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Monte Carlo Study of the Eguchi-Kawai Model

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The Eguchi-Kawai model is studied by using the Monte Carlo technique with the gauge groups SU(N), $N = 2$, 3, 4, 5, and 10. Clear evidence of spontaneous breaking of the Z_N symmetry is observed in the weak-coupling region for $N = 10$.

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Recently Eguchi and Kawai[1] reported the very exciting possibility of reducing the dynamical degrees of freedom of lattice gauge theory in the large-N limit. They proposed to study the theory defined by the partition function

$$Z_{\mathrm{EK}} = \int \prod_{\mu} dU_\mu \exp\{\beta \sum_{\mu \neq \nu = 1}^{d} \mathrm{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\} \qquad (1)$$

instead of the standard Wilson theory[2]

$$Z_{\mathrm{W}} = \int \prod_{x} \prod_{\mu} dU_{x,\mu} \exp\{\beta \sum_{x} \sum_{\mu \neq \nu = 1}^{d} \mathrm{Tr} U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\nu}^\dagger\}, \qquad (2)$$

where U_μ is an $N \times N$ unitary matrix and μ denotes a unit vector in the μ direction. d is the space-time dimensionality. They claimed that in the large-N limit the Wilson loop amplitudes defined in the theory (1) obey the same infinite set of identities, the so-called Schwinger-Dyson equations, as in the standard model. Since these identities uniquely specify the theory, the Wilson gauge theory (2) reduces to its simpler version, Eq. (1). In this paper I study the Eguchi-Kawai (EK) model (1) by using the Monte Carlo method.

It is interesting to note that one can always rewrite the Wilson action (2) in the form of Eq. (1). Indeed, introducing $nN \times nN$ matrices U^μ (n being the number of sites in the lattice) which have nonvanishing elements $U^\mu{}_{x,i;y,j} = (U^\mu{}_x)_{ij}$ only when $y = x + \mu$, the partition function in Eq. (2) can be expressed as[3]

$$Z_{\mathrm{W}} = \int' \prod_{\mu} dU_\mu \exp\{\beta \sum_{\mu \neq \nu = 1}^{d} \mathrm{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\}. \qquad (3)$$

The prime means that the integration $\int' \prod_\mu dU_\mu$ should not be extended to the whole manifold of $nN \times nN$ unitary matrices, but only to an appropriate submanifold. Their argument indicates that it is plausible that the very large number of degrees of freedom in an ordinary lattice theory is replaced by the degrees of freedom in U(N) matrices of the EK model. (The crucial question remains whether an unrestricted integration over the whole group manifold can reproduce the asymptotic properties of the original Wilson theory.)

In Ref. 1 EK considered the theory having the U(N) symmetry. However, in the large-N limit SU(N) and U(N) theories are expected to have the same limit. Thus I consider the SU(N) model where the Z_N center plays the role of the U(1) subgroup of U(N).

The SU(N) EK model is invariant under the Z_N phase transformation $U_\mu \to e^{(2\pi l/N)i} U_\mu$ ($l = 0, 1, \ldots, N-1$). The major assumption in the EK derivation is that this symmetry is not spontaneously broken for any value of the coupling constant in the large-N limit. In particular this means that the expectation value of the trace of products of U matrices in which U_μ and $U_\mu^\dagger$ appear a different number of times must vanish identically. This point is crucial to obtain the same Schwinger-Dyson equations in both theories. Very recently it has been argued[4] that this symmetry is broken spontaneously. The authors of Ref. 4 studied the EK model by the method of steepest descent and showed that in the weak-coupling region the Z_N-symmetric saddle point becomes unstable. They also performed the preliminary Monte Carlo run for $N=5$ and found some indication of symmetry breaking at large values of β.

In this paper I study the EK model by developing a new numerical method which is capable of effectively handling good statistical data for gauge groups with large N. This method works quite well in the EK model so that I easily obtain the result with high statistics (I have checked the validity of this method up to $N=20$). I find clear evidence of spontaneous breakdown of Z_N symmetry in the weak-coupling region and clarify the nature of the phase transition.

