begin{bmatrix} 0 & I_{n_0} \\ \bar{X}_{00} & \bar{Y}_{00} \end{bmatrix} \quad (9g)$$

where I_n denotes the $n \times n$ identity matrix

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix} = \begin{bmatrix} A_{00} & * \\ * & -A_{00}^T \end{bmatrix} - \sum_{i=1}^{2} T_{0ii}T_{iii}^{-1}T_{i0i}$$

$$\begin{bmatrix} * & -S_{si} \\ -Q_{si} & * \end{bmatrix} = T_{00i} - T_{0ii}T_{iii}^{-1}T_{i0i}$$

$$T_{00i} = \begin{bmatrix} A_{00} & -S_{00i} \\ -Q_{00i} & -A_{00}^T \end{bmatrix}$$

$$T_{0ii} = \begin{bmatrix} A_{0i} & -S_{0ii} \\ -Q_{0ii} & -A_{i0}^T \end{bmatrix}$$

$$T_{i0i} = \begin{bmatrix} A_{i0} & -S_{0ii}^T \\ -Q_{0ii}^T & -A_{0i}^T \end{bmatrix}, \qquad i = 1, 2.$$

The following lemma show the relation between the solutions X and Y and the zeroth-order solutions $\bar{X}_{lm}$ and $\bar{Y}_{lm}, lm = 00, 10, 20, 11, 21, 22$.

Lemma 2: Suppose that

$$\det\Gamma = \det \begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & -[(S_{s1}\bar{Y}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s1}\bar{Y}_{00})^T] \\ -[(S_{s2}\bar{X}_{00})^T \otimes I_{n_0} + I_{n_0} \otimes (S_{s2}\bar{X}_{00})^T] & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T \end{bmatrix} \neq 0 \quad (10)$$

where $\hat{A}_s := A_s - S_{s1}\bar{X}_{00} - S_{s2}\bar{Y}_{00}$ and the matrix $\hat{A}_s$ is stable.

Under Assumptions 1–3, the GCMARE (6) admits unique solutions X and Y such that these matrices possess the power series expansion at $\mu = 0$. That is

$$X = \bar{X} + O(\mu) = \begin{bmatrix} \bar{X}_{00} & 0 & 0 \\ \bar{X}_{10} & \bar{X}_{11} & 0 \\ \bar{X}_{20} & 0 & 0 \end{bmatrix} + O(\mu) \quad (11a)$$

$$Y = \bar{Y} + O(\mu) = \begin{bmatrix} \bar{Y}_{00} & 0 & 0 \\ \bar{Y}_{10} & 0 & 0 \\ \bar{Y}_{20} & 0 & \bar{Y}_{22} \end{bmatrix} + O(\mu). \quad (11b)$$

Proof: Since the proof can be done by using the similar technique in [8], it is omitted. ■

The condition (10) is established by applying the implicit function theorem. Namely, if the condition (10) holds, we get the asymptotic structure of (11). It may be noted that although this condition seems to be conservative for the control designer, the existence of the asymptotic structure of (11) is not guaranteed if such condition does not hold.

It is well known that the GCMARE (6) could have several positive definite solutions and even some indefinite solutions [10]. However, since the implicit function theorem admits unique solution at the neighborhood of $\mu = 0$, the uniqueness of these solutions with the form (11) is guaranteed if the optimal solutions exist.

IV. Successive Approximation Method

Introducing the error matrices E and F, the solutions of (11) can be changed as follows.

$$X = \bar{X} + \mu E = \bar{X} + \mu \begin{bmatrix} E_{00} & \sqrt{\alpha}X_{10}^T & \sqrt{\alpha}^{-1}X_{20}^T \\ E_{10} & E_{11} & \sqrt{\alpha}^{-1}E_{21}^T \\ E_{20} & \sqrt{\alpha}E_{21} & E_{22} \end{bmatrix} \quad (12a)$$

$$Y = \bar{Y} + \mu F = \bar{Y} + \mu \begin{bmatrix} F_{00} & \sqrt{\alpha}Y_{10}^T & \sqrt{\alpha}^{-1}Y_{20}^T \\ F_{10} & F_{11} & \sqrt{\alpha}^{-1}F_{21}^T \\ F_{20} & \sqrt{\alpha}F_{21} & F_{22} \end{bmatrix} \quad (12b)$$

where $X_{i0} = \bar{X}_{i0} + \mu E_{i0}$ and $Y_{i0} = \bar{Y}_{i0} + \mu F_{i0}, i = 1, 2$.

