Lattice QCD Calculation of Full Pion Scattering Lengths

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Full pion four-point functions at zero momentum are calculated with Kogut-Susskind quarks on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$ in quenched lattice QCD, employing the wall source method without gauge fixing. For both $I = 0$ and 2 channels, results for the s-wave scattering lengths are in agreement with the prediction of current algebra and PCAC (partial conservation of axial-vector current).

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While a lot of numerical techniques have been devised to study hadron physics based on lattice QCD, the variety of physics explored is so far quite limited. The quantities which have been computed are only those which can be extracted from connected hadron two-point functions, or a restricted class of multipoint functions whose calculation can be reduced to that of connected two-point functions. As a result, even for the simplest case of pion four-point functions, only the diagrams relevant for the $I = 2$ channel have been evaluated [1,2], with the physically more interesting $I = 0$ and 1 channels left unexplored.

The difficulty stems from the necessity of calculating quark propagators connecting two arbitrary points of the lattice, which requires $L^3 \times T$ inversions of the quark matrix with the conventional method of point source on a lattice of size $L^3 \times T$. The problem can be solved partially with the use of the wall source [3] and calculating quark propagators for T walls. This method has the additional advantage that Fierz-rearranged contributions, which complicates the analysis for two pion sources placed on the same time slice [2], can be avoided. In this Letter we report on a calculation of the full pion four-point functions at zero momentum using this method in quenched lattice QCD with the Kogut-Susskind quark action on a $12^3 \times 20$ lattice at $\beta = 6/g^2 = 5.7$. We obtain a clear signal for attraction for the $I = 0$ channel and that for repulsion for $I = 2$. The results for the scattering lengths show agreement with the prediction of current algebra and PCAC (partial conservation of axial-vector current) [4], allowing for systematic uncertainties in the calculation.

Let us consider scattering of two Nambu-Goldstone pions with zero momentum $\pi(t) = \sum_{\mathbf{x}} (-1)^{t+|\mathbf{x}|} \bar{\chi}(\mathbf{x},t) \times \chi(\mathbf{x},t)$ in the Kogut-Susskind fermion formalism. The diagrams contributing to the four-point function $\langle \pi(t_1) \times \pi(t_2) \pi(t_3) \pi(t_4) \rangle$ are shown in Fig. 1. The direct (D) and crossed (C) diagrams are calculable for arbitrary values of the time t_3 and t_4 using only two wall sources placed at the fixed time slices t_1 and t_2. This allows a calculation of the $I = 2$ scattering length, which has been attempted previously [2]. The rectangular (R) and vacuum (V) diagrams, necessary for the $I = 0$ channel, require additional quark propagators connecting the time slices t_3 and t_4.

We calculate these diagrams from T quark propagators corresponding to wall sources at each time slice on an $L^3 \times T$ lattice, which are defined by

$$D_{n',n''} G_t(n'') = \sum_{\mathbf{x}} \delta_{n',(\mathbf{x},t)}, \quad 1 \le t \le T, \tag{1}$$

with D the quark matrix for the Kogut-Susskind fermion. For the rectangular diagram we then write

$$C^R(t_1,t_2,t_3,t_4) = \sum_{\mathbf{x}_2,\mathbf{x}_3} \langle \mathrm{Re\,Tr}[G_{t_1}^\dagger(\mathbf{x}_2,t_2) G_{t_4}(\mathbf{x}_2,t_2) G_{t_4}^\dagger(\mathbf{x}_3,t_3) G_{t_1}(\mathbf{x}_3,t_3)] \rangle . \tag{2}$$

Using the relations $G_t(n'') = \sum_{\mathbf{x}} D^{-1}_{n'',(\mathbf{x},t)}$ and $G^\dagger(n;n') = (-1)^{n+n'} G(n';n)$, we easily see that (2) yields the rectangular amplitude of the four-point function up to terms for which the quark loop does not close at the time slices t_1 or t_4.

These terms create gauge-variant noise. One way to suppress the noise, employed in all recent work with wall sources, is to fix gauge configurations to some gauge and choose a particular wall source so that it emits only the Nambu-Goldstone pion [3]. We take an alternative way, which in fact was used in the initial work of extended sources [5], of carrying out the gauge field average

