Monte Carlo Renormalization-Group Study of the Four-Dimensional Z_2 Gauge Theory

A. Gonzalez-Arroyo
Departamento de Física Teórica C-XI, Universidad Autónoma de Madrid, Cantoblanco, Madrid, Spain

and

M. Okawa and Y. Shimizu
National Laboratory for High Energy Physics (KEK), Tsukuba, Ibaraki 305, Japan
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The four-dimensional Z_2 gauge theory is studied by Monte Carlo renormalization-group methods. The renormalization-group flows are discontinuous at the first-order phase-transition point and are not associated with a discontinuity fixed point. Evidence for critical fixed points within the metastable regions is presented.

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The lattice formulation[1] of gauge field theories provides a most powerful tool to study the nonperturbative phenomena in these theories. The simplest model is the Z_2 gauge model which has therefore received special attention from very early times.[2] The four-dimensional model with Wilson action is self-dual and the location of its phase-transition point can be computed analytically. The result is exactly the same as for the nearest-neighbor two-dimensional Ising model. The analogy between these two models also shows up in the renormalization-group flow predicted by the Migdal-Kadanoff approximation.[3] However, Monte Carlo simulations[4] revealed that the phase transition of the four-dimensional Z_2 gauge model is first order, instead of second order as for the Ising model. The situation became more intriguing after some evidence for long-range correlations at the first-order phase transition was presented.[5]

The purpose of the present paper is to study the behavior of the Z_2 gauge theory by Monte Carlo renormalization-group (MCRG) methods.[6] The standard theory of renormalization-group (RG) analysis at first-order phase transitions[7,8] is based on the existence of a *discontinuity* fixed point with vanishing correlation length. The signals for such a fixed point are a continuous RG flow at the transition point and an eigenvalue exponent equal to the dimensionality of the model. This situation is verified for a number of models[9] including the $d=2$ Ising model at $T<T_c$ and vanishing magnetic field. Blöte and Swendsen,[10] however, have pointed out that this pattern is not general. They have evaluated the eigenvalue exponents of the three-state Potts model in three and four dimensions and found that these quantities are discontinuous at the transition point. In this case, the RG flow is expected to be driven toward a different fixed point in each phase. In this paper we will explicitly evaluate the renormalized coupling constants for the Z_2 gauge theory and show that the RG flows are discontinuous at the transition point and are not associated with the discontinuity fixed point. We will also present evidence for critical fixed points within the metastable regions.

Before entering into the description of our results we want to mention the usefulness of studying the first-order transitions by the MCRG method. The continuum limit of the lattice gauge theories is taken at the second-order transition points. In many cases, however, it is very difficult to distinguish weak first-order transitions from second-order ones by the standard Monte Carlo simulations. We expect that the MCRG method is capable of determining the order of the transitions, since we show below that the patterns of the RG flows depend crucially on the order of the transitions.

The partition function of the $d=4$ Z_2 gauge theory is

$$Z=\mathrm{Tr}_{U_l}\exp\{S(U)\}, \tag{1}$$

where the trace is taken over all link variables $U_l=\pm1$. The probability of each configuration is determined by the gauge-invariant action functional $S(U)$,

$$S(U)=\textstyle\sum_\alpha\beta_\alpha W^\alpha(U). \tag{2}$$

In this paper we will restrict ourselves to the first five interactions; W^1, single plaquette; W^2, double plaquette; W^3, bent double plaquette; W^4, twisted bent double plaquette; and W^5, planar 2×2 loop. The first term W^1 represents the well-known Wilson action.

Now one can define a RG transformation. This involves the definition of a new set of variables $U_{l'}^{(1)}$ which are functions of the U_l. One can write

$$Z=\mathrm{Tr}_{U_{l'}^{(1)}}\exp\{S^{(1)}(U^{(1)})\}, \tag{3}$$

where $\exp\{S^{(1)}\}$ is the new probability distribution induced by (1) for the new variables $U_{l'}^{(1)}$. These variables are labeled by the links l' of a sublattice that is obtained by our skipping every second point in each direction and that contains $1/2^4$ of the original number of variables.

We will choose a RG transformation such that $U_{l'}^{(1)}$ is $V_{n',\mu}/|V_{n',\mu}|$, where link l' is parametrized as l'

$=(n',\mu)$, and $V_{n',\mu}$ is constructed from the sum over paths of length 2 and 4 connecting the corresponding sites on the sublattice[11]:

$$V_{n',\mu}=U_{n,\mu}U_{n+\mu,\mu}+\sum_{\pm\nu(\nu\perp\mu)}U_{n,\nu}U_{n+\nu,\mu}U_{n+\nu+\mu,\mu}U_{n+2\mu,\nu}. \quad (4)$$

This construction preserves the necessary property of gauge invariance. Thus, $S^{(1)}$ is a gauge-invariant functional and we will parametrize it as in (2) with new coupling parameters $\beta^{(1)}$.

