Lattice QCD Calculation of the Kaon B Parameter with the Wilson Quark Action

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The kaon B parameter is calculated in quenched lattice QCD with the Wilson quark action. The mixing problem of the $\Delta s = 2$ four-quark operators is solved nonperturbatively with full use of chiral Ward identities, and this method enables us to construct the weak four-quark operators exhibiting good chiral behavior. We find $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.69(7)$ (where NDR denotes naive dimensional regularization) at the lattice cutoff scale of $a^{-1} = 2.7{-}4.3$ GeV. [S0031-9007(98)06929-4]

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Reliable knowledge of the K^0-$\bar{K}^0$ transition matrix element B_K is indispensable for further advancement in CP violation phenomenology, and much effort has been expended towards this end using lattice QCD. Successful calculations of B_K so far achieved [1,2] exclusively employ the Kogut-Susskind quark action that respects chiral U(1) symmetry. Whereas the verification that both Wilson and Kogut-Susskind (KS) quark actions yield the identical result is an important step to give full credit to the lattice QCD calculation, the attempts made with the Wilson quark action have not yielded much success [3–5]: the Wilson action that explicitly breaks chiral symmetry causes mixing among four-quark operators of different chiral structure, and hence ensuring the correct chiral behavior of the $\Delta s = 2$ operators is a substantially more complicated problem. Early studies have shown that the mixing problem is not adequately treated by perturbation theory, leading to an "incorrect answer" for the matrix element [3]. Attempts were then made to solve the mixing problem nonperturbatively with the aid of chiral perturbation theory [4]. Unfortunately, they were not successful since the calculation contains large systematic uncertainties arising from higher order effects that survive the continuum limit. More recently a proposal has been made [5] to improve the chiral behavior of the $\Delta s = 2$ operator with the use of nonperturbative renormalization (NPR) [6], and encouraging results have been reported [7].

In this Letter we propose an alternative nonperturbative method to solve the operator mixing problem using chiral Ward identities [8]. This method fully incorporates the chiral properties of the Wilson action, yielding the $\Delta s = 2$ operator that shows good chiral behavior. No effective theories are invoked to estimate the matrix element. The resulting B_K we obtained shows good agreement with the value from the KS quark action. We shall also revisit the perturbative method.

Let us consider a set of weak operators in the continuum $\{\hat{O}_i\}$ which closes under chiral rotation $\delta^a \hat{O}_i = i c^a_{ij} \hat{O}_j$. These operators are given by linear combinations of a set of lattice operators $\{O_\alpha\}$, as $\hat{O}_i = \sum_\alpha Z_{i\alpha} O_\alpha$. We choose the mixing coefficients $Z_{i\alpha}$ such that the Green functions of $\{\hat{O}_i\}$ with quarks in the external states satisfy the chiral Ward identity to $O(a)$. This identity can be derived in a standard manner [8] and takes the form

$$-2\rho Z_A \left\langle \sum_x P^a(x) \hat{O}_i(0) \prod_k \tilde{\psi}(p_k) \right\rangle + c^a_{ij} \left\langle \hat{O}_j(0) \prod_k \tilde{\psi}(p_k) \right\rangle - i \sum_l \left\langle \hat{O}_i(0) \prod_{k \neq l} \tilde{\psi}(p_k) \delta^a \tilde{\psi}(p_l) \right\rangle + O(a) = 0, \quad (1)$$

where p_k is the momentum of the external quark, Z_A and $\rho = (m - \delta m)/Z_A$ are constants to be determined from the Ward identities for the axial vector currents [9], and P^a is the pseudoscalar density of flavor a.

