Monte Carlo Study of the Quenched Eguchi-Kawai Model

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The quenched Eguchi-Kawai model is studied by using the Monte Carlo technique. The data indicate that in the large-N limit the quenched Eguchi-Kawai model is identical to the Wilson theory throughout the whole range of coupling constants. The first-order phase transition is observed at $\beta/N = 0.29 \pm 0.02$ for the gauge group U(20).

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Recently there has been great progress in investigating the large-N limit of the U(N) [or SU(N)] lattice gauge theories. Eguchi and Kawai[1] claimed that in this limit we could replace the usual Wilson action by the reduced model which has only one site (one space-time point) with a periodic boundary condition for gauge fields. Although their model works quite well in the strong-coupling region, Bhanot, Heller, and Neuberger[2] argued that their model cannot reproduce the Wilson action in the weak-coupling region. In fact, a precise Monte Carlo simulation was performed by the present author[3] who showed that the model undergoes a phase transition before it reaches the continuum limit. To save the model the authors of Ref. 2 proposed the quenched version of the Eguchi-Kawai model (QEK model) which is supposed to have good behavior in weak coupling. A general quenching prescription was proposed by Parisi.[4] His method has wide applicability to general field theories having the U(N) [or SU(N)] internal symmetries. A detailed analysis was made by Gross and Kitazawa.[5] Among other things they showed that the QEK model is equivalent to the usual Wilson theory to any finite order in perturbation theory. The same problem is also considered by several authors.[6] In this paper the QEK model is studied by using the Monte Carlo method.

The action of the QEK model is obtained from that of the original model,

$$S_{\rm EK} = \sum_{\mu\neq\nu=1}^{4} {\rm tr} U_\mu U_\nu U_\mu{}^\dagger U_\nu{}^\dagger, \tag{1}$$

by substituting the link variable U_μ by $V_\mu{}^\dagger D_\mu V_\mu$,

$$S_{\rm QEK} = \sum_{\mu\neq\nu=1}^{4} {\rm tr}(V_\mu{}^\dagger D_\mu V_\mu)(V_\nu{}^\dagger D_\nu V_\nu) \times (V_\mu{}^\dagger D_\mu{}^* V_\mu)(V_\nu{}^\dagger D_\nu{}^* V_\nu), \tag{2}$$

where D_μ is the $N\times N$ diagonal matrix given by

$$D_\mu = {\rm diag}[\exp(i\theta_\mu{}^1), \ldots, \exp(i\theta_\mu{}^N)] \tag{3}$$

and V_μ is the unitary matrix. We calculate the expectation value of the physical quantity O first fixing the variable $\theta_\mu{}^i$,

$$\overline{O}(\theta_\mu{}^i) = \frac{\int dV_\mu O \exp[\beta S_{\rm QEK}(V_\mu, \theta_\mu{}^i)]}{\int dV_\mu \exp[\beta S_{\rm QEK}(V_\mu, \theta_\mu{}^i)]}. \tag{4}$$

Then we average $\overline{O}(\theta_\mu{}^i)$ over the quenched variable $\theta_\mu{}^i$ with the weight function $F(\theta_\mu{}^i)$

$$\langle O\rangle \equiv \frac{\int d\theta_\mu{}^i F(\theta_\mu{}^i)\overline{O}(\theta_\mu{}^i)}{\int d\theta_\mu{}^i F(\theta_\mu{}^i)}. \tag{5}$$

Equations (2)–(5) define the QEK model. In the large-N limit we expect that $\langle O\rangle$ does not depend on the weight function $F(\theta_\mu{}^i)$. However, at finite N (of order $N=20$), as we shall see later, $\overline{O}(\theta_\mu{}^i)$ depends crucially on $\theta_\mu{}^i$ in the strong and intermediate coupling. Thus to discuss the model definitely, we have to specify $F(\theta_\mu{}^i)$. In this paper the following two cases are considered:

$$F_1(\theta_\mu{}^i) = \prod_\mu \prod_{i>j} \sin^2[\tfrac{1}{2}(\theta_\mu{}^i - \theta_\mu{}^j)], \tag{6}$$

$$F_2(\theta_\mu{}^i) = 1. \tag{7}$$

$F_1(\theta_\mu{}^i)$ is naturally obtained if we parametrize U_μ as $U_\mu = V_\mu{}'^\dagger D_\mu V_\mu{}'$ and express the Haar measure dU in terms of $\theta_\mu{}^i$ and $V_\mu{}'$,

$$dU = F_1(\theta_\mu{}^i) d\theta_\mu{}^i\, dV_\mu{}' \tag{8}$$

[note that in Eq. (8) $V_\mu{}'$ has only $N^2 - N$ degrees of freedom but in Eq. (4) there is no such restriction]. $F_2(\theta_\mu{}^i)$ is convenient for the strong-coupling expansion. In Ref. 5, the equivalence of the QEK model and the Wilson theory is proved with this weight $F_2(\theta_\mu{}^i)$.