Substituting (12) into the GCMARE (6) and using

$$A^T\bar{X} + \bar{X}^TA + Q_1 - \bar{X}^TS_1\bar{X} - \bar{X}^TS_2\bar{Y} - \bar{Y}^TS_2\bar{X} = 0 \quad (13a)$$

$$A^T\bar{Y} + \bar{Y}^TA + Q_2 - \bar{Y}^TS_2\bar{Y} - \bar{Y}^TS_1\bar{X} - \bar{X}^TS_1\bar{Y} = 0 \quad (13b)$$

we have

$$\mathbf{G}_1(E, F) := D^TE + E^TD - L^TF - F^TL + Y^TG_2Y - \mu(E^TS_1E + E^TS_2F + F^TS_2E) = 0 \quad (14a)$$

$$\mathbf{G}_2(E, F) := D^TF + F^TD - M^TE - E^TM + X^TG_1X - \mu(F^TS_2F + F^TS_1E + E^TS_1F) = 0 \quad (14b)$$

where

$$D = A - S_1\bar{X} - S_2\bar{Y} = \begin{bmatrix} D_{00} & D_{x01} & D_{y02} \\ D_{x10} & D_{x11} & 0 \\ D_{y20} & 0 & D_{y22} \end{bmatrix}$$

$$L = S_2\bar{X} = \begin{bmatrix} L_{x00} & 0 & 0 \\ 0 & 0 & 0 \\ L_{x20} & 0 & 0 \end{bmatrix}$$

$$M = S_1\bar{Y} = \begin{bmatrix} M_{y00} & 0 & 0 \\ M_{y10} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_{00} = A_{00} - S_{001}\bar{X}_{00} - S_{002}\bar{Y}_{00} - S_{011}\bar{X}_{10} - S_{022}\bar{Y}_{20}$$

$$D_{x01} = A_{01} - S_{011}\bar{X}_{11}$$

$$D_{x11} = A_{11} - S_{111}\bar{X}_{11}$$

$$D_{y02} = A_{02} - S_{022}\bar{Y}_{22}$$

$$D_{y22} = A_{22} - S_{222}\bar{Y}_{22}$$

$$D_{x10} = A_{10} - S_{011}^T\bar{X}_{00} - S_{111}\bar{X}_{10}$$

$$D_{y20} = A_{20} - S_{022}^T\bar{Y}_{00} - S_{222}\bar{Y}_{20}$$

$$L_{x00} = S_{002}\bar{X}_{00} + S_{022}\bar{X}_{20}$$

$$L_{x20} = S_{022}^T\bar{X}_{00} + S_{222}\bar{X}_{20}$$

$$M_{y00} = S_{001}\bar{Y}_{00} + S_{011}\bar{Y}_{10}$$

$$M_{y10} = S_{011}^T\bar{Y}_{00} + S_{111}\bar{Y}_{10}$$

$$G_1 = B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^T$$

$$G_2 = B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2^T.$$

When the parameters ε_i are known, the GCMARE (14) can be solved by using the existing results [6]–[8]. However, the exact values are unknown for the considered problem in this paper. Hence, we apply the successive approximation to the (14). Firstly, we shall prove existence of the implicit function and construct asymptotic solutions of the GCMARE (14).

Lemma 3: Under Assumptions 1–3, if the condition (10) is met, the GCMARE (14) has unique solutions E and F such that these matrices possess a power series expansion at $\mu = 0$. That is

$$E := E(\mu, \alpha) = \sum_{n=0}^{\infty} \mu^n E^{(n)}(\alpha) \tag{15a}$$

$$F := F(\mu, \alpha) = \sum_{n=0}^{\infty} \mu^n F^{(n)}(\alpha) \tag{15b}$$

where

$$E^{(n)}(\alpha) = \begin{bmatrix} E_{00}^{(n)} & \sqrt{\alpha} E_{10}^{(n-1)T} & \sqrt{\alpha}^{-1} E_{20}^{(n-1)T} \\ E_{10}^{(n)} & E_{11}^{(n)} & \sqrt{\alpha}^{-1} E_{21}^{(n)T} \\ E_{20}^{(n)} & \sqrt{\alpha} E_{21}^{(n)} & E_{22}^{(n)} \end{bmatrix}$$