I prepare a table of 400 SU(2) matrices A_{ij}. Then, choosing one SU(2) matrix randomly from the table, I form the SU(N) matrix B_{mn} which is obtained from the $N\times N$ unit matrix by the replacements $B_{ii}=A_{11}$, $B_{ij}=A_{12}$, $B_{ji}=A_{21}$, $B_{jj}=A_{22}$. i and j $(i\neq j)$ take some values from 1 to N. Then a given link variable U_μ is multiplied by this matrix B. The change is accepted or rejected according to the standard Monte Carlo algorithm. This procedure is applied $N(N-1)$ times [i.e., for all possible values of the set (i,j)] to a given link variable before proceeding to the next. A sequential pass through all variables U_μ represents one Monte Carlo iteration. The SU(2) matrices in the table are selected randomly from the entire group but weighted toward the identity. This weighting is coupling dependent and selected so that the probability for accepting a change is always greater than 50%. After each fifty iterations all link variables are normalized in order to eliminate any accumulation of roundoff errors.

To reduce the computation time, I use the following trick: I define $V_{\mu\nu}$ by $V_{\mu\nu}=U_\mu U_\nu U_\mu^\dagger$. Under the change of variable $U_\mu\to BU_\mu$, $V_{\mu\nu}$ changes as $V_{\mu\nu}\to BV_{\mu\nu}B^\dagger$. Then in addition to the variable U_μ, I store $V_{\mu\nu}$ in the memory. Since the matrix B has only four nontrivial elements, if I store both $V_{\mu\nu}$ and U_ν, the computation time of the change of the action $\mathrm{Tr}V_{\mu\nu}U_\nu^\dagger$ grows only linearly in N.

Using the above method I have studied the EK model for $N=2$, 3, 4, 5, and 10. The first quantity which I measure is the internal energy

$$E=N^{-1}\langle \mathrm{Re\,Tr}U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\rangle, \tag{4}$$

where the expectation is with the weight of Eq. (1). A strong-coupling expansion yields

$$E(\beta)=\frac{1}{N^2}+\frac{N^4+10N^2-6}{N^2(N^2-1)}\frac{\beta}{N}, \tag{5}$$

where I set the space-time dimension $d=4$. Figures 1(a) and 1(b) show the internal energy as a function of coupling β/N for the SU(5) and SU(10) models. In Fig. 1(a), each point is an average over 200 iterations in the simulation of a rapid thermal cycle. For SU(10), the plotted points are averages over 100 iterations. In the graphs, I also plot the curves of the weak-coupling expansion[4]

$$E(\beta)=1-\frac{1-1/N}{8}\frac{N}{\beta}. \tag{6}$$

Equation (6) is calculated for the vacuum in which the Z_N symmetry is not spontaneously broken. As N becomes large it agrees with the results obtained from (2). For small β/N, the Monte Carlo results are in good agreement with the strong-coupling expansion (5). The $N=2$, 3, and 4 runs do not show any special structure from strong to weak coupling. In these cases, the Monte Carlo data agree with the curve (6) in the weak-coupling

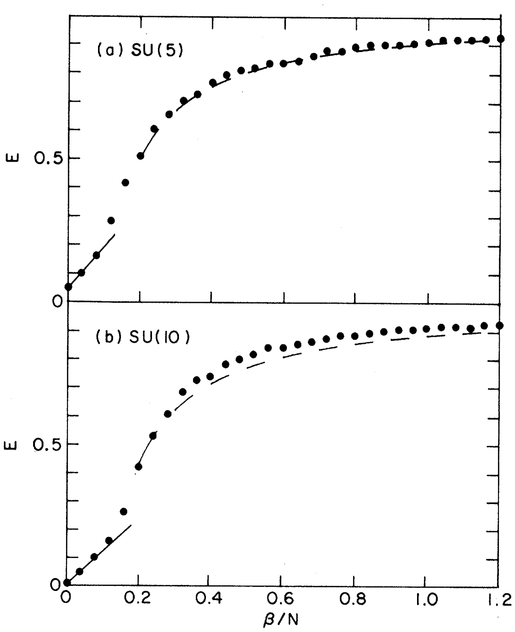

FIG. 1. The internal energies per plaquette as functions of β/N for the gauge group (a) SU(5) and (b) SU(10). The strong- and weak-coupling curves are from Eqs. (5) and (6).

region $0.4 \leqslant \beta/N \leqslant 1.2$. However, as N becomes large the data deviate from this curve as shown in Fig. 1. We do not find any indication of the first-order phase transition which was discovered in the standard SU(N) ($N \geqslant 4$) gauge theories[5] at $\beta/N \simeq 0.3$. I note that $\beta/N \simeq 0.3$ is in the weak-coupling region of the EK model.