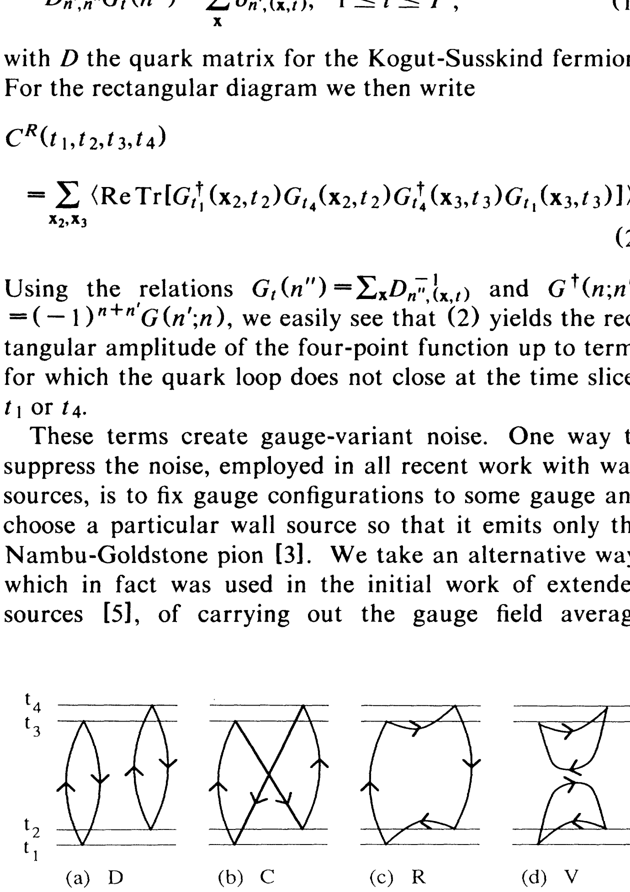

FIG. 1. Diagrams contributing to pion four-point functions.

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without gauge fixing. Wall sources are advantageous since the summation of the gauge-variant terms over spatial sites of the wall further reduces gauge-variant noise in addition to cancellations among configurations. We found in our simulation that the method works well for the four-point function within practically manageable statistics.

The vacuum diagram can be calculated in a similar manner, and the method also applies to the direct and crossed diagrams. Let us note that the present method yields the four-point function for all possible values of $t_1, \ldots, t_4$. Choosing $t_1 \neq t_2$ one can therefore avoid the rearrangement of quark lines of two pions due to color Fierz transformation, which occurs when one takes [2] $t_1 = t_2$. Such a rearrangement leads to a mixing of the direct and crossed amplitudes and in some cases switches the initial pions to non-Nambu-Goldstone states, making a separate extraction of the two amplitudes difficult.

Let us now outline the procedure to obtain $\pi\pi$ scattering lengths from the four-point function. The s-wave $\pi\pi$ scattering length a_0 is related to the energy level E of the lightest two-pion state with vanishing momenta in a cubic box of length L as [6]

$$E - 2m_\pi = -\frac{4\pi a_0}{m_\pi L^3}\left[1 + c_1 \frac{a_0}{L} + c_2 \left(\frac{a_0}{L}\right)^2\right] + O(1/L^6), \quad (3)$$

where $c_1 = -2.837\,297$ and $c_2 = 6.375\,183$. The energy E is extracted from the large t behavior of the pion four-point function calculated with $t_1 = 0$, $t_2 = 1$, $t_3 = t$, and $t_4 = t+1$. In practice we calculate the energy shift $\delta E = E - 2m_\pi$ from the ratio

$$R^X(t) = \frac{C^X(0,1,t,t+1)}{C_\pi(0,t)C_\pi(1,t+1)}, \quad X = D, C, R, \text{ and } V, \quad (4)$$

with $C_\pi(t',t) = \sum_{\mathbf{x}} \langle \mathrm{Tr}|G_{t'}(\mathbf{x},t)|^2 \rangle$ the pion two-point function. The amplitudes which project out the $I=0$ and 2 isospin eigenstates are given by

$$R_{I=0}(t) = R^D(t) + \frac{N_f}{2} R^C(t) - 3N_f R^R(t) + \frac{3}{2} R^V(t), \quad (5)$$

$$R_{I=2}(t) = R^D(t) - N_f R^C(t), \quad (6)$$

where the factor of $N_f = 4$ is inserted to correct for the four flavor degrees of freedom of the Kogut-Susskind fermion.

We should note that the procedure above, when applied to the Kogut-Susskind quark action, involves several systematic uncertainties arising from the nondegeneracy of pions between the Nambu-Goldstone and other channels at a finite lattice spacing that affects both the relation between the ratio $R(t)$ and the energy shift δE and that between δE and scattering lengths (3). The point was discussed in Ref. [2] which showed that (a) the coefficient of the $O(L^{-5})$ term in the relation (3) is invalidated by the effect, and (b) $O(t^2)$ terms in $R(t)$ are not correct. Thus a proper extraction of scattering lengths requires that the $O(L^{-5})$ contribution be small and that δE be extracted from the region linear in t of $R(t)$.