$$F^{(n)}(\alpha) = \begin{bmatrix} F_{00}^{(n)} & \sqrt{\alpha} F_{10}^{(n-1)T} & \sqrt{\alpha}^{-1} F_{20}^{(n-1)T} \\ F_{10}^{(n)} & F_{11}^{(n)} & \sqrt{\alpha}^{-1} F_{21}^{(n)T} \\ F_{20}^{(n)} & \sqrt{\alpha} F_{21}^{(n)} & F_{22}^{(n)} \end{bmatrix}$$

$$E^{(0)} := \begin{bmatrix} E_{00}^{(0)} & \sqrt{\alpha} \bar{X}_{10}^{T} & \sqrt{\alpha}^{-1} \bar{X}_{20}^{T} \\ E_{10}^{(0)} & E_{11}^{(0)} & \sqrt{\alpha}^{-1} E_{21}^{(0)T} \\ E_{20}^{(0)} & \sqrt{\alpha} E_{21}^{(0)} & E_{22}^{(0)} \end{bmatrix}$$

$$F^{(0)} := \begin{bmatrix} F_{00}^{(0)} & \sqrt{\alpha} \bar{Y}_{10}^{T} & \sqrt{\alpha}^{-1} \bar{Y}_{20}^{T} \\ F_{10}^{(0)} & F_{11}^{(0)} & \sqrt{\alpha}^{-1} F_{21}^{(0)T} \\ F_{20}^{(0)} & \sqrt{\alpha} F_{21}^{(0)} & F_{22}^{(0)} \end{bmatrix}, \quad n = 1, 2, \ldots.$$

Proof: We also apply the implicit function theorem to the partitioned GCMARE (14). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\mu = 0$. Since the detailed proof is the same as the proof that is given by [8], it is omitted. ∎

Secondly, we shall establish the methodology for solving the GCMARE (14) for the matrix solutions E and F. Since (14) must hold for $\mu = 0$, the matrices $E^{(0)}$ and $F^{(0)}$ satisfy the following GCMARE:

$$D^T E^{(0)} + E^{(0)T} D - L^T F^{(0)} - F^{(0)T} L + \bar{Y}^T G_2 \bar{Y} = 0 \tag{16a}$$

$$D^T F^{(0)} + F^{(0)T} D - M^T E^{(0)} - E^{(0)T} M + \bar{X}^T G_1 \bar{X} = 0. \tag{16b}$$

On the other hand, substituting the matrices E and F into the (14) and equating successively coefficients of equal powers of μ, we get the following linear equations:

$$D^T E^{(n+1)} + E^{(n+1)T} D - L^T F^{(n+1)} - F^{(n+1)T} L + \mathbf{Q}_1^{(n)} = 0 \tag{17a}$$

$$D^T F^{(n+1)} + F^{(n+1)T} D - M^T E^{(n+1)} - E^{(n+1)T} M + \mathbf{Q}_2^{(n)} = 0 \tag{17b}$$

where

$$\begin{aligned}
\mathbf{Q}_1^{(0)} &= E^{(0)T} S_1 E^{(0)} + \bar{Y}^T G_2 F^{(0)} + F^{(0)T} G_2 \bar{Y} \\
\mathbf{Q}_2^{(0)} &= F^{(0)T} S_2 F^{(0)} + \bar{X}^T G_1 E^{(0)} + E^{(0)T} G_1 \bar{X} \\
\mathbf{Q}_1^{(1)} &= E^{(0)T} S_1 E^{(1)} + E^{(1)T} S_1 E^{(0)} + E^{(0)T} S_2 F^{(1)} \\
&\quad + E^{(1)T} S_2 F^{(0)} + F^{(0)T} S_2 E^{(1)} + F^{(1)T} S_2 E^{(0)} \\
&\quad + \bar{Y}^T G_2 F^{(1)} + F^{(1)T} G_2 \bar{Y} + F^{(0)T} G_2 F^{(0)} \\
\mathbf{Q}_2^{(1)} &= F^{(0)T} S_2 F^{(1)} + F^{(1)T} S_2 F^{(0)} + F^{(0)T} S_1 E^{(1)} \\
&\quad + F^{(1)T} S_1 E^{(0)} + E^{(0)T} S_1 F^{(1)} + E^{(1)T} S_1 F^{(0)} \\
&\quad + \bar{X}^T G_1 E^{(1)} + E^{(1)T} G_1 \bar{X} + E^{(0)T} G_1 E^{(0)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}_1^{(n)} &:= \bar{Y}^T G_2 F^{(n)} + F^{(n)T} G_2 \bar{Y} + \sum_{k=0}^{n-1} F^{(k)T} G_2 F^{(n-1-k)} \\
&\quad + \sum_{k=0}^{n} E^{(k)T} S_1 E^{(n-k)} + \sum_{k=0}^{n} E^{(k)T} S_2 F^{(n-k)} \\
&\quad + \sum_{k=0}^{n} F^{(k)T} S_2 E^{(n-k)}, \qquad n \geq 2
\end{aligned} \tag{18a}$$

