$I = 2$ Pion Scattering Length from Two-Pion Wave Function [*]

CP-PACS Collaboration : S. Aoki [a], M. Fukugita [b], K-I. Ishikawa [c], N. Ishizuka [a,d], Y. Iwasaki [a], K. Kanaya [a], T. Kaneko [e], Y. Kuramashi [a,d], M. Okawa [c], A. Ukawa [a,d], T. Yamazaki [a†] and T. Yoshié [a,d]

[a] Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

[b] Institute for Cosmic Ray Research, University of Tokyo, Tanashi, Tokyo 188-8502, Japan

[c] Department of Physics, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan

[d] Center for Computational Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8577, Japan

[e] High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan

We present a report on a calculation of scattering length for $I = 2$ S-wave two-pion system from two-pion wave function. Calculations are made with an RG-improved action for gluons and improved Wilson action for quarks at $a^{-1} = 1.207(12)$ GeV on $16^3 \times 80$, $20^3 \times 80$ and $24^3 \times 80$ lattices. We investigate the validity of necessary condition for application of Lüscher's formula through the wave function. We find that the condition is satisfied for lattice volumes $L \geq 3.92$ fm for the quark mass range $m_\pi^2 = 0.273 - 0.736$ GeV^2. We also find that the scattering length can be extracted with a smaller statistical error from the wave function than with a time correlation function used in previous studies.

By now the standard procedure employed for calculating the scattering length of hadrons from lattice QCD is to use the finite-size method proposed by Lüscher, which relates the finite volume shift of energy eigenvalues ΔE to the scattering lengths [1]. The formula reads $a_0/(L\pi) = -x - A \cdot x^2 - B \cdot x^3 + O(x^4)$, where $x = \Delta E \cdot 2 m_\pi L^2/(4\pi)^2$, and $A = -8.9136$ and $B = 95.985$ are geometrical constants. The energy shift ΔE is extracted from the asymptotic time behavior of two-pion correlation functions. In this way a number of calculations has been made for the $I = 2$ S-wave pion scattering length [2,3,4,5,6,7,8,9,10,11].

It should be noted that the condition $R < L/2$ is assumed for the two-pion interaction range R and the lattice volume L^3 in the derivation of the Lüscher's formula [1]. So far there have been studies of the lattice volume dependence of the scattering length, but no direct investigation of

the interaction range R. It is very important that we examine the validity of the necessary condition for the Lüscher's formula in our current lattice simulations. This can be done by investigating the two-pion wave functions, which is one of the purpose of this article.

Once one has access to the wave functions, one can try to extract the scattering length from them. This is the second, and perhaps more interesting, purpose of this article. Preliminary results of the present work was presented at Lattice'03 [12].

Our idea is based on the derivation of the Lüscher's formula. Lüscher found that the two-pion wave function $\phi(\vec{x})$ on a finite periodic box L^3 satisfies an effective Schrödinger equation $(\triangle + k^2)\phi(\vec{x}) = \int \mathrm{d}^3 y\, U_k(\vec{x}, \vec{y})\phi(\vec{y})$, where $\vec{x}$ is the relative coordinate of the two pions and k^2 is related to the two-pion energy eigenvalue through $E = 2\sqrt{m_\pi^2 + k^2}$. The function $U_k(\vec{x}, \vec{y})$ is the Fourier transform of the modified Bethe-Salpeter kernel introduced in Ref. [1]. It is non-local and

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[*] presented by N. Ishizuka
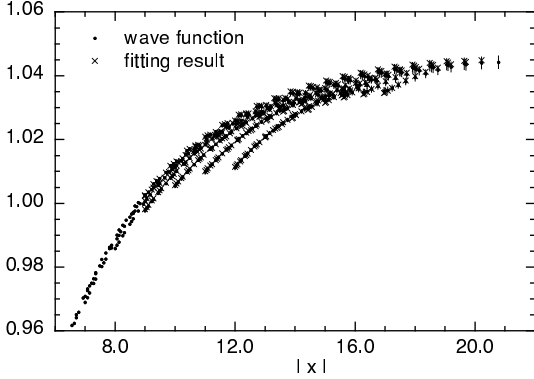[†] Present address : RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA

Figure 1. Wave function on 24^3 lattice at $t = 52$ and $m_\pi^2 = 0.273$ GeV^2.

Figure 2. Ratio $U(\vec{x})$ on 24^3 lattice at $t = 52$.

generally depends on energy.