We want to determine the evolution of the couplings $\beta\to\beta^{(1)}\to\beta^{(2)}$ under such a RG transformation. Notice that in the MCRG approach it is straightforward to evaluate vacuum expectation values of any renormalized operators, since configurations of the renormalized link variables $U^{(a)}$ are generated by the standard Monte Carlo method with initial action S. Then our problem is how to reconstruct $S^{(a)}$ from these expectation values. Recently, the present authors have proposed to use Schwinger-Dyson equations to study this problem.[12,13] These equations involve vacuum expectation values of renormalized operators and explicitly depend on renormalized coupling constants. Thus we can use these equations to estimate the values of the renormalized action. In the following paragraphs we will explain how one proceeds for the case of Z_2 gauge theory.

Let us consider one particular link l_0 of the lattice and let $W^{\alpha}_{l_0}$ be the part of W^{α} involving U_{l_0} linearly. We can obtain

$$\langle W^{\alpha}_{l_0}\rangle=-\langle W^{\alpha}_{l_0}\exp\{-2S_{l_0}\}\rangle \quad (5)$$

by performing a change of variables $U_{l_0}\to -U_{l_0}$ in the expression for the expectation value. The functional S_{l_0} is equal to $\sum_\alpha\beta_\alpha W^{\alpha}_{l_0}$. Notice that (5) is a set of expressions of the form $f^{\alpha}(\beta)=0$ where f^{α} are nonlinear functions. One can solve these equations by a Newton method. This is an iterative method producing a sequence $\tilde\beta_\alpha(N)$ of approximations. The $(N+1)$-th iteration is given by

$$\tilde\beta_\alpha(N+1)=\tilde\beta_\alpha(N)+\tfrac{1}{2}\langle W^{\alpha}_{l_0}W^{\gamma}_{l_0}\rangle^{-1}\times[\langle W^{\gamma}_{l_0}\rangle+\langle W^{\gamma}_{l_0}\exp\{-2\tilde S_{l_0}(N)\}\rangle], \quad (6)$$

where $\tilde S_{l_0}(N)$ is equal to $\sum_\alpha\tilde\beta_\alpha(N)W^{\alpha}_{l_0}$. One can continue the iterations until $\tilde\beta(N+1)\simeq\tilde\beta(N)$ within errors.

We have studied the Z_2 gauge model at the phase-transition point $\beta_1=0.440687$ (other $\beta_\alpha=0$). The lattice contains 16^4 points. After the system reached equilibrium, we carried on several short runs to obtain the first few iterations of (6) with rough errors. Then, a long run was performed to increase precision in the determination of the couplings. It is important to make separate runs for the ordered and disordered phases. If both phases are allowed to coexist, the correlations between discontinuous thermodynamic densities depend on the relative composition of the mixture rather than on the value of these correlations for each individual phase. In this case, one eigenvalue exponent becomes equal to the dimensionality of the lattice, but this fact has nothing to do with the existence of a discontinuity fixed point.[14]

In Table I, parts (A) and (B), we give the values of the couplings for ordered and disordered phases, respectively. The final Monte Carlo (MC) run used in this determination consisted of 5000 MC steps per link in each case. Data were taken every 5 MC steps. The errors in the couplings are determined by our partitioning the total sample into several groups and studying the dispersion within these groups. The same data are used to determine the value of the original coupling and in this way the ability of our method can be tested. We have included up to two iterations of the RG transformation leading to a lattice with 4^4 points.

TABLE I. The renormalized coupling constants $\beta_\alpha^{(a)}$ starting from (A) $\beta_1=0.440687$ in the ordered phase; (B) $\beta_1=0.440687$ in the disordered phase; (C) $\beta_1=0.428$ in the ordered metastable phase; (D) $\beta_1=0.453$ in the disordered metastable phase. a refers to the blocking level and α distinguishes different interactions.

