Boundary slopes of non-orientable Seifert surfaces for knots

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Abstract

We study non-orientable Seifert surfaces for knots in the 3-sphere, and examine their boundary slopes. In particular, it is shown that for a crosscap number two knot, there are at most two slopes which can be the boundary slope of its minimal genus non-orientable Seifert surface, and an infinite family of knots with two such slopes will be described. Also, we discuss the existence of essential non-orientable Seifert surfaces for knots.

Key words: boundary slope, non-orientable Seifert surface, knot
1991 MSC: 57M25

1 Introduction

For a knot K in the 3-sphere S^3, we mean by a *Seifert surface* a connected compact surface with boundary K. Usually, a Seifert surface is assumed to be orientable, but we allow non-orientable one in this paper. It is well known that

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any knot has an orientable Seifert surface. Also any knot has a non-orientable one as well. For example, we obtain it by adding a small half-twisted band to an orientable one.

Let $N(K)$ be a tubular neighborhood of K and let $E(K) = \mathrm{cl}(S^3 - N(K))$ be the exterior. A *slope* is the isotopy class of an essential unoriented simple closed curve on the torus $\partial E(K)$. The slopes on $\partial E(K)$ are parameterized by $\mathbb{Q} \cup \{1/0\}$ in the usual way, using a meridian-longitude system of K (see [13]).

For a Seifert surface F, it can be assumed that $F \cap N(K)$ is an annulus. (Then $F \cap E(K)$ is also referred to as a Seifert surface for K.) Then $F \cap \partial E(K)$ is an essential loop in $\partial E(K)$, and hence it defines a slope, which is called the *boundary slope* of F. By the homological argument, we see that the boundary slope of an orientable Seifert surface is always 0, whereas that of a non-orientable Seifert surface is an even integer.

If F is a non-orientable Seifert surface for K, then a new non-orientable Seifert surface F' for K can be obtained by adding a small half-twisted band to F locally. Thus any even integer can be the boundary slope of some non-orientable Seifert surface for K.

The *genus* $g(K)$ of a knot K is the minimal number of the genera of orientable Seifert surfaces for K. Clark [3] defined the *crosscap number* $cr(K)$ of K to be the minimal number of the first Betti numbers of non-orientable Seifert surfaces for K. (For the trivial knot, it is defined to be 0.) It is not easy to determine the crosscap number of a given knot in general. See [11,14,15]. A non-orientable Seifert surface for a knot is said to be minimal genus if its first Betti number equals the crosscap number of the knot.

In this paper, we focus on the boundary slopes of minimal genus non-orientable Seifert surfaces for knots.

Theorem 1 *For a crosscap number one knot, the boundary slope of its minimal genus non-orientable Seifert surface is unique.*

It is easy to see that the figure eight knot has crosscap number two and bounds two once-punctured Klein bottles with boundary slopes 4 and -4. Also the $(-2, 3, 7)$-pretzel knot bounds such with boundary slopes 16 and 20. See Section 3.

Theorem 2 *For a crosscap number two knot K, the boundary slope of its minimal genus non-orientable Seifert surface F is a multiple of four, and there are at most two slopes which can be the boundary slope of F. If there are two, α and β, then $|\alpha - \beta| = 4$ or 8. Furthermore, if $|\alpha - \beta| = 8$, then K is the figure eight knot and $\{\alpha, \beta\} = \{-4, 4\}$.*

As an earlier result, it was shown in [14] that a crosscap number two, genus one knot is a doubled knot, and that the boundary slope of a minimal genus non-orientable Seifert surface for such a knot is 4 or -4. Also, a crosscap number two composite knot is a connected sum of two 2-cabled knots, and the boundary slope of its minimal genus non-orientable Seifert surface is unique [15]. The proof of Theorem 2 is the main part of this paper, and is based on the analysis of graphs of intersections coming from two once-punctured Klein bottles bounded by a knot (cf. [4]).

Theorem 3 *There exists an infinite family of crosscap number two knots such that each of the knots bounds two minimal genus non-orientable Seifert surfaces whose boundary slopes have distance* 4.

In fact, we expect that our family gives all such knots.

In the case of higher crosscap numbers, we could not give an upper bound for the number of boundary slopes of minimal genus non-orientable Seifert surfaces, but it is not hard to give examples which admit some such slopes.