Next I consider the expectation value $\langle \mathrm{Tr}\, U_\mu \rangle$ of the trace of the link variable U_μ. This quantity plays a role of the order parameter. If the Z_N symmetry is not spontaneously broken, $\langle \mathrm{Tr} U_\mu \rangle$ must be identically zero. Then a measurement of $\langle \mathrm{Tr} U_\mu \rangle$ may be used to detect whether the symmetry is broken or not. I parametrize it as

$$N^{-1}\langle \mathrm{Tr} U_\mu \rangle = P_\mu \exp i\theta_\mu . \tag{7}$$

In Fig. 2(a) I show the average $\frac{1}{4}\sum_{\mu=1}^{4} P_\mu$ of the absolute values of $\langle \mathrm{Tr} U_\mu \rangle$ as a function of the coupling β/N for the SU(10) model. It is clear that in the weak coupling the Z_N symmetry is spontaneously broken. For $N = 2$, 3, and 4 we do not see any sign of symmetry breakdown. For SU(5) I find some indication of the nonvanishing of $\langle \mathrm{Tr} U_\mu \rangle$. But the evidence is not so clear.

Once the symmetry is broken, the phase factor θ_μ has the N stable points

$$\theta_\mu = (2\pi/N) l_\mu, \quad l_\mu = 0, 1, \ldots, N-1. \tag{8}$$

In the SU(10) model, the fluctuation of the phase factor θ_μ is very small and each phase θ_μ ($\mu = 1, \ldots, 4$) appears to settle at one of the stable points. For example in one experiment I observed

$$l_1 = 9, \quad l_2 = 8, \quad l_3 = 2, \quad l_4 = 7. \tag{9}$$

In Figs. 2(b) and 2(c), I show the real and imaginary parts of $\langle \mathrm{Tr}\, U_1 \rangle$ in this simulation. In the SU(5) case, however, θ_μ fluctuates about several points (8) and does not appear to have a definite value.

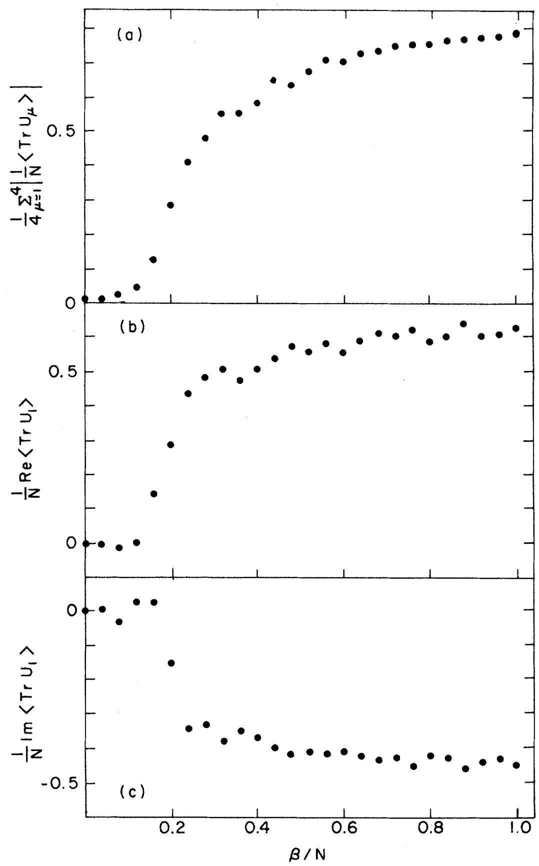

FIG. 2. The Monte Carlo data of $(1/N)\langle \mathrm{Tr}\, U_\mu \rangle$ as functions of β/N for the SU(10) model. (a) The average $\frac{1}{4}\sum_{\mu=1}^{4} |N^{-1}\langle \mathrm{Tr}\, U_\mu \rangle|$ of the absolute values of $N^{-1}\langle \mathrm{Tr}\, U_\mu \rangle$. (b) and (c) The real and imaginary parts of $N^{-1}\langle \mathrm{Tr}\, U_1 \rangle$.