$\beta_\alpha^{(a)}$	$\alpha=1$	2	3	4	5
(A) $a=0$	0.4423(15)	0.0003(3)	−0.0003(2)	0.0004(4)	−0.0001(4)
$a=1$	0.3411(70)	−0.0008(18)	0.0041(10)	0.0410(19)	0.0133(18)
$a=2$	0.848(77)	0.004(21)	−0.073(15)	0.197(20)	0.038(16)
(B) $a=0$	0.4410(2)	−0.0001(1)	0.0000(1)	0.0000(1)	−0.0001(1)
$a=1$	0.1447(1)	0.0000(1)	0.0061(1)	0.0040(1)	−0.0002(1)
$a=2$	0.0265(1)	0.0007(1)	0.0008(1)	−0.0001(1)	0.0000(1)
(C) $a=0$	0.4273(9)	0.0003(4)	−0.0001(1)	0.0002(2)	−0.0001(2)
$a=1$	0.1967(31)	0.0018(4)	0.0166(5)	0.0207(8)	0.0068(3)
$a=2$	0.303(43)	0.0260(25)	−0.0059(65)	0.0725(93)	0.0093(13)
(D) $a=0$	0.4531(4)	−0.0001(1)	0.0000(1)	0.0000(2)	−0.0001(1)
$a=1$	0.1707(1)	0.0013(1)	0.0113(1)	0.0088(1)	0.0000(1)
$a=2$	0.0367(4)	0.0010(1)	0.0012(5)	−0.0001(1)	0.0000(1)

TABLE II. The largest eigenvalues $\lambda^{(a)}$ of the T matrices.

$\lambda^{(a)}$	(A)	(B)	(C)	(D)
$a=1$	4.75(23)	4.63(27)	6.84(38)	10.61(59)
$a=2$	1.21(4)	0.12(2)	3.74(29)	0.74(10)

From Table I we conclude that the RG flow is different for both phases. In the disordered (confined) phase the couplings seem to approach zero. In the ordered phase $\beta_1^{(a)}$ behaves nonuniformly: first decreases and then increases.[15] We have also studied the largest eigenvalues $\lambda^{(a)}$ of the transition matrices $T_{\alpha,\gamma}^{(a)} = \partial\beta_\alpha^{(a)}/\partial\beta_\gamma^{(a-1)}$ by Swendsen's method.[6] The results are shown in Table II, parts (A) and (B), where a clearly different value shows up for the second iteration of both phases. Furthermore, this eigenvalue is markedly different from $16=2^4$, 4 being the dimensionality of the lattice. Again the conclusion is that no discontinuity fixed point exists and the RG flow is discontinuous at the transition point.

To study the nature of the transition further we have performed several Monte Carlo runs in the metastable phase. Figure 1 shows the internal energy $E=\langle W^1\rangle/N_P$ and the specific heat $\partial E/\partial\beta_1 = N_P(\langle E^2\rangle - \langle E\rangle^2)$ as functions of β_1 (other $\beta_\alpha=0$). Here N_P is the number of plaquettes on the original lattice. At each point we have performed 3500 MC steps per link and data were taken every 5 MC steps. In the ordered (disordered) metastable region the system quickly evolved toward the stable phase at $\beta_1=0.427$ (0.454). On the other hand, at $\beta_1=0.428$ and 0.453, the system remained in the corresponding metastable phases. Then at these values of β_1 we have evaluated the renormalized couplings. The results are shown in Table I, parts (C) and (D). The largest eigenvalue of the T matrices is displayed in Table II, parts (C) and (D).

By comparing the data shown in Tables I and II, and Fig. 1, we have observed the following facts: (a) The observables E, $\partial E/\partial\beta_1$, and the RG flows continue smoothly into the metastable state in the individual phase. However, small changes in the original couplings are amplified by the RG transformations. (b) The specific heat increases very rapidly as we penetrate into the metastable regions, implying the increase of the plaquette-plaquette correlation length. We found that the correlations $\langle W^\alpha W^\gamma\rangle_{\text{conn}}$ also increase in the metastable regions. (c) Two RG flows in the metastable states first approach each other and then separate. In fact, for the first RG iteration $(a=1)$ the results shown in Table I, parts (C) and (D), are quite close to each other. (d) The largest eigenvalue of T matrix grows very fast in the metastable regions.