Theorem 4 *For any integer $n \geq 3$, there exist infinitely many knots K with $cr(K) = n$ such that there are at least n (when n is even) or $n - 1$ (when n is odd) slopes which can be the boundary slope of its minimal genus non-orientable Seifert surface.*

Any minimal genus orientable Seifert surface for a knot K is essential (that is, both incompressible and boundary incompressible) in the exterior $E(K)$. Also we can show that any minimal genus non-orientable Seifert surface for a knot is incompressible. But there exists a knot (e.g. 7_4 in the knot table [13]) whose minimal genus non-orientable Seifert surface cannot be essential [1]. As far as we know, the next question seems to be unknown.

Question 5 *Does every knot have an essential non-orientable Seifert surface?*

In this direction, it is proved in [5] that alternating knots have essential non-orientable Seifert surfaces by using the checkerboard surfaces.

We have a partial answer to this question.

Theorem 6 *Every knot whose crosscap number is at most two has an essential non-orientable Seifert surface.*

Acknowledgements

We would like to thank Makoto Ozawa for his helpful conversations.

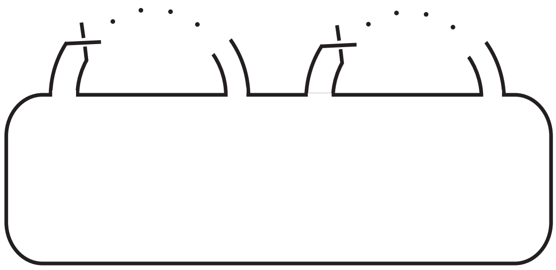

Fig. 1.

2 Crosscap number one case

In this section, we prove Theorem 1.

Proof of Theorem 1 Let K be a crosscap number one knot and let A be a Möbius band spanned by K. Then K is a cable knot of the center line of A. More precisely, K is a $(2,p)$-cable knot of some knot (possibly, the unknot) for an odd integer p (see [3, Proposition 2.2]). Then it is easy to see that the boundary slope of A is equal to $2p$.

Since $2p$ is the only slope which yields a reducible manifold for a $(2,p)$-cable knot [6, Corollaries 7.3, 7.4], we have the uniqueness of the boundary slopes of Möbius bands spanned by K. This completes the proof of Theorem 1. □

3 Crosscap number two case

Let K be a crosscap number two knot and let F be a once-punctured Klein bottle with $\partial F = K$.

Lemma 7 *The boundary slope of F is a multiple of four.*

PROOF. First F can be expressed as a disk with two non-orientable bands. That is, each band has an odd number of half-twists as shown in Figure 1. We can assume that crossings of bands are as in Figure 2 and that each band has twists near its end and it is flat elsewhere. Figure 3 shows an example.

Let K' be a simple loop obtained by pushing K into F slightly. Then the linking number of K and K' is equal to the boundary slope of F. Each crossing on bands contributes $4 \pmod 4$, and each set of half-twists on bands contributes $2 \pmod 4$. Therefore the linking number is a multiple of four. □

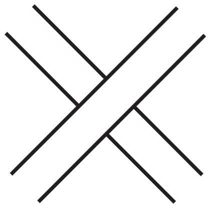

Fig. 2.

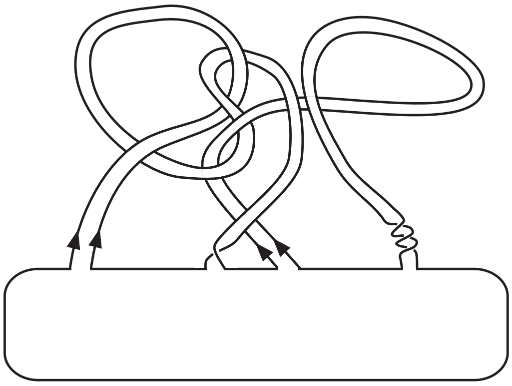

Fig. 3.

For a surface S ($\neq S^2, D^2$) properly embedded in $E(K)$, S is said to be *incompressible* in $E(K)$ if for each disk $D \subset E(K)$ with $D \cap S = \partial D$, there is a disk $D' \subset S$ with $\partial D = \partial D'$, and *boundary incompressible* in $E(K)$ if for each disk $E \subset E(K)$ with $E \cap S = \alpha$ and $E \cap \partial E(K) = \beta$ (where $\alpha \cup \beta = \partial E$, $\alpha \cap \beta = \partial\alpha = \partial\beta$), α cuts off a disk from S. See [9].