$$\begin{aligned}
\mathbf{Q}_2^{(n)} &:= \bar{X}^T G_1 E^{(n)} + E^{(n)T} G_1 \bar{X} + \sum_{k=0}^{n-1} E^{(k)T} G_1 E^{(n-1-k)} \\
&\quad + \sum_{k=0}^{n} F^{(k)T} S_2 F^{(n-k)} + \sum_{k=0}^{n} F^{(k)T} S_1 E^{(n-k)} \\
&\quad + \sum_{k=0}^{n} E^{(k)T} S_1 F^{(n-k)}, \qquad n \geq 2.
\end{aligned} \tag{18b}$$

It should be noted that the successive approximations (17) are independent of the small unknown parameter μ. Moreover, the approach used in this paper is quite different because the proposed successive approximations (17) are based on not the Newton's method [7] and the recursive algorithm [8], [9] but Maclaurin series expansions. Thus, the desired approximation strategies are obtained by solving the linear equations directly without any iterative computations.

Using the successive approximation (17), we now give the N-order approximate Nash strategy (19)

$$\begin{aligned}
u_{1\mathrm{app}}^{(N)} &= -R_{11}^{-1} B_1^T \left[\bar{X} + \sum_{n=0}^{N-1} \bar{\mu}^{n+1} E^{(n)}(\hat{\alpha}) \right] x \\
&= -R_{11}^{-1} B_1^T X^{(N)} x, \qquad N = 1, 2, \ldots
\end{aligned} \tag{19a}$$

$$\begin{aligned}
u_{2\mathrm{app}}^{(N)} &= -R_{22}^{-1} B_2^T \left[\bar{Y} + \sum_{n=0}^{N-1} \bar{\mu}^{n+1} F^{(n)}(\hat{\alpha}) \right] x \\
&= -R_{22}^{-1} B_2^T Y^{(N)} x, \qquad N = 1, 2, \ldots
\end{aligned} \tag{19b}$$

where $\hat{\alpha} := \bar{\varepsilon}_2 / \bar{\varepsilon}_1$.

Theorem 1: Assume that

$$\bar{\varepsilon}_j - \sigma_j \bar{\mu}^{N+1} \leq \varepsilon_j \leq \bar{\varepsilon}_j + \sigma_j \bar{\mu}^{N+1}, \quad j = 1, 2, \qquad N = 1, 2, \ldots. \tag{20}$$

Under Assumptions 1–3, the use of the high-order approximate strategy (19) results in $J_i(u_{i\mathrm{app}}^{(N)}, u_{j\mathrm{app}}^{(N)})$ satisfying

$$J_i\left(u_{i\mathrm{app}}^{(N)}, u_{j\mathrm{app}}^{(N)}\right) = J_i(u_i^*, u_j^*) + O(\bar{\mu}^{N+1}), \quad i = 1, 2; \quad N = 1, 2, \ldots \tag{21}$$

where $J_i(u_i^*, u_j^*), i = 1, 2$ are the optimal equilibrium values of the cost functions (2a).

Proof: We prove only the case $i = 1$. The proof of the case $i = 2$ is similar. When $u_{1\mathrm{app}}^{(N)}$ is used, the value of the performance index is

$$J_1\left(u_{1\mathrm{app}}^{(N)}, u_{2\mathrm{app}}^{(N)}\right) = \frac{1}{2}x(0)^T W_{1e}^{(N)} x(0) \tag{22}$$

where $W_{1e}^{(N)}$ is the positive semidefinite solution of the following multiparameter algebraic Lyapunov equation (MALE)

$$\begin{aligned}&\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right)^T W_{1e}^{(N)}\\ &\quad + W_{1e}^{(N)}\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right)\\ &\quad + Q_1 + X_e^{(N)}S_{1e}X_e^{(N)} + Y_e^{(N)}G_{2e}^{\mu}Y_e^{(N)} = 0\end{aligned} \tag{23}$$

with $A_e = \Phi_e^{-1}A, S_{ie} = \Phi_e^{-1}S_i\Phi_e^{-1}, G_{ie}^{\mu} = \Phi_e^{-1}G_i^{\mu}\Phi_e^{-1}$.