There is an interesting interpretation of our data which we want to mention. We can explain the growth of the specific heat by assuming the existence of critical points with infinite correlation length in both ordered and disordered metastable regions. In the multiple-coupling-constant space, there is a first-order transition surface, the point $\beta_1=0.440687$ being just one point on this surface. We then expect that in the metastable regions there are two critical surfaces, the critical points, we assume, being located on these surfaces. Presumably these surfaces run approximately parallel to the first-order transition surface and coincide with the boundaries of metastable regions, where metastable states disappear. Since these surfaces map onto themselves under RG transformations, we expect that there exist some critical fixed points situated on these surfaces. Notice the observed nonuniform behaviors of our RG flows shown in Fig. 2. Since our simulations were not done at the critical points, the RG flows are expected first to approach the fixed points and then to deviate from them along the

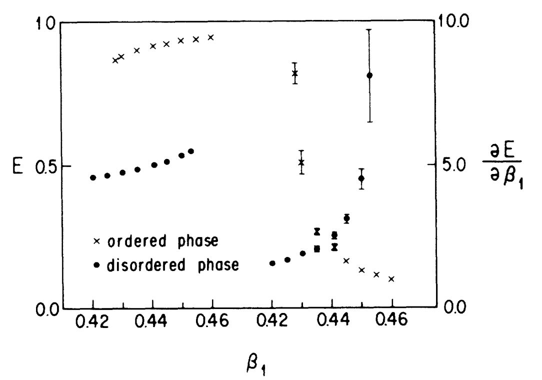

FIG. 1. The internal energy E and the specific heat $\partial E/\partial\beta_1$ as functions of β_1.

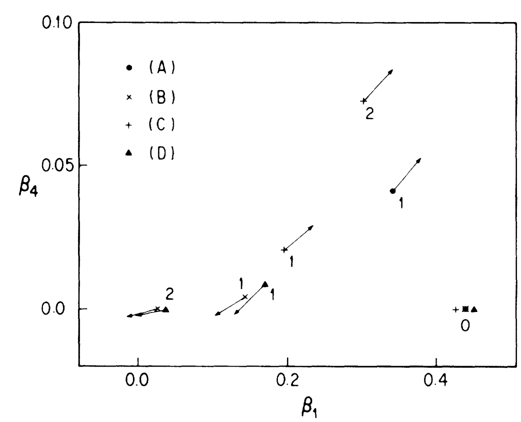

FIG. 2. The renormalized coupling $\beta_\alpha^{(a)}$ and the eigenvector $x_\alpha^{(a)}$ of the largest eigenvalue of $T_{\alpha,\gamma}^{(a)}$ projected into the β_1-β_4 plane. The indices 0–2 refer to the RG iteration. The eigenvectors are normalized to $\sum_\alpha x_\alpha^{(a)2}=1$ and are shown as arrows starting from each $\beta_\alpha^{(a)}$. $\Delta\beta_\alpha^{(a)}=0.04$ corresponds to $x_\alpha^{(a)}=1$. Other components of $x_\alpha^{(a)}$ $(\alpha=2,3,5)$ are, in general, smaller than $x_1^{(a)}$ and $x_4^{(a)}$.

lines of the renormalized trajectories. Our data are consistent with the hypothesis of critical fixed points.

A remarkable property of our data is that the flows of both phases tend to align after performance of one RG transformation, thus suggesting that the renormalized trajectories are approximately overlapping. Figure 2 also shows that the eigenvector $x_a^{(a)}$ corresponding to the largest eigenvalue, which gives the direction of the renormalized trajectory, is approximately the same for both phases. A possible explanation of these facts is that two fixed points are very close or coincide. In these cases the position of the fixed point should not differ much from the couplings given in Table I part (C) or (D), with $a=1$. The largest eigenvalues are quite large in both phases. Given the sharp dependence on coupling, it is not excluded that they are equal (and equal to 2^4). We note that two fixed points can coincide only if the two critical surfaces merge, and this could only occur at the stable transition surface itself. Therefore, in order to have a single fixed point (instead of two very close fixed points), the first-order transition surface must change into a second-order one. We have tried to find second-order transition in the multicoupling Z_2 gauge theory by direct Monte Carlo simulation and by the mean-field method; however, up to now, we do not have any evidence.

We now summarize our conclusions. There is no discontinuity fixed point for the four-dimensional Z_2 gauge theory. Rather, the RG flows are discontinuous at the first-order transition. The hypothesis of critical fixed points within the metastable regions can explain our data and the long-range correlations of Ref. 5. These results together with those of Blöte and Swendsen may represent a general situation alternative to that of the discontinuity fixed point.

The numerical calculation for the present work was carried out on a Hitachi model HITAC S810/10 vector computer at KEK. We are grateful to the Data division for continuous support.

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[15]As was first found by R. Cordery, R. Gupta, and M. Novotny, Phys. Lett. **128B**, 1318 (1983), the vacuum expectation values of the operators $\langle W_{l_0}^{a} \rangle$ do not show any nonuniform behavior. In the disordered phase they approach zero and in the ordered phase they increase uniformly as we iterate the RG transformation. To estimate the pattern of the RG flows it is essential to map the flows in the coupling-constant space.