Lemma 8 *$F \cap E(K)$ is incompressible and boundary incompressible in $E(K)$.*

PROOF. For simplicity, we denote $F \cap E(K)$ by F. Assume that F is compressible in $E(K)$, and let D be a compressing disk for F. Note that ∂D is orientation-preserving in F.

If ∂D is non-separating in F, then compression along D gives a disk bounded by K. This means that K is unknotted, a contradiction. If ∂D is separating in F, then compression along D gives a projective plane in $E(K)$, which is impossible. Therefore, F is incompressible in $E(K)$.

Next, assume that F is boundary compressible, and let E be a boundary compressing disk for F. That is, $\partial E = \alpha \cup \beta$, $\alpha \subset F$ and $\beta \subset \partial E(K)$, where α is an essential arc in F. Let r denote the boundary slope of F, and let us consider r-surgery $K(r)$. Let $\widehat{F}$ be the Klein bottle in $K(r)$ obtained by capping ∂F off by a meridian disk of the attached solid torus V. If β bounds a disk E'

on $\partial E(K)$ together with a subarc of ∂F, then $E \cup E'$ gives a compressing disk for F in $E(K)$ (after pushing off from $\partial E(K)$). This contradicts the incompressibility of F. Hence the union of β and a subarc of ∂F forms a longitude of V. Since the total space $K(r)$ is orientable, α must be orientation-reversing on F. Thus the core knot K_r of V can be isotoped to an orientation-reversing loop on $\widehat{F}$ using E.

Now, K_r is an orientation-reversing loop on $\widehat{F}$. Then $V \cap \widehat{F}$ is a Möbius band, and hence $B = \mathrm{cl}(\widehat{F} - V \cap \widehat{F})$ is also a Möbius band. Thus $E(K)$ contains a properly embedded Möbius band B. Let s be the boundary slope of B. Then s-surgery $K(s)$ contains a projective plane. Hence $K(s)$ is real projective 3-space P^3 or a reducible manifold with P^3 summand. In the latter case, s must be integral by [8]. In the former case, K is not a torus knot by [10]. Then the cyclic surgery theorem [4] implies that s is integral. This contradicts that K has crosscap number two. Hence F is boundary incompressible. $\square$

Now, we consider two once-punctured Klein bottles P and Q bounded by K. But we use the same notations P and Q for $P \cap E(K)$ and $Q \cap E(K)$ hereafter. By Lemma 7, we can assume that the boundary slopes of P and Q are $4k$ and 4ℓ, respectively. As usual, let $\Delta = \Delta(4k, 4\ell) = 4|k - \ell|$ denote the minimal geometric intersection number of those boundary slopes on $\partial E(K)$.

We may assume that P and Q intersect transversely. By the incompressibility of P and Q (Lemma 8), we can assume that no circle component of $P \cap Q$ bounds a disk in P or Q. We can further assume that ∂P intersects ∂Q in exactly Δ points. Let $\widehat{P}$ be the Klein bottle obtained by capping ∂P off by a disk. Define $\widehat{Q}$ similarly.

Let G_P be the graph in $\widehat{P}$ obtained by taking as the (fat) vertex the disk $\widehat{P} - \mathrm{Int}P$ and as the edges the arc components of $P \cap Q$ in $\widehat{P}$. Similarly, G_Q is the graph in $\widehat{Q}$. Note that both of G_P and G_Q have only one vertex of degree Δ.

Number the points of $\partial P \cap \partial Q$ $1, 2, \ldots, \Delta$ in sequence along ∂P. Remark that the labels $1, 2, \ldots, \Delta$ appear in the same order along ∂Q (with a suitable direction). This comes from the fact that both of ∂P and ∂Q have integral slopes.

A *trivial loop* in a graph is a length one cycle which bounds a disk face of the graph.

The next lemma follows from Lemma 8.

Lemma 9 *Neither G_P nor G_Q contains trivial loops.*

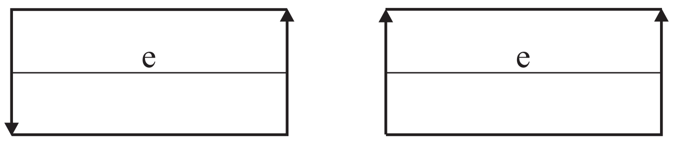

Fig. 4.