In the derivation of the formula it is assumed that $\int \mathrm{d}^3 y \, U_k(\vec{x}, \vec{y})\phi(\vec{y}) \neq 0$ only for $|\vec{x}| < R < L/2$. In other words the volume has to be sufficiently large, so that the boundary condition does not distort the structure of the two-pion interaction. Out of this region the wave function satisfies the Helmholtz equation $(\triangle + k^2)\phi(\vec{x}) = 0$. The general solution of Helmholtz equation on a finite periodic box L^3 can be written as

$$\phi(\vec{x}) = \sum_{\vec{p} \in \Gamma} \mathrm{e}^{i\vec{p}\cdot\vec{x}}/(p^2 - k^2) \qquad (1)$$

up to an overall constant, where $\Gamma = \{ \vec{p} \mid \vec{p} = (2\pi)/L \cdot \vec{n} , \ \vec{n} \in \mathcal{Z}^3 \}$. Expanding the general solution (1) in terms of spherical Bessel $j_l(x)$ and Neumann $n_l(x)$ functions for $R < |\vec{x}| < L/2$, we obtain $\phi(\vec{x}) = \alpha_0(k) \cdot j_0(k|\vec{x}|) + \beta_0(k) \cdot n_0(k|\vec{x}|) + \cdots$, where neglected terms are contributions from states with angular momentum $l \geq 4$. The expansion coefficients $\alpha_0(k)$ and $\beta_0(k)$ yield the scattering phase shift in infinite volume by $\tan\delta_0(k) = \beta_0(k)/\alpha_0(k)$. In particular for the lowest energy state of the two-pion, it gives the scattering length by $\beta_0(k)/\alpha_0(k) = a_0 k + \mathrm{O}(k^3)$. The constant $\beta_0(k)$ is also related to $\alpha_0(k)$ geometrically and the relation leads to the Lüscher's formula.

We define the two-pion wave function by $\phi(\vec{x}, t) = \sum_{\mathbf{R}, \vec{X}} \langle \pi(R(\vec{x}) + \vec{X}, t)\pi(\vec{X}, t)S \rangle$, where $\mathbf{R}$ is an element of cubic group, and summation over $\mathbf{R}$ and $\vec{X}$ projects out the $\mathbf{A}^+$ sector of the cubic group and that of the zero center of mass momentum. In order to enhance signals we use a source constructed with two wall sources given by $S = W(t_0)W(t_0 + 1)$, with $W(t)$ the wall source
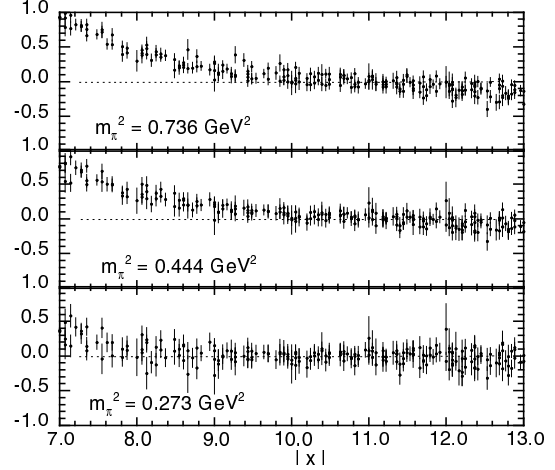
at t, in Coulomb gauge fixed configurations.

We work in quenched lattice QCD employing an RG-improved action for gluons at $\beta = 2.334$ and an improved Wilson action for quarks at $C_{SW} = 1.398$. The corresponding lattice cutoff is estimated as $1/a = 1.207(12)$ GeV from m_ρ. The volumes of lattices (number of configuration) are $16^3 \times 80$ (1200), $20^3 \times 80$ (1000) and $24^3 \times 80$ (506) which correspond to 2.61^3, 3.26^3 and 3.92^3 fm^3 in physical units. Quark masses are chosen to be $m_\pi^2 = 0.273, 0.351, 0.444, 0.588$ and 0.736 GeV^2. Quark propagators are solved with the Dirichlet boundary condition imposed in the time direction and the periodic boundary condition in the space directions. The sources are set at $t_0 = 12$.

The wave function on a 24^3 lattice at $t = 52$ and $m_\pi^2 = 0.236$ GeV^2 is plotted in Figure 1, where the horizontal axis is $x = |\vec{x}|$. We find that the statistical error is very small.

