$I = 2$ Pion Scattering Length and Phase Shift with Wilson Fermions [*]

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We present preliminary results of scattering length and phase shift for I=2 S-wave $\pi\pi$ system with the Wilson fermions in the quenched approximation. The finite size method presented by Lüscher is employed, and calculations are carried out at $\beta = 5.9$ on a $24^3 \times 60$ and $32^3 \times 60$ lattice.

1. Introduction

Lattice calculations of scattering lengths and phase shifts for the pion system is an important step for understanding of strong interactions beyond the hadron mass spectrum.

There are already several studies of scattering lengths in literature. For the $I = 2$ process calculations has been carried out with the staggered [1,2] and the Wilson fermion actions [2–4]. A recent work by JLQCD [4] carried out the calculation at several lattice cutoffs and obtained the value in the continuum limit. For the $I = 0$ process, which is difficult due to the presence of disconnected contributions, a pioneering attempt has been made in Ref. [2].

For the scattering phase shift, in contrast, there is only one calculation for $I = 2$ [5], and the data are still too noisy to obtain reliable results.

In this article we report on a high statistics calculation of the $I = 2$ S-wave pion scattering length and phase shift. We work in quenched lattice QCD employing the standard plaquette action for gluons and the Wilson fermion action for quarks. Simulations are made at $\beta = 5.9$ using 200 and 173 configurations on a $24^3 \times 60$ and $32^3 \times 60$ lattice. Quark propagators are solved with the Dirichlet boundary condition imposed in

the time direction and the periodic boundary condition in the space directions. Quark masses are chosen to be the same as in the previous study of quenched hadron spectroscopy by CP-PACS [6] (*i.e.*, $m_\pi/m_\rho = 0.491, 0.593, 0.692, 0.752$) for both lattice sizes.

2. Method

The energy eigenvalue W_p of an S-wave $\pi\pi$ system with momentum $\vec{p}$ and $-\vec{p}$ in a finite periodic box of a size L^3 is shifted from twice the pion energy $2 \cdot E_p$ by finite-size effects. Lüscher derived a relation between the energy shift $\Delta W_p = W_p - 2 \cdot E_p$ and the scattering phase shift $\delta(p)$, which takes the form [7]

$$\nu_n \frac{\tan \delta(p)}{\pi L p} = -x_p - A_n x_p^2 - B_n x_p^3 + O(x_p^4) \quad (1)$$

where $p^2 = n \cdot (2\pi/L)^2$, $(n = 0, 1, \cdots 6)$, $x_p = \Delta W_p \cdot (2 E_p L^2)/(16\pi^2) = O(1/L)$, and ν_n, A_n, and B_n are geometrical constants. The scattering length is given by $a_0 = (\tan \delta(p)/p)_{p \to 0}$.

In order to obtain the energy eigenvalue W_p we construct $\pi\pi$ 4-point functions $G_{pk}(t) = \langle 0|\Omega_p(t)\Omega_k(0)|0\rangle$. Here $\Omega_p(t)$ is an interpolating field for the S-wave $\pi\pi$ system at time t given by $\Omega_p(t) = \sum_R \pi(R(\vec{p}), t)\pi(-R(\vec{p}), t)$ where R is an element of the cubic group. In numerical calculations we construct the source operator $\Omega_k(0)$

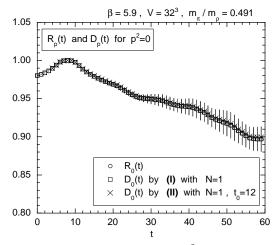Figure 1. $R_p(t)$ and $D_p(t)$ for $p^2 = 0$.

Figure 2. $R_p(t)$ and $D_p(t)$ for $p^2 = (2\pi/L)^2$.

using the noisy source method with U(1) random numbers.

Since the 4-point function $G_{pk}(t)$ contains many exponential terms due to the Maiani-Testa no-go theorem [8], the extraction of energy eigenvalues from $G_{pk}(t)$ is non-trivial. The following two methods are known [9] :

(I) Diagonalization of $G_{pk}(t)$ at each t, where the momenta p and k are regarded as matrix indices. The eigenvalues take the form $\lambda_p(t) = C_p \exp(-W_p t) \cdot [1 + O(\exp(-\Delta_p t)]$, where $C_p > 0$ and Δ_p is the distance of W_p from the other eigenvalues W_k. In this method the fitting range of t should be taken large so that the $O(\exp(-\Delta_p t))$ terms can be ignored.