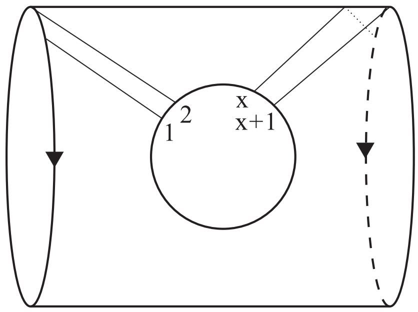

Fig. 5.

Although P and Q are non-orientable, we can establish a parity rule. In fact, this is a natural generalization of the usual parity rule [4].

Assign arbitrary orientations to ∂P and ∂Q. Let e be an edge in G_P. Since e is an arc properly embedded in P, a regular neighborhood D of e in P is a disk in P. Then $\partial D = a \cup b \cup c \cup d$, where a and c are arcs in ∂P with induced orientations from ∂P. On D, if a and c have opposite directions as illustrated in Figure 4, then e is called *positive*, otherwise *negative*.

Similarly, define the sign of edges in G_Q. Then we have the following rule.

Lemma 10 (Parity rule) *An edge e is positive (negative resp.) in G_P if and only if e is negative (positive resp.) in G_Q.*

PROOF. This follows from the fact that $E(K)$ is orientable and $\partial E(K)$ is a torus. $\square$

Lemma 11 *G_P and G_Q contain at most two negative edges. Furthermore, if there are two negative edges, then they are not parallel.*

PROOF. Assume that G_P contains three negative edges. Since there are only two isotopy classes of negative edges in G_P, there exist two negative edges e_1 and e_2 that are parallel in G_P. We can assume that e_1 has the labels $\{1, x\}$ and e_2 has $\{2, x+1\}$. See Figure 5. Here, two end circles of the cylinders are identified through a suitable involution to form the Klein bottle $\widehat{P}$.

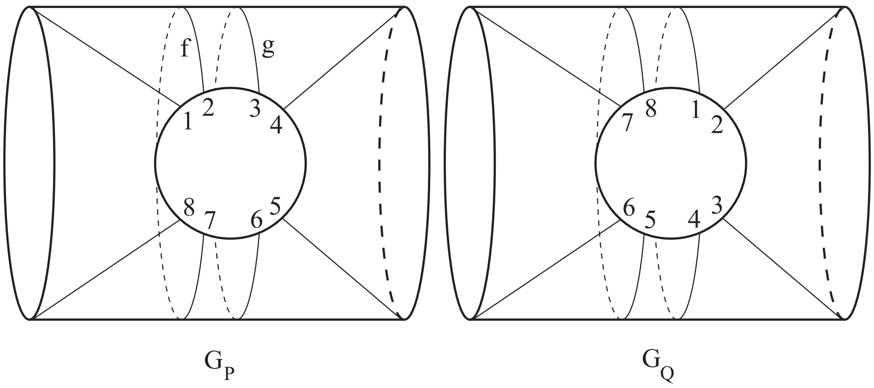

Fig. 6.

By the parity rule, e_1 and e_2 are positive in G_Q. There are two isotopy classes of positive edges in G_Q, but G_Q cannot have two non-isotopic positive edges. (By Lemma 9, there are no trivial loops.) Therefore, e_1 and e_2 are parallel in G_Q. Then the labels would appear in a wrong order along ∂Q.

The latter conclusion immediately follows from the above argument. $\square$

Lemma 12 $\Delta \leq 8$.

PROOF. By Lemma 11 and the parity rule, each of G_P and G_Q contains at most four edges. Since the vertex of G_P, say, has degree Δ, we have $\Delta \leq 8$. $\square$

Thus $\Delta = 0, 4$ or 8.

First, we prove that if $\Delta = 8$ then the knot is the figure eight knot.

Consider the standard (minimal crossing) diagram of the figure eight knot. Then the two checkerboard surfaces give once-punctured Klein bottles bounded by the figure eight knot. One has the boundary slope 4, and the other has -4. Then we have a pair of graphs as illustrated in Figure 6.

Proposition 13 *If $\Delta = 8$ then K is the figure eight knot, and moreover the boundary slope of a once-punctured Klein bottle bounded by K is 4 or -4.*

PROOF. Assume that $\Delta = 8$. By Lemma 11 and the parity rule, each of G_P and G_Q contains exactly two positive edges and two negative edges. Then there is only one configuration for the pair $\{G_P, G_Q\}$ as shown in Figure 6. In addition, $P \cap Q$ contains no circle component, since each face of G_P and G_Q is a disk. From now on, it is convenient if we denote by P and Q the original once-punctured Klein bottles bounded by K.