For numerical calculation we used 160 quenched configurations separated by 1000 pseudo-heat-bath sweeps on a $12^3 \times 20$ lattice at the gauge coupling constant $\beta = 5.7$ and the quark mass $m_q = 0.01$ in lattice units. The lattice size must be large enough so that we can employ as weak a coupling as possible to avoid finite lattice spacing effects, yet it should not be too large so as not to spoil a detection of a small energy shift of $O(L^{-3})$; our parameters are a reasonable compromise with the present computing resource. In order to avoid contamination from pions propagating backward in time we take the Dirichlet boundary condition in time for quark propagators. The periodic boundary condition is employed for the spatial directions. A necessary condition for the applicability of the formula (3) is that the lattice size L is sufficiently large so that finite-size effects for the single pion state are negligible. This point was checked, albeit indirectly, with a calculation of pion mass which gave $m_\pi = 0.290(3)$ for our $L = 12$, to be compared with the value 0.2876(7) for $L = 24$ [7].

In Fig. 2 the individual ratios R^X ($X = D$, C, R, and V)

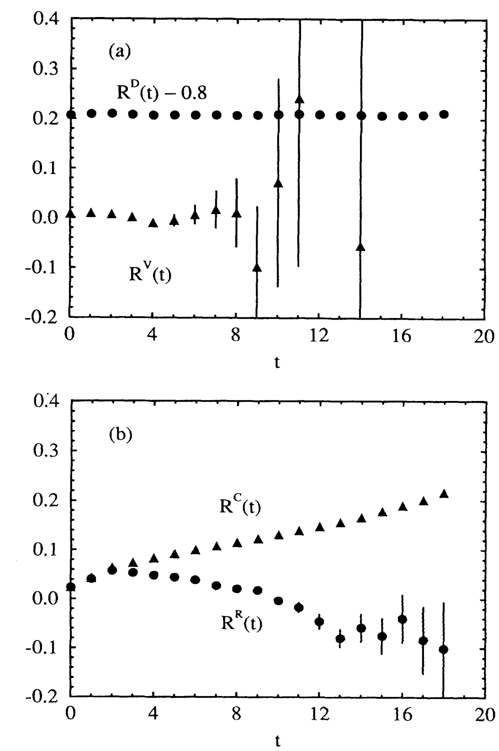

FIG. 2. Amplitude ratios $R^X(t)$ as defined in (4) for the diagrams in Fig. 1 as functions of t. (a) Direct diagram shifted by 0.8 (circles) and vacuum diagram (triangles); (b) crossed (triangles) and rectangular (circles) diagrams.

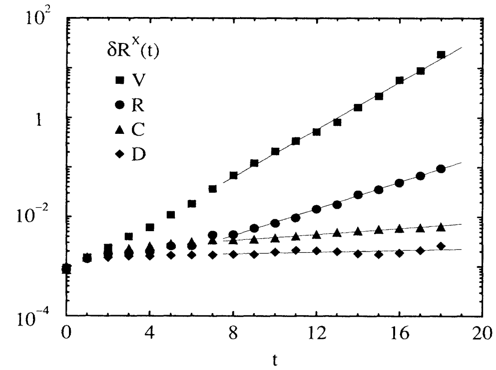

FIG. 3. Errors of ratios $R^X(t)$ as functions of t. Solid lines are single exponential fits over $8 \leq t \leq 18$.

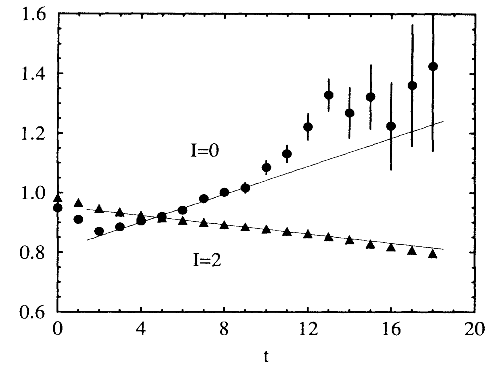

FIG. 4. $\pi\pi$ four-point function at zero momenta in the $I=0$ and 2 channels divided by the square of pion propagator. Solid lines are linear fits for $4 \leq t \leq 9$.

are plotted as functions of t. Errors are estimated by the single elimination jack-knife procedure. The direct amplitude is quite flat with the value close to unity, showing that the interaction is weak in this channel. The crossed amplitude, on the other hand, increases linearly, which means a repulsion in the $I=2$ channel. These features are known through the work of Ref. [2].

The amplitudes for the rectangular and vacuum diagrams represent the main result of the present work. After an initial increase up to $t \sim 3$ the rectangular amplitude exhibits a linear decrease up to $t \sim 10$. This means an attractive force between the two pions. Furthermore the absolute value of the slope is quite similar to that of the crossed amplitude. These results are what is expected from current algebra and PCAC. It should be mentioned that the close values of the rectangular and crossed amplitudes for small t is not an accident; the analytical expressions for the two amplitudes coincide at $t=0$. Hence they should behave similarly until the asymptotic two-pion state is reached. We also note that the statistical quality of $R^R(t)$ is good and similar to those of $R^D(t)$ and $R^C(t)$ up to $t \sim 10$. This demonstrates the practical applicability of the method of wall source without gauge fixing used here.