In order to estimate the two-pion interaction range R we construct a ratio $U(\vec{x}) = (\triangle + k^2)\phi(\vec{x}, t)/(k^2\phi(\vec{x}, t))$, where k^2 is obtained from the two-pion time correlation function. This ratio is expected to vanish out of the two-pion interaction range $R < x$. The ratio $U(\vec{x})$ on a 24^3 lattice at $t = 52$ for several quark masses are plotted in Fig. 2. We find a region $U(\vec{x}) = 0$ for $x < L/2$ for all quark masses within the statistical errors. The interaction range R tends to be larger for larger quark mass. We also find that $R \sim 10$ (1.6 fm) in the worst case of our parameters (the largest quark mass, $m_\pi^2 = 0.736$ GeV^2). These results mean that the necessary condition for the appli-
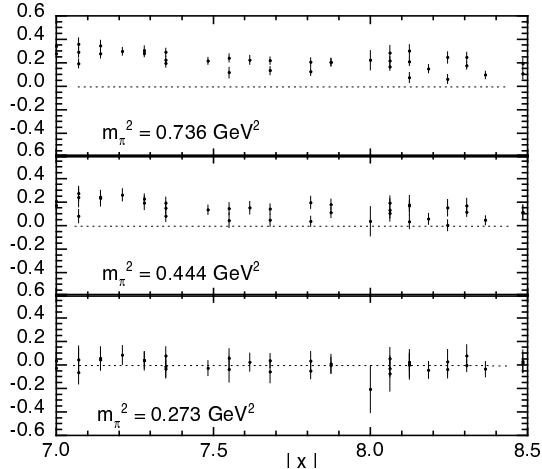
Figure 3. Ratio $U(\vec{x})$ on 16^3 lattice at $t = 52$.

Figure 4. $I = 2$ pion scattering length a_0/m_π $(1/\text{GeV}^2)$ obtained from time correlation function (from T), constant fit of $V(\vec{x})$ (from V) and fitting wave function with (1) (from ϕ).

cation of the Lüscher's formula is satisfied on the 24^3 lattice for all our quark masses.

The ratio $U(\vec{x})$ on a 16^3 lattice at $t = 52$ for several quark masses are shown in Fig. 3. There are deviations of $U(\vec{x})$ from zero for $x < L/2$ for larger quark masses. These deviation disappear for the lightest quark mass, however. This trend is also shown for the 20^3 lattice. In the following we analyze only data for which we can clearly find the region $U(\vec{x}) = 0$ for $x < L/2$.

We estimate the scattering length by substituting the momentum k^2 into the the Lüscher's formula. The momentum k^2 is obtained by the following two methods. In one of them we extract the momentum k^2 by a constant fit of the ratio $V(\vec{x}) = -\triangle\phi(\vec{x},t)/\phi(\vec{x},t)$ for $R < x$. In the other method k^2 is obtained by fitting the wave function $\phi(\vec{x},t)$ for $R < x$ with the function (1).

An example of fitting the wave function for a 24^3 lattice at $t = 52$ and $m_\pi^2 = 0.273$ GeV^2 is plotted by cross symbols in Fig. 1. The fit works very well.

Finally we compare the scattering length on the three lattice volumes in Fig.4. We also plot the results obtained from the two-pion time correlation function on the 16^3 and 20^3 lattices, for which we can not clearly find the region $U(\vec{x}) = 0$ for $x < L/2$ (data points enclosed by box in Fig. 4). We observe that the scattering length obtained from the three methods are consistent within the statistical errors. Further the statistical errors of those from our new methods are smaller than those from the two-pion time correlation function.
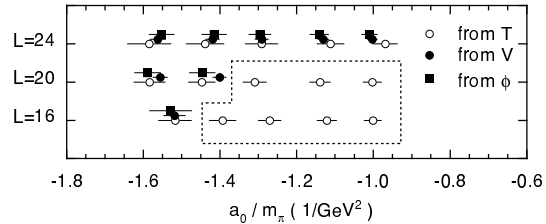
We also find no significant volume dependence for all quark masses including the data points enclosed by the box in Fig. 4. The necessary condition of the Lüscher's formula is not satisfied for these data. However, the effects of deformations of the two-pion interaction on the scattering length due to finite size effect is apparently small compared with the statistical errors.

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