Let f and g be the positive edges parallel in G_P (see Figure 6), and let D_1 be the disk representing the parallelism of f and g in P.

Then a thin regular neighborhood B_1 of D_1 in S^3 gives a 2-string trivial tangle $(B_1, B_1 \cap K)$. Hereafter, we fix an orientation of K, and assume that any subarc of K has an induced orientation from K. Then D_1 respects this orientation, that is, there is an orientation of D_1 which induces compatible orientations to $B_1 \cap K$.

Let $B_2 = \mathrm{cl}(S^3 - B_1)$. Then $(B_2, B_2 \cap K)$ is also a 2-string tangle.

Claim 14 *The strings of the tangle $(B_2, B_2 \cap K)$ are parallel in B_2.*

Proof of Claim 14 Let $D_2 = Q \cap B_2$. Then D_2 is a disk whose boundary consists of two strings of the tangle $(B_2, B_2 \cap K)$ and two arcs in ∂B_2. Thus we have the conclusion. $\Box$

We remark that D_2 also respects the orientation of K.

If the two strings of $(B_2, B_2 \cap K)$ are unknotted in B_2, then $(B_2, B_2 \cap K)$ is a trivial tangle, and therefore K is a 2-bridge knot.

We know that the surgered manifolds $K(4k)$ and $K(4\ell)$ are not hyperbolike, because they contain Klein bottles. (Recall that a closed orientable 3-manifold is hyperbolike if it is irreducible, atoroidal, and is not a Seifert fibered manifold whose orbifold is a 2-sphere with at most three cone points [7].) By the classification of Dehn surgeries on 2-bridge knots by Brittenham and Wu [2, Theorem 1.1], we have the desired conclusion that K is the figure eight knot, and the boundary slopes of P and Q are 4 and -4.

Thus we assume that the two strings of $(B_2, B_2 \cap K)$ are knotted in B_2.

Let $D_3 \subset D_2$ be the disk giving the parallelism between two parallel positive edges in G_Q. Therefore, D_3 respects the orientation of K. Let $B_3 \subset B_2$ be a thin regular neighborhood of D_3 in S^3, and let $B_4 = \mathrm{cl}(S^3 - B_3)$.

Clearly, the tangle $(B_3, B_3 \cap K)$ is a trivial 2-string tangle, and the two strings of $(B_4, B_4 \cap K)$ are parallel by Claim 14, since G_P and G_Q have the same form.

If two strings of $(B_4, B_4 \cap K)$ are unknotted, then K is 2-bridge, and hence we have the desired conclusion as above.

Otherwise, each string of $(B_4, B_4 \cap K)$ is knotted in B_4. But this does not happen as seen in Figure 7. $\Box$

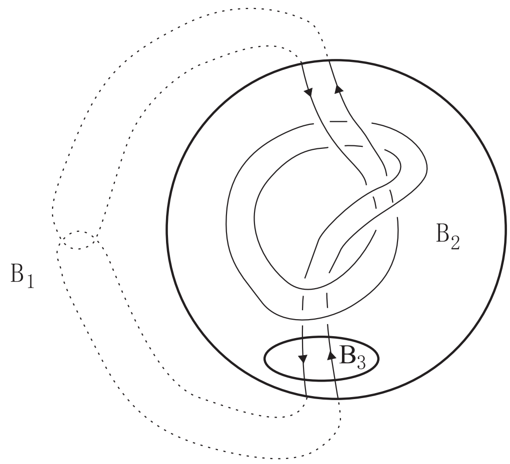

Fig. 7.

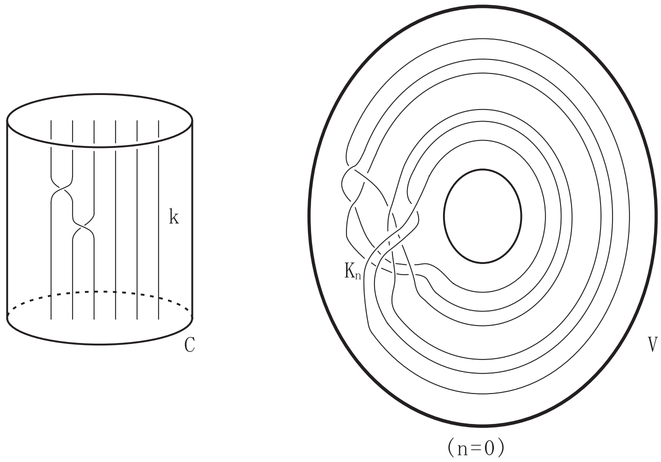

Fig. 8.