The four-quark operator relevant to B_K is given by $\hat{O}_{VV+AA} = VV + AA$ where $V = \bar{s}\gamma_\mu d$ and $A = \bar{s}\gamma_\mu \times \gamma_5 d$. Then, $\hat{O}_{VV+AA} = VV + AA$ and $\hat{O}_{VA} = VA$ form a minimal set of the operators that closes under $\lambda^3 = \mathrm{diag}(1, -1, 0)$ chiral rotation. Taking account of CPS symmetry (note that we take $m_d = m_s$ in this article) [3], mixing of these operators is written $\hat{O}_{VV+AA}/2 = Z_{VV+AA}(O_0 + z_1 O_1 + \cdots + z_4 O_4)$ and $\hat{O}_{VA} = Z_{VA} \times z_5 O_5$, where the six lattice operators O_i are given in the Fierz eigenbasis by $O_0 = (VV + AA)/2$, $O_1 = (SS + TT + PP)/2$, $O_2 = (SS - TT/3 + PP)/2$, $O_3 = (VV - AA)/2 + (SS - PP)$, $O_4 = (VV - AA)/2 -$

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$(SS - PP)$, and $O_5 = VA$ with $S = \bar{s}d$, $P = \bar{s}\gamma_5 d$, and $T = \bar{s}[\gamma_\mu, \gamma_\nu]d/2$ $(\mu < \nu)$.

We consider the four external quarks having an equal momentum p, and denote by Γ_{VV+AA} and Γ_{VA} the sum of the Green functions on the left-hand side of (1) with external quark legs amputated. Using the projection operator P_i for the Fierz eigenbasis corresponding to O_i, we write $\Gamma_{VV+AA}/Z_{VV+AA} = \Gamma_5 P_5$ and $\Gamma_{VA}/Z_{VA} = \Gamma_0 P_0 + \Gamma_1 P_1 + \cdots + \Gamma_4 P_4$. Writing $\hat{O}_{VV+AA,VA}$ in (1) in terms of lattice operators, we obtain six equations for the five coefficients $z_1, \ldots, z_5$: $\Gamma_i = c_0^i + c_1^i z_1 + \cdots + c_5^i z_5 = O(a)$ for $i = 0, \ldots, 5$. This gives an overconstrained set of equations, and we may choose any five equations to exactly vanish to solve z_i: the remaining equation should automatically be satisfied to $O(a)$. We choose four equations to be those for $i = 1, \ldots, 4$, since $O_1, \ldots, O_4$ are absent in the continuum. The choice of the fifth equation, $i = 0$ or 5, is more arbitrary. We have confirmed that either $\Gamma_0 = 0$ or $\Gamma_5 = 0$ leads to a consistent result to $O(a)$ for $z_1, \ldots, z_4$ in the region $pa \lesssim 1$. In the present analysis we choose $\Gamma_5 = 0$. The overall factor Z_{VV+AA} is determined by the NPR method [6]. We convert the matrix elements on the lattice into those of the modified minimal subtraction ($\overline{\text{MS}}$) scheme in the continuum using naive dimensional regularization (NDR) renormalized at $\mu = 2$ GeV [10]:

$$B_K(\text{NDR}, \mu) = \left\{1 + \frac{\alpha_s(\mu)}{4\pi}\left[-4\log\left(\frac{\mu}{p}\right) - \frac{14}{3} + 8\log 2\right]\right\} \times \frac{\langle \bar{K}^0 | \hat{O}_{VV+AA} | K^0 \rangle}{\frac{8}{3} |\langle 0 | \hat{A} | K^0 \rangle|^2}, \quad (2)$$

where p denotes the momentum at which the mixing coefficients are evaluated.

For comparative purposes we also calculate B_K with one-loop perturbative mixing coefficients [11] after applying a finite correction to convert into the NDR scheme together with the tadpole improvement with $\alpha_{\overline{\text{MS}}}(1/a)$.

We remark here that the equations obtained in Ref. [5] correspond to $\Gamma_i = 0$ for $i = 1, \ldots, 4$ in which the first and the third terms in the Ward identity (1) are dropped. The NPR method satisfies the full Ward identities only in the limit of large external virtualities [5–7].