The quantity which we measure is the internal energy E of the QEK model,

$$E = N^{-1}\langle {\rm Re\,tr}(V_\mu{}^\dagger D_\mu V_\mu)(V_\nu{}^\dagger D_\nu V_\nu) \times (V_\mu{}^\dagger D_\mu{}^* V_\mu)(V_\nu{}^\dagger D_\nu{}^* V_\nu)\rangle. \tag{9}$$

In the strong-coupling case E behaves like

$$E = (\beta/N) + O(\beta^2) \tag{10}$$

as N becomes large. The weak-coupling expansion is given by

$$E_{\rm sym} = 1 - \frac{1-1/N}{8}\,\frac{N}{\beta}\,. \tag{11}$$

The original Eguchi-Kawai model (1) is invariant under the global U(1) transformation $U_\mu \to \exp(i \times \theta_\mu) U_\mu$. Equation (11) is also derived from Eq. (1) provided that this symmetry is not spontaneously broken. Actually in the original model this U(1) symmetry is broken spontaneously and the weak-coupling expansion reads[7]

$$E_{\rm asy} = 1 - \frac{1-1/N^2}{12}\,\frac{N}{\beta}\,. \tag{12}$$

As N becomes large, only $E_{\rm sym}$ agrees with the results obtained from the usual Wilson action.

The Monte Carlo simulation is performed in the following way: We first select the $\theta_\mu{}^i$ variables randomly from 0 to 2π with the weight F_1 or F_2. Then the average over V_μ variables is performed by using the Monte Carlo method proposed in the previous paper.[3] After the system reaches equilibrium the expectation value, Eq. (4), is measured. We then reselect the $\theta_\mu{}^i$ variables. Typically we change the variables $\theta_\mu{}^i$ five times to calculate E [see Eqs. (5) and (9)]. To reduce the computation time we define $V_{\mu\nu}$ by $V_{\mu\nu} = V_\mu V_\nu{}^\dagger D_\nu V_\nu V_\mu{}^\dagger$. Under the change of the variable $V_\mu \to BV_\mu$, $V_{\mu\nu}$ changes as $V_{\mu\nu} \to BV_{\mu\nu}B^\dagger$. Then in addition to the link variables V_μ, we store $V_{\mu\nu}$ in the memory. In this way the computation time of the change of the action $\mathrm{Tr} D_\mu V_{\mu\nu} \times D_\mu{}^* V_{\mu\nu}{}^\dagger$ grows only linearly in N.

First we consider the model with the weight function $F_1(\theta_\mu{}^i)$. Figure 1 shows the internal energy as a function of the coupling β/N for $N = 20$.

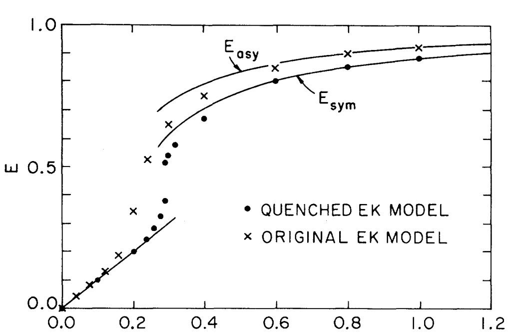

FIG. 1. The internal energy of the QEK model with the weight $F_1(\theta_\mu{}^i)$ as a function of β/N for the gauge group U(20). The internal energy of the original EK model is also plotted. The strong- and weak-coupling curves are from Eqs. (10)–(12).

Each point is an average over 30–50 iterations. A given link variable V_μ is hit 380 [$=N(N-1)$] times before proceeding to the next. The acceptance rate is always greater than 50%. Also plotted is the internal-energy result of the original Eguchi-Kawai model. It is clear that in the intermediate- and weak-coupling regions the original and the quenched models have completely different behaviors. In the weak-coupling region the QEK model fits the curve of Eq. (11) very well. The model has the same phase structure as the large-N limit of the standard Wilson action. It undergoes the first-order phase transition near $\beta/N = 0.29$. Figure 2 shows the results of two long runs at this point with different initial conditions. We observe the latent heat clearly. From the measurement of the latent heat at other points, I am led to quote $\beta/N = 0.29 \pm 0.02$ as the critical coupling. This point is a little bit lower than the critical point $\beta_c/N = 0.33$ measured in the SU(6) Wilson theory.[8] Perhaps this is the finite-N effect caused by the weight factor $F_1(\theta_\mu{}^i)$.[9]

For comparison the U(10) group was studied. Although measurements of the latent heat are difficult in this case, I observed an abrupt change of the internal energy as a function of β/N from strong to weak coupling within the interval $\beta/N = 0.26 \pm 0.02$. I also performed the rapid thermal cycle. The hysteresis loop is clearly seen near $\beta/N = 0.26$.