Proof of Theorem 2 By Lemma 7, the boundary slope of a minimal genus non-orientable Seifert surface for K is a multiple of four, and for any two such boundary slopes the distance Δ between them is 4 or 8 by Lemma 12. If K bounds two once-punctured Klein bottles whose boundary slopes have $\Delta = 8$, then K is the figure eight knot, and $\{-4, 4\}$ are the only possibilities of such boundary slopes by Proposition 13. If $\Delta = 4$, then there are only two boundary slopes that are consecutive multiples of four. The proof of Theorem 2 is complete. $\square$

Proof of Theorem 3 Let k be the braid $\sigma_1\sigma_2^{-1}$ of six strings contained in a cylinder C. For an integer n, glue the top and bottom of C with $(2n+1)\pi$ rotation to obtain a standard solid torus V in S^3 and a knot K_n. See Figure 8.

As shown in Figure 9, K_n bounds two once-punctured Klein bottles whose boundary slopes have distance 4. It is easy to see that K_n is a closed positive

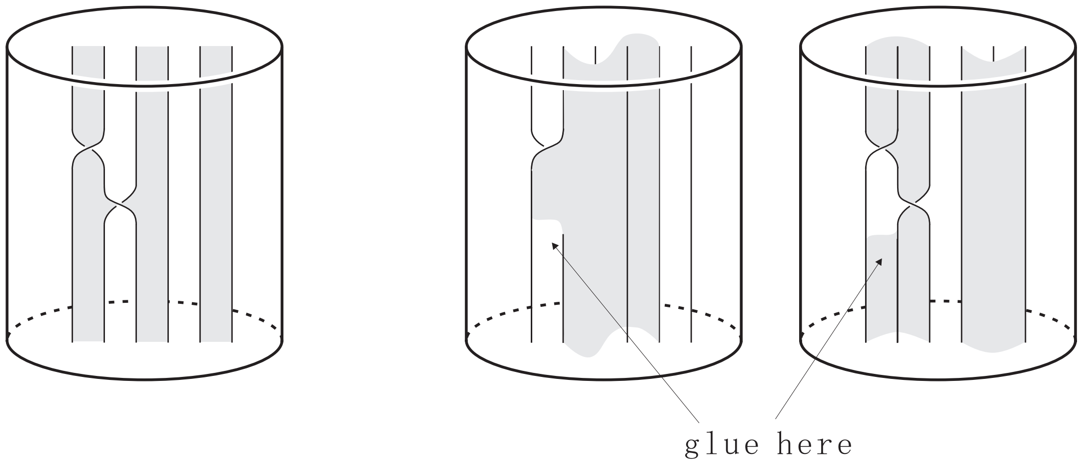

Fig. 9.

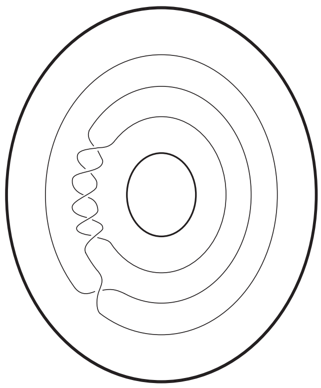

Fig. 10.

or negative braid, and therefore fibered. Then an easy Euler characteristic calculation shows that K_n has genus $15n + 5$ when $n \geq 0$ or $15|n| - 10$ when $n < 0$. Thus our family gives infinitely many knots.

Finally we prove $cr(K_n) = 2$. In fact, K_0 and K_{-1} are the $(-2, 3, 7)$-pretzel knot and its mirror image, and so they are crosscap number two. The other K_n is a satellite knot whose companion is a $(2, 2n+1)$ torus knot and pattern is as shown in Figure 10.