We made calculations with the Wilson quark action and the plaquette gluon action at $\beta = 5.9$–6.5 in quenched QCD using a Fujitsu VPP500/80 at KEK. Table I summarizes our run parameters. Gauge configurations are generated with the five-hit pseudoheat-bath algorithm with 2000($\beta = 5.9$ and 6.1), 5000($\beta = 6.3$), or 8000($\beta = 6.5$) sweep intervals apart. The physical size of the lattice is chosen to be $La \approx 2.4$ fm with the lattice spacing determined from $m_\rho = 770$ MeV. Four values of the hopping parameter are adopted at each β. We interpolate the result to $m_s a/2$, which is determined from $m_K/m_\rho = 0.648$, for degenerate d and s quark masses. Errors are estimated by the single elimination jackknife method for all measured quantities.

TABLE I. Parameters of simulations.

β	5.9	6.1	6.3	6.5
$L^3 \times T$	$24^3 \times 64$	$32^3 \times 64$	$40^3 \times 96$	$48^3 \times 96$
#conf.	300	100	50	24
K	0.15862	0.15428	0.15131	0.14925
	0.15785	0.15381	0.15098	0.14901
	0.15708	0.15333	0.15066	0.14877
	0.15632	0.15287	0.15034	0.14853
K_c	0.15986(3)	0.15502(2)	0.15182(2)	0.14946(3)
a^{-1} (GeV)	1.95(5)	2.65(11)	3.41(20)	4.30(29)
$\alpha_{\overline{\text{MS}}}(1/a)$	0.1922	0.1739	0.1596	0.1480
$m_s a/2$	0.0294(14)	0.0198(16)	0.0144(17)	0.0107(16)
$p^{*2}a^2$	0.9595	0.5012	0.2988	0.2056

Our calculations are carried out in two steps. We first calculate z_i and Z_{VV+AA} using the quark Green functions having finite space-time momenta. Quark propagators are solved in the Landau gauge for the point source located at the origin with the periodic boundary condition. We next extract B_K from the ratio $\langle \bar{K}^0(t = T)\hat{O}_{VV+AA}(t')K^0(t = 1)\rangle/(8/3)/\langle \bar{K}^0(t = T)\hat{A}(t')\rangle/\langle \hat{A}(t')K^0(t = 1)\rangle$, each Green function projected onto the zero spatial momentum, by fitting a plateau seen as a function of t'. For this calculation quark propagators are solved without gauge fixing employing the wall source placed at the edge where the Dirichlet boundary condition is imposed in the time direction. We obtain B_K at $m_s/2$ by quadratically interpolating the data at the four hopping parameters.

We plot in Fig. 1 a typical result for the mixing coefficients as a function of the external quark momenta. The plot shows, as desired, only weak dependence of z_i on momentum in the range $0.1 \lesssim p^2a^2 \lesssim 1.0$. This enables us to evaluate the mixing coefficients with small errors at the scale $p^* \approx 2$ GeV, which always falls within the range of a plateau for our runs at $\beta = 5.9$–6.5.

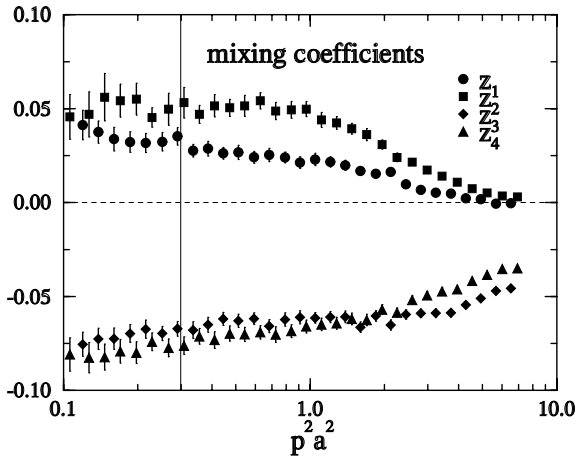

FIG. 1. Mixing coefficients $z_1, \ldots, z_4$ plotted as a function of external momentum squared $(pa)^2$ for $K = 0.15034$ at $\beta = 6.3$. Vertical line corresponds to $p^* \approx 2$ GeV.