We estimate the N dependence of the critical coupling $\beta_c(N)$; we parametrize it as

$$\beta_c(N)/N = A + (B/N), \tag{13}$$

where the B/N term corresponds to the leading-order correction of the $1/N$ expansion (in the QEK model, generally, the higher-order correction

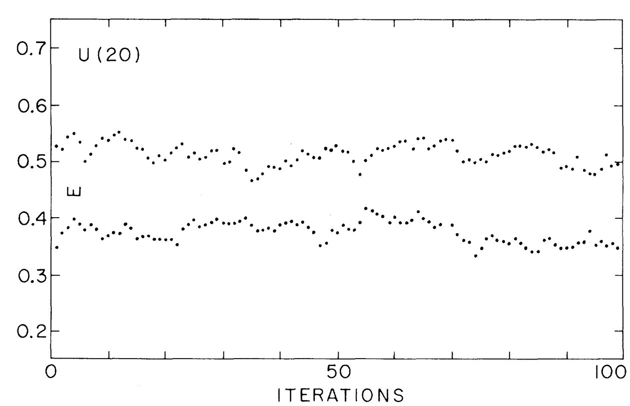

FIG. 2. Two Monte Carlo runs at $\beta/N = 0.29$.

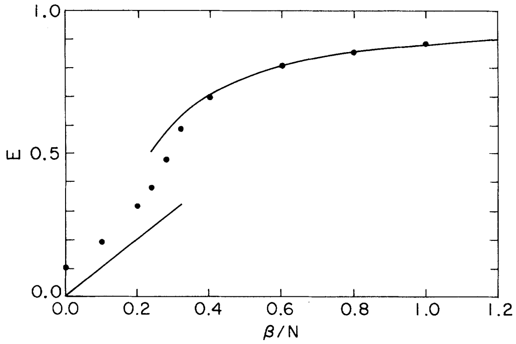

FIG. 3. The internal energy of the QEK model with the weight $F_2(\theta_\mu{}^i)$ as a function of β/N for the gauge group U(20).

starts from order $1/N$ and not $1/N^2$). Using values $\beta_c(10)=0.26$ and $\beta_c(20)=0.29$ we obtain

$$A=0.32,\quad B=-0.6. \tag{14}$$

Within the experimental error this limiting value 0.32 is consistent with the critical coupling observed in the usual Wilson theory.

Next we consider the model with $F_2(\theta_\mu{}^i)=1$. Figure 3 shows the internal energy as a function of the coupling β/N for $N=20$. In the weak-coupling region it also fits well with the curve of Eq. (11). We do not find any indication of the phase transition in this case. The curve is smooth and approaches $E=2(N-1)/(N^2-1)$ at $\beta=0$.

In weak coupling Eq. (4) does not depend crucially on the $\theta_\mu{}^i$ variables and the above two models lead to the same result. In strong and intermediate coupling, however, $\overline{O}(\theta_\mu{}^i)$ has a large dependence on the variables $\theta_\mu{}^i$. It is important to choose a suitable weight function to study these regions [for example, $F_1(\theta_\mu{}^i)$ is capable of detecting the first-order phase transition in the U(20) group].

Although in the critical region the approach to large N is not so fast, this experiment indicates that as N goes to infinity the QEK model becomes identical to the standard Wilson theory throughout the whole range of coupling constants. To draw a more conclusive result the evaluation of the Wilson loop is desirable. In the QEK model this is not so difficult because $(U_\mu)^n \to V_\mu{}^\dagger(D_\mu)^n V_\mu$. This is an interesting problem which I am presently investigating.

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[9]While this work was in progress, the author received the preprint of G. Bhanot, U. M. Heller, and H. Neuberger (to be published) in which the critical region of the QEK model was also studied by the Monte Carlo method. Instead of taking the weight factor (6), the authors fixed the set $\{\theta_\mu{}^i\}$ ($\mu=1,4$) to be $\{\theta_\mu{}^i\}=\{\alpha_\mu+2\pi i/N\}$. For finite N, their model is not the same as the present one. They found the first-order phase transition near $2\beta/N=0.63$. (The author would thank H. Neuberger for discussion on this point.)