The vacuum amplitude is very small up to $t \sim 5$–6 beyond which signals are lost. The small value of the amplitude is consistent with the Okubo-Zweig-Iizuka rule and chiral perturbation theory which predicts the vanishing of the amplitude in leading order. The rapid loss of signal for large t can be understood from the analytical argument [8] that errors for disconnected amplitudes should be roughly independent of t, and hence grows exponentially as $e^{2m_\pi t}$ in the ratio $R^V(t)$. The argument also implies that the ratios for the direct and crossed diagrams have errors independent of t while that for the rectangular diagram increases as $e^{m_\pi t}$. The magnitude of errors is quantitatively consistent with these expectations as shown in Fig. 3. Fitting the errors $\delta R^X(t)$ by a single exponential $\delta R^X(t) = c_X e^{\mu_X t}$ over $8 \leq t \leq 18$ we find $\mu_X = 0.55$, 0.31, 0.07, and 0.02 for $X = V$, R, C, and D, to be compared with $m_\pi = 0.290(3)$. Thus while signals for the vacuum amplitude are masked by an exponentially growing error in our data, it is reasonable to suppose that the vacuum amplitude remains small for large t. We therefore neglect $R^V(t)$ in the rest of the analysis.

In Fig. 4 we show the ratio $R_I(t)$ projected onto the $I=0$ and 2 isospin channels. To extract the energy shift δE_I for each channel we employ a linear form $Z_I(1 - \delta E_I t)$ with the fitting range chosen to be $4 \leq t \leq 9$, and ignore higher order terms. The fitted values of δE_I and the results for the s-wave scattering length a_0 in lattice units obtained with (3) and $m_\pi = 0.290(3)$ are summarized in Table I. The errors quoted for a_0 are statistical only. Possible sources of systematic errors are the uncertainties in the $O(L^{-5})$ and higher term of (3), and scaling violations due to a fairly large lattice spacing of our simulation ($a^{-1} \approx 1$ GeV at $\beta = 5.7$ if determined from the ρ meson mass) [9]. Concerning the former effect the $O(L^{-5})$ term contributes 10% in the $I=0$ channel, while it is negligibly small ($< 1\%$) for $I=2$.

In Table I we also list the values predicted by current

TABLE I. $\pi\pi$ scattering length in lattice units calculated on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$ in quenched QCD. The energy shift δE_I and the wave function factor Z_I are also given. $m_\pi = 0.290(3)$ and $f_\pi = 0.132(3)$ are used to estimate the current algebra value Eq. (7).

	Lattice result	Current algebra
$a_0(I=0)$	1.57(25)	1.16(5)
$\delta E_{I=0}$	−0.0291(37)	
$Z_{I=0}$	0.807(15)	
$a_0(I=2)$	−0.301(28)	−0.331(15)
$\delta E_{I=2}$	0.00813(82)	
$Z_{I=2}$	0.955(5)	

algebra and PCAC [4],

$$a_0(I=0)=\frac{1}{32\pi}\frac{7m_\pi}{f_\pi^2}\,,\quad a_0(I=2)=-\frac{1}{32\pi}\frac{2m_\pi}{f_\pi^2}\,,\qquad (7)$$

where we substituted the value of m_π above and the pion decay constant $f_\pi=0.132(3)$ extracted on the same set of configurations. We observe that the simulation results are consistent with (7) within 1-1.5 standard deviations, which we find quite reasonable in view of the systematic uncertainties discussed above. A result for $a_0(I=2)$ in agreement with (7) was previously obtained [2] both for Kogut-Susskind and Wilson quark actions (with the assumption that the contribution of the direct diagram is negligible for the former case).

The agreement with the current algebra result is not unexpected since Ward identities for U(1) chiral symmetry of the Kogut-Susskind action is sufficient to show that the leading chiral behavior holds also on the lattice under some continuity assumptions [2,9]. Nonetheless we find it particularly encouraging that the $I=0$ pion scattering length which involves the essential difficulties of four-point functions can be calculated with the technique of wall sources without gauge fixing using only a modest computing power (we used about 320 hours on HITAC S820/80 with the peak speed of 2.4 GFLOPS for this work). This raises a prospect that the method may be successfully used to tackle other important processes which have hitherto resisted attempts for quantitative calculations, of which the prime examples include the $K\to\pi\pi$ decay and the disconnected two-point functions of flavor singlet mesons.

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