In Fig. 2 we compare the mixing coefficients evaluated at the scale p^* (filled symbols) with the perturbative values obtained with $\alpha_{\overline{\rm MS}}(1/a)$ (open symbols) as a function of lattice spacing. A large value of z_2 determined by the Ward identities sharply contrasts with the one-loop perturbative result, $z_2 = 0$. For the other coefficients, the perturbative calculations agree with the nonperturbative ones in sign and rough orders of magnitude: they differ in quantitative details, however.

Let us examine the chiral property of the operator $\hat{O}_{VV+AA}$ by calculating the ratio $\langle \bar{K}^0|\hat{O}_{VV+AA}|K^0\rangle/(8/3)/|\langle 0|\hat{P}|K^0\rangle|^2$, which vanishes at $m_q = 0$ in the continuum. In Fig. 3 we show the results at $m_q = 0$ obtained by a quadratic extrapolation of data in $m_q = (1/K - 1/K_c)/2$, where WI stands for our method using chiral Ward identities and PT for tadpole-improved one-loop perturbation theory (numbers are given in Table II). The pseudoscalar density $\hat{P}$ in the denominator is renormalized perturbatively for both cases. The advantage is clearly seen with use of the Ward identities, the ratio becoming consistent with zero at the lattice spacing $m_\rho a \lesssim 0.3 (a \lesssim 0.08$ fm). In the perturbative approach, the chiral behavior is recovered only after extrapolation to the continuum limit, where we adopted a linear dependence on a expected for the Wilson quark action in the extrapolation shown in Fig. 3.

Our final results for $B_K(\mathrm{NDR}, 2\ \mathrm{GeV})$ are presented in Fig. 4 as a function of lattice spacing (see Table II for numerical details). The method based on WI gives a value well convergent from a lattice spacing of $m_\rho a \approx 0.3$. Unfortunately the large errors do not allow us to take a linear extrapolation to the continuum limit. We may instead take a constant fit of the three results at smaller lattice spacings ($a^{-1} = 2.7$–4.3 GeV) and find $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.69(7)$, which is our best estimate for the WI method.

Since the origin of the large error is traced to that of the mixing coefficients, we attempt to develop an alternative method, in which the denominator of (2) is estimated with the vacuum saturation of $\hat{O}_{VV+AA}$ constructed by the WI method (we refer to this as the $\mathrm{WI}_{\mathrm{VS}}$ method). Since the large error of the WI method arises from the mixing coefficients, we expect with the $\mathrm{WI}_{\mathrm{VS}}$ method that the fluctuations in the numerator are largely canceled by those in the denominator. In fact, errors are substantially reduced with the $\mathrm{WI}_{\mathrm{VS}}$ method as apparent in Fig. 4. The cost is that the correct chiral behavior of the denominator is not respected at a finite lattice spacing due to the contributions of the pseudoscalar matrix element. This contribution brings the $\mathrm{WI}_{\mathrm{VS}}$ result to disagree with WI at a finite lattice spacing, but the discrepancy tends to vanish in the continuum limit. A linear extrapolation in a yields $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.562(64)$.