(II) Diagonalization of $M_{pk}(t_0, t) \equiv [G(t_0)^{-1/2} G(t)G(t_0)^{-1/2}]_{pk}$ at each t where t_0 is fixed at some small value. The eigenvalues are given by $\lambda_p(t) = \exp(-W_p(t - t_0))$ where $O(\exp(-\Delta_p t))$ terms are absent.

In the actual diagonalization we have to cut off the set of momenta. Here we expect that the components of $G_{pk}(t)$ or $M_{pk}(t_0, t)$ for $p, k \leq q$ are dominant for the eigenvalue $\lambda_q(t)$ in large t region, while the components $p, k > q$ are less important. With this expectation, using both methods we calculate the energy eigenvalue for the ground state $(n = 0)$ and the first excited state $(n = 1)$ for $V = 24^3$, and also that of the second excited state $(n = 2)$ for $V = 32^3$. The cut-off dependence is investigated by varying the number of momenta $N \geq n$ for $N = 0, 1, 2, 3$.
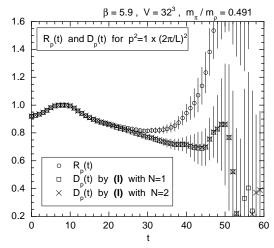
3.   Results

In order to examine the effects of diagonalization, we calculate two ratios defined by $R_p(t) \equiv G_{pp}(t)/[G_p^\pi(t)]^2$ and $D_p(t) \equiv \lambda_p(t)/[G_p^\pi(t)]^2$, where $G_p^\pi(t)$ is the pion propagator with momentum p. If $G_{pp}(t)$ or $\lambda_p(t)$ behaves as a single exponential function, we can obtain the energy shift ΔW_p easily from the ratio $R_p(t)$ or $D_p(t)$ by a single exponential fit.

In Fig. 1 we compare the ratios for $p^2 = 0$. The two-pion source is placed at $t = 8$. The momentum cut-off is set at $N = 1$ and both diagonalizations are employed. The signals are very clear and diagonalizations do not affect the result. We also checked the cut-off dependence by taking $N = 2$ and confirmed that it is negligible. In previous calculations of scattering lengths[1–4] the ratio $R_0(t)$ was used to extract the energy shift ΔW_0. Our calculation demonstrates the reliability of these calculations.

We compare the ratios for $p^2 = (2\pi/L)^2$ in Fig. 2. The momentum cutoff is set at $N = 1$ and $N = 2$. The method (I) is used. In contrast to the $p^2 = 0$ case, the diagonalization is very effective. The cut-off dependence is negligible, however. We also made the same analysis for method (II) and confirmed that the results are independent of the method.

The analysis here, and additional one for $p^2 = (2\pi/L)^2 \cdot 2$ for $V = 32^3$, lead us to conclude that (i) The momentum cut-off should be taken $N \geq n$
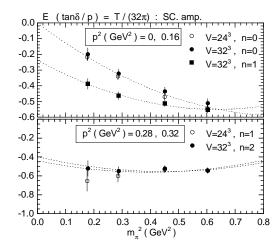
Figure 3. Results for scattering amplitude.

for the energy shift ΔW_p ($p^2 = (2\pi/L)^2 \cdot n$), and (ii) both method (I) and (II) give same results.

In Fig. 3 we plot our results of the scattering amplitude $T/(32\pi) = E_p \cdot (\tan\delta(p)/p)$ obtained by substituting our data for ΔW_p into (1). The signals are clear in the small momentum region (upper figure in Fig. 3) but become noiser for larger momenta (lower figure). In order to obtain the scattering length and phase shift for various momenta at the physical pion mass, we extrapolate our data with the following fitting assumption : $T/(32\pi) = A_{10} \cdot (m_\pi^2) + A_{20} \cdot (m_\pi^2)^2 + A_{01} \cdot (p^2) + A_{02} \cdot (p^2)^2 + A_{11} \cdot (m_\pi^2 p^2)$. The fit curves are also plotted in Fig. 3.

In Fig. 4 we compare our results of the phase shift $\delta(p)$ at physical pion mass obtained with the fit above with experiments [10,11]. The simulation points are plotted by the large square symbols. Our results for $\delta(p)$ are 40% smaller in magnitude than those of experiments, and our result of scattering length, $a_0 m_\pi = -0.0266(16)$, differs from the ChPT prediction given by $a_0 m_\pi = -0.044$ [12].

A possible cause of the discrepancy is finite lattice spacing effects. The JLQCD results for scattering length [4] show sizable scaling violation, and hence that of the scattering phase shift cannot be considered small. Further calculations nearer to the continuum limit or calculations with improved actions are desirable.

Figure 4. Comparison of our results of scattering phase shift $\delta(p)$ and experiments.

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