Subtracting (6a) from (23), we find that $V_{1e}^{(N)} = W_{1e}^{(N)} - X_e, X_e := \Phi_e X$ satisfies the following MALE:

$$\begin{aligned}&\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right)^T V_{1e}^{(N)}\\ &\quad + V_{1e}^{(N)}\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right)\\ &\quad + \left(X_e^{(N)} - X_e\right)S_{1e}\left(X_e^{(N)} - X_e\right)\\ &\quad + Y_eS_{2e}\left(X_e^{(N)} - X_e\right) + \left(X_e^{(N)} - X_e\right)S_{2e}Y_e\\ &\quad + Y_e^{(N)}G_{2e}^{\mu}Y_e^{(N)} - Y_eG_{2e}^{\mu}Y_e = 0.\end{aligned} \tag{24}$$

Using the relations $X_e^{(N)} - X_e = O(\bar{\mu}^{N+1}), Y_e^{(N)} - Y_e = O(\bar{\mu}^{N+1})$, $\bar{\mu} - \mu = O(\bar{\mu}^{N+1})$ and $\sqrt{\alpha} - \sqrt{\hat{\alpha}} = O(\bar{\mu}^N)$, we can change the form of (24) into (25)

$$\begin{aligned}&\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right)^T V_{1e}^{(N)}\\ &\quad + V_{1e}^{(N)}\left(A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)}\right) + O(\bar{\mu}^{N+1}) = 0.\end{aligned} \tag{25}$$

It is easy to verify that $V_{1e}^{(N)} = O(\bar{\mu}^{N+1})$ because $A_e - S_{1e}X_e^{(N)} - S_{2e}Y_e^{(N)} = \Phi_e^{-1}[D + O(\bar{\mu})]$ is stable by using the standard Lyapunov theorem [11] for sufficiently small $\bar{\mu}$. Consequently, the equality (21) holds. ■

Although ε_i is unknown, we can design the high-order approximate strategy that achieves the $O(\bar{\mu}^{N+1})$ approximation for the equilibrium value of the cost functional.

Using the similar technique of the Proof of Theorem 1, the following conditions are satisfied.

Theorem 2: Under Assumptions 1–3, the following result holds.

$$J_i\left(u_i, u_{j\mathrm{app}}^{(N)}\right) = J_i(u_i, u_j^*) + O(\bar{\mu}^{N+1}). \tag{26}$$

Proof: Since the proof can be done by using the above technique, it is omitted. ■

Finally, by using the similar manner that has been established in [2], the main result is easily derived.

Theorem 3: Under Assumptions 1–3, the use of the high-order strategies (19) results in (27)

$$J_i\left(u_{i\mathrm{app}}^{(N)}, u_{j\mathrm{app}}^{(N)}\right) \le J_i\left(u_i, u_{j\mathrm{app}}^{(N)}\right) + O(\bar{\mu}^{N+1}). \tag{27}$$

Proof: Let us rewrite an inequality (27) as

$$\begin{aligned}&J_i\left(u_{i\mathrm{app}}^{(N)}, u_{j\mathrm{app}}^{(N)}\right) - J_i\left(u_i, u_{j\mathrm{app}}^{(N)}\right)\\ &= J_i\left(u_{i\mathrm{app}}^{(N)}, u_{j\mathrm{app}}^{(N)}\right) - J_i(u_i^*, u_j^*) + J_i(u_i^*, u_j^*) - J_i(u_i, u_j^*)\\ &\quad + J_i(u_i, u_j^*) - J_i\left(u_i, u_{j\mathrm{app}}^{(N)}\right).\end{aligned} \tag{28}$$

Using (21), (5), and (26), the proof of (27) completes. The other case is similar. ■

It should be noted that our results include the existing strategies as a special case because it is easy to obtain the strategies that have been introduced in [2].

V. Conclusion

The linear quadratic Nash games for infinite horizon MSPS have been studied. The high-order approximate strategy that is based on the successive approximation has been proposed without the exact information of the singular perturbation parameters. As a result, even if the singular perturbation parameter μ are not too small and partially unknown, the proposed higher order approximations strategies attain the desired performance that does not depend on any path compared with the existing result [2], [3].

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