This linear extrapolation, however, involves a systematic uncertainty arising from the chiral symmetry breaking term $c_P|\langle 0|P|K^0\rangle|^2$ in the denominator, where $c_P = \sum_{i=1}^4 f_i z_i$ with f_i as the coefficients of vacuum saturation. The perturbative contribution to c_P starts at a two-loop and is of $O(g^4(1/a))$; the divergence of the matrix element $\langle 0|P|K^0\rangle \propto [g^2(1/a)]^{-4/11}$. Hence, $c_P|\langle 0|P|K^0\rangle|^2$ receives contributions of the form $[g^2(1/a)]^{14/11}$ which diminishes only as a fractional power of $1/\log a$. To assess the systematic error associated with this effect, we estimate the two-loop contribution to c_P by squaring the typical magnitude of the one-loop terms in z_i: e.g., $|z_i^{\text{one-loop}}[\alpha_{\overline{\rm MS}}(1/a)]| \lesssim 0.08$ at $\beta = 5.9$ from Fig. 2. We also estimate $|\langle 0|P|K^0\rangle|/|\langle 0|A|K^0\rangle| \approx 6$ which yields $c_P|\langle 0|P|K^0\rangle|^2/|\langle 0|A|K^0\rangle|^2 \lesssim 0.25$. Since $\alpha_{\overline{\rm MS}}(1/a)^{14/11}$ decreases by 30% between $\beta = 5.9$–6.5, over which a decreases by a factor of 2, this fraction should reduce to $\approx$0.1 after taking the continuum limit. Taking account of

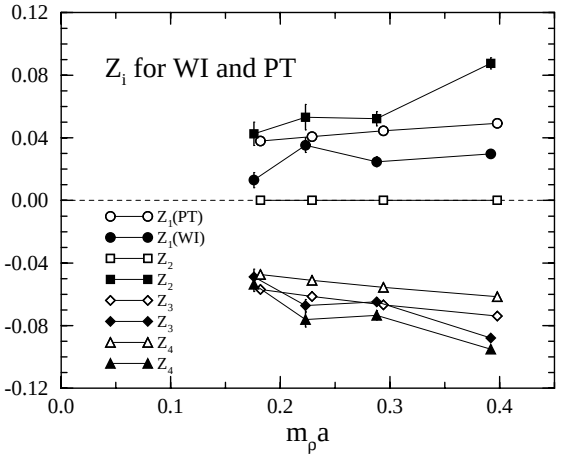

FIG. 2. Comparison of the mixing coefficients $z_1, \ldots, z_4$ evaluated at $p^* \approx 2$ GeV using the Ward identity (WI, solid symbols) and perturbative (PT, open symbols) methods. The coefficients are plotted as a function of $m_\rho a$.

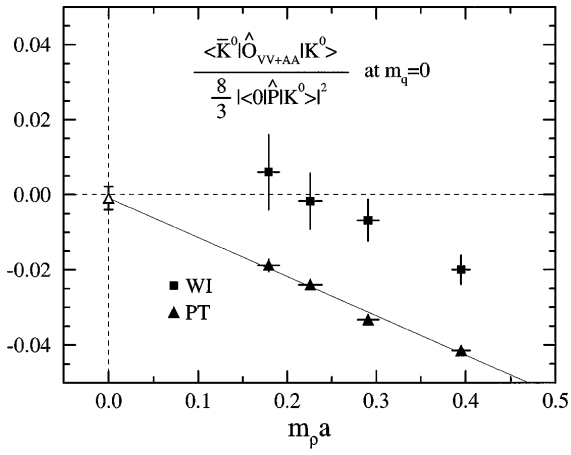

FIG. 3. Test of the chiral behavior of $\langle \bar{K}^0|\hat{O}_{VV+AA}|K^0\rangle/(8/3)/|\langle 0|\hat{P}|K^0\rangle|^2$ at $m_q = 0$ for the WI and PT methods. The operators are renormalized at 2 GeV in the NDR scheme. The solid line is a linear extrapolation to the continuum limit.

TABLE II. $\langle \bar{K}^0|\hat{O}_{VV+AA}|K^0\rangle/(8/3)/|\langle 0|\hat{P}|K^0\rangle|^2$ in the chiral limit and $B_K(\mathrm{NDR}, 2\ \mathrm{GeV})$ for WI, $\mathrm{WI}_{\mathrm{VS}}$, and PT methods as a function of β.