This pattern knot is hyperbolic, since the complement is homeomorphic to that of the $(-2, 4, 5)$-pretzel link [12]. Thus the torus decomposition of the exterior of K_n $(n \neq 0, -1)$ consists of one hyperbolic piece and a torus knot exterior. Hence K_n is not a 2-cabled by the uniqueness of the torus decomposition, and therefore its crosscap number is not one. This completes the proof of Theorem 3. $\square$

Obviously, the above construction can be generalized in such a way that V is knotted in S^3. We conjecture that this generalized construction gives all crosscap number two knots that admit two minimal genus non-orientable Seifert surfaces whose boundary slopes have distance 4. Note that the $(-2,3,7)$-pretzel knot and its mirror image are the only hyperbolic knots arisen by our construction. We remark that there is a supporting evidence for this conjecture. When $\Delta = 4$, we can conclude that there is only one possible configuration for the pair $\{G_P, G_Q\}$. In fact, our construction is based on the graph pair.

4 Higher crosscap number case

To prove Theorem 4, we use the next result about the additivity of crosscap numbers.

Lemma 15 (Murakami-Yasuhara [11]) *For any non-trivial knots $K_1, K_2, \ldots, K_n$ with $cr(K_i) \leq 2$ $(i = 1, 2, \ldots, n)$, we have*

$$cr(K_1 \sharp K_2 \sharp \ldots \sharp K_n) = cr(K_1) + cr(K_2) + \ldots + cr(K_n).$$

Proof of Theorem 4 Let $n \geq 3$ be an integer. Assume $n = 2m + 2$. Let K_p $(p \geq 0)$ be the knot constructed in the proof of Theorem 3, where the top and bottom of the cylinder C are glued with $(2p + 1)\pi$ rotation and V is unknotted in S^3. Recall that $cr(K_p) = 2$ and K_p has genus $15p + 5$. Let $K = K_p \sharp(\sharp^m 4_1))$, where 4_1 denotes the figure eight knot. Then $cr(K) = 2(m + 1) = n$ by Lemma 15. Let x and $x + 4$ be the boundary slopes of once-punctured Klein bottles bounding K_p. Then the boundary slope of minimal genus non-orientable Seifert surface for K can be $x - 4m, x - 4m + 4, x - 4m + 8, \ldots, x + 4m, x + 4m + 4$. Thus K has at least n such boundary slopes. Also different values of p yield distinct knots K.

Assume $n = 2m + 1$. Let T_p be the $(2, p)$ torus knot for an odd integer $p \geq 3$, and F the $(-2, 3, 7)$-pretzel knot. Let $K = T_p \sharp F \sharp(\sharp^{m-1} 4_1)$. Then $cr(K) = 3 + 2(m - 1) = n$ by Lemma 15. The boundary slope of minimal genus non-orientable Seifert surface for K can be $2p + 20 - 4m, 2p + 24 - 4m, 2p + 28 - 4m, 2p + 32 - 4m, \ldots, 2p + 12 + 4m, 2p + 16 + 4m$. Thus K has at least $n - 1$ such boundary slopes, and different values of p yield distinct knots K again. $\square$

5 Essential non-orientable Seifert surfaces

It is well known that a minimal genus orientable Seifert surface for a knot is essential in the knot exterior.

Lemma 16 *A minimal genus non-orientable Seifert surface for a knot is incompressible in the knot exterior.*

PROOF. Let K be a non-trivial knot in S^3, and let S be a minimal genus non-orientable Seifert surface for K. We use the same notation S for $S \cap E(K)$.

Assume that S is compressible in $E(K)$, and let D be a compressing disk for S.

If ∂D is separating in S, then compression along D gives two surfaces S_1 and S_2, where $\partial S_1 = K$ and S_2 is closed. Then S_2 is orientable, and hence S_1 is non-orientable. It is easy to see that $\beta_1(S_1) \leq \beta_1(S) - 2$, where β_1 denotes the first Betti number. This contradicts the minimality of S. Therefore ∂D is non-separating in S.

Let S' be the resulting surface obtained by compressing S along D. Since $\beta_1(S') = \beta_1(S) - 2$, S must be orientable by the minimality of S. But if we add a small half-twisted band to S', then we obtain a non-orientable Seifert surface S'' for K with $\beta_1(S'') = \beta_1(S') + 1 < \beta_1(S)$. This contradicts the minimality of S. $\square$

Finally, we prove Theorem 6.

Proof of Theorem 6 Let K be a non-trivial knot in S^3 with $cr(K) \leq 2$, and let S be a minimal genus non-orientable Seifert surface for K. By Lemma 16, S is incompressible in $E(K)$.

If S is boundary compressible, then the argument in the proof of Lemma 8 shows that K bounds a disk (if $cr(K) = 1$), or a Möbius band (if $cr(K) = 2$). In either case, this is a contradiction. $\square$

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