Finally I coupled an external source h to the system:

$$Z = \int \prod_\mu dU_\mu \exp[\beta \sum_{\mu \neq \nu = 1}^{4} \mathrm{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + hN \sum_\mu \mathrm{Re}\, \mathrm{Tr} U_\mu]. \tag{10}$$

I performed simulations at $\beta/N = 0.1$ and 0.5 in the SU(10) model, varying cyclically the strength of the external source. In Figs. 3(a) and 3(b) I plot the result obtained at $\beta/N = 0.5$. Figure 3(a) shows the real part of $N^{-1}\langle \mathrm{Tr}\, U_1 \rangle$ whereas Fig. 3(b) shows the imaginary part of it. The other link variables behave in the same way. I start from the initial condition $U_\mu = 1$ ($\mu = 1, 4$) at $h = 2$. Then I decrease the strength of the source down to $h = -2$ and increase it to the original value. Each point represents the average over 100 iterations.

It is interesting to observe that although we apply only a real external field h, $\langle \mathrm{Tr} U_\mu \rangle$ eventually become complex when we pass the point $h = 0$: The external source drives the symmetry breakdown to a stable point with $\mathrm{Re}\langle \mathrm{Tr} U_\mu \rangle$ positive, but it is not strong enough to fix $\mathrm{Im}\langle \mathrm{Tr} U_\mu \rangle$. In any case, Fig. 3(a) shows the large jump at $h = 0$ implying the Z_N symmetry breaking. On the other hand we do not see any special structure in the hysteresis cycle at $\beta/N = 0.1$. This is shown in Fig. 3(c). In this case the imaginary part of $\langle \mathrm{Tr} U_\mu \rangle$ is essentially zero. This is consistent because now there is no stable point in the phase

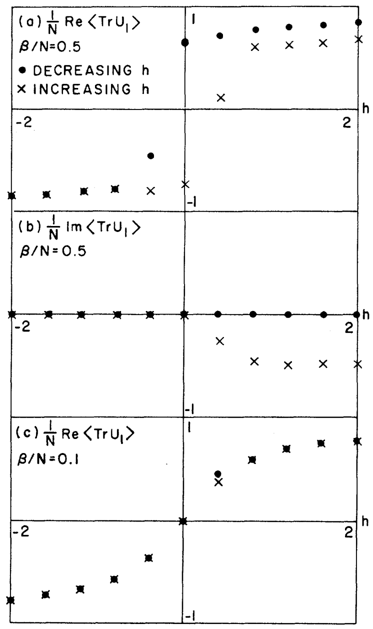

FIG. 3. Hysteresis cycles of $N^{-1}\langle \mathrm{Tr}\,U_\mu\rangle$ as functions of the external sources h. (a) The re[illegible] (b) the imaginary parts of $N^{-1}\langle \mathrm{Tr}\,U_1\rangle$ at β/N [illegible] The result of $N^{-1}\,\mathrm{Re}\langle \mathrm{Tr}\,U_1\rangle$ at $\beta/N = 0.$[illegible]

factor θ_μ.

This experiment clearly [illegible]at in weak coupling the EK model und[illegible] spontaneous symmetry breakdown. The order parameter $\langle \mathrm{Tr}U_\mu\rangle$ is continuous at the transition point. The order of the transition might be determined with more extended simulations.

Although the original EK model has its own interest since the Z_N symmetry is spontaneously broken, we cannot identify it with the Wilson action in the large-N limit. It has been pointed out[4,6] that a suitably formulated quenched version of the same model may have good behavior in the weak-coupling region. This is an interesting possiblity, which I am presently investigating.

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