	$\frac{\langle \bar{K}^0|\hat{O}_{VV+AA}|K^0\rangle}{\frac{8}{3}|\langle 0|\hat{P}|K^0\rangle|^2}$ at $m_q = 0$		$B_K(\mathrm{NDR}, 2\ \mathrm{GeV})$		
β	WI	PT	WI	$\mathrm{WI}_{\mathrm{VS}}$	PT
5.9	−0.0200(39)	−0.0415(8)	+0.38(6)	+0.168(20)	−0.468(14)
6.1	−0.0068(55)	−0.0333(10)	+0.68(11)	+0.288(29)	−0.225(22)
6.3	−0.0017(74)	−0.0240(12)	+0.69(12)	+0.342(33)	−0.000(21)
6.5	+0.006(10)	−0.0188(17)	+0.72(18)	+0.360(52)	+0.156(40)
$a = 0$		−0.0009(31)		+0.562(64)	+0.639(76)

uncertainties in the choice of coupling constant and the mixing coefficients at the two-loop level, we estimate the chiral symmetry breaking contribution of the pseudoscalar density that survives the continuum limit to be $\lesssim 20\%$. We conclude $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.56(6)(11)$ for the $\mathrm{WI}_{\mathrm{VS}}$ method.

Intriguing in Fig. 4 is the fact that the perturbative calculation (PT), which gives the completely "wrong value" at $a \neq 0$, yields the correct result for B_K when extrapolated to the continuum $a = 0$. This is a long extrapolation from negative to positive, but the linearly extrapolated value $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.639(76)$ is reasonable compared with those obtained with the WI or $\mathrm{WI}_{\mathrm{VS}}$ method. We note that this long extrapolation may bring an error larger than quoted in the extrapolated value due to systematic effects of $O(ag^2(1/a))$ and $O(g^4(1/a))$. The estimation of these systematic errors, however, is too complicated because the matrix elements of the mixing operators have quite different absolute values.

Each of the results from the above three methods suffers from statistical and systematic errors of 10–20% which are comparable in magnitude. Although the $\mathrm{WI}_{\mathrm{VS}}$ and the PT methods have the advantage of small statistical errors, we recognize that this is offset by the difficulty to control large systematic errors when attempting a continuum extrapolation. We thus conservatively take the result of the WI method $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.69(7)$ at $a^{-1} = 2.7$–4.3 GeV as our final estimate of the present work. This value is compared with a JLQCD calculation with the KS action, $B_K(\mathrm{NDR}, 2\ \mathrm{GeV}) = 0.628(42)$ at the continuum limit [2], where we expect that the two values should agree up to $O(a)$.

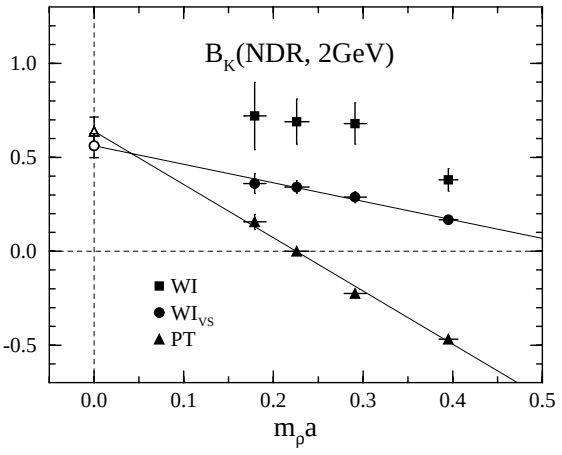

FIG. 4. K^0-$\bar{K}^0$ matrix element B_K(NDR,2 GeV) plotted as a function of the lattice spacing for the WI, $\mathrm{WI}_{\mathrm{VS}}$, and PT methods. The solid lines show linear extrapolations to the continuum limit.

In conclusion, our analysis for B_K demonstrates the effectiveness of the method using the chiral Ward identities for constructing the $\Delta s = 2$ operator with the correct chiral property. We have shown that both Wilson and KS actions give virtually the identical answer for B_K in their continuum limit. The application of this method to B_B is also straightforward.

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