Integrated Volatility Measuring from Unevenly Sampled Observations.

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1 Estimators of integrated volatility

Let p_t be logarithmic asset price

$$dp(t) = \mu(t)dt + \sigma(t)dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian Motion, $\mu(t), \sigma(t)$ are random time dependent functions. The diffusion is observed at $\{t_i\}_{i=0}^N$. In this paper, we compare the estimators of integrated volatility $\int_0^T \sigma^2(t)dt$.

1.1 Quadratic variation of evenly sampled observations through linear interpolation

The transaction data which are unevenly spaced, are not directly used. After creating evenly spaced data $\{p\,(iT/m)\}_{i=0}^m$ from $\{p\,(t_i)\}_{i=0}^N$ through linear interpolation, the volatility is measured by the following estimator,

$$\hat{\sigma}^2(m) = \sum_{i=1}^m \left(p\left(\frac{iT}{m}\right) - p\left(\frac{(i-1)\,T}{m}\right)\right)^2. \tag{2}$$

This estimator is downward biased.

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1.2 Fourier estimator of [MM02]

To avoid the interpolation bias, [MM02] proposed the method without any data manipulation by using Fourier series.

$$\hat{\sigma}_F^2 = 2\pi a_0(\sigma^2) \tag{3}$$

where

$$a_0(\sigma^2) = \lim_{n\to\infty} \frac{\pi}{n-1} \sum_{s=2}^{n} \frac{1}{2}(a_k^2(dp) + b_k^2(dp)), \tag{4}$$

$$a_k(dp) = \frac{1}{\pi} \int \cos(kt)dp(t), \tag{5}$$

$$b_k(dp) = \frac{1}{\pi} \int \sin(kt)dp(t), \tag{6}$$

and n is Nyquist frequency $N/2$.

1.3 Quadratic variation of unevenly sampled observations

Another method using unevenly sampled observations $\{p\,(t_i)\}_{i=0}^{N}$:

$$\hat{\sigma}^2 = \sum_{i=1}^{N} (p(t_i) - p(t_{i-1}))^2, \tag{7}$$

has a nice property that if $\sup_{i\geq 1}\,(t_i - t_{i-1}) \to 0$,

$$\lim_{N\to\infty} \hat{\sigma}^2 = \int_0^T \sigma^2(t)dt. \tag{8}$$

See e.g. [ABDL03]. This estimator is simple but as efficient as Fourier estimator.

1.4 Monte Carlo simulations

We generate proxy for continuous observation by discretizing following equations with a time step of one second,

$$dp(t) = \mu(t)dt + \sigma(t)dW(t),$$
$$d\log\sigma_t = -k\log\sigma_t dt + \gamma dW_t$$

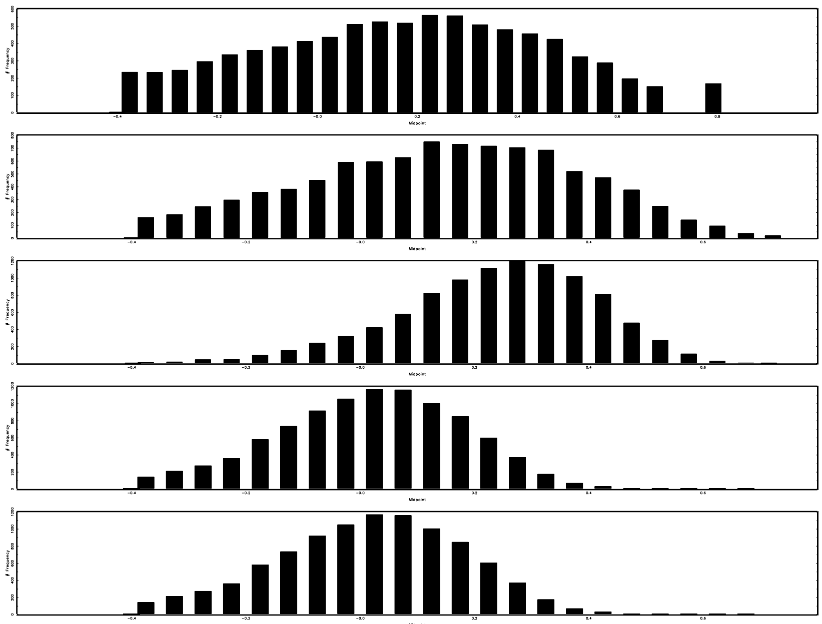

Figure 1: 10min vs 5min vs 2min vs FE vs QV. The distributions are compared with 10,000 replications.

	computational time
Fourier estimator	1116.42″
Quadratic variation	0.25″

Table 1: Computational time (seconds)

where W_s is standard Brownian Motion. The waiting times are drawn from an exponential distribution with mean 45 seconds according to [BR02]. See [ER98] for the modeling of waiting time. Figure 1 reports the distributions of

$$1 - \frac{\hat{\sigma}^2}{\int_0^T \sigma^2(t)dt}.$$

Table report the computational time of FE and QV.

2 Cross-volatility

$$dp_j(t) = \mu_j(t)dt + \sum_{k=1}^{d} \sigma_{jk}(t)dW_k(t), \tag{9}$$

Volatility matrix is defined by

$$\Omega_{(jk)}\left(t\right) = \sum_{i=1}^{d} \sigma_{ji}\sigma_{ki}.$$

Our target is $\int_0^T \Omega\left(t\right) dt$.

2.1 Linear interpolation

$$\hat{\Omega}_{(jk)}(m) = \sum_{i=1}^{m} \left(p_j \left(\frac{iT}{m} \right) - p_j \left(\frac{\left(i-1\right)T}{m} \right) \right) \left(p_k \left(\frac{iT}{m} \right) - p_k \left(\frac{\left(i-1\right)T}{m} \right) \right).$$

What occurs on linear interpolation bias?

2.2 Fourier estimator

$$\hat{\Omega}_{F(jk)} = 2\pi a_0(\Omega_{(jk)})$$

where

$$a_0(\Omega_{(jk)}) = \lim_{n\to\infty} \frac{\pi}{n-1} \sum_{s=2}^{n} \frac{1}{2}(a_s(dp_j)a_s(dp_k) + b_s(dp_j)b_s(dp_k)), \tag{10}$$

$$a_k(dp_i) = \frac{1}{\pi} \int \cos(kt)dp_i(t), \tag{11}$$

$$b_k(dp_i) = \frac{1}{\pi} \int \sin(kt)dp_i(t), \tag{12}$$

where $n = [N/2]$.

2.3 One-side linear Interpolation

The jth and kth diffusion of (9) are observed at $\{t_i\}_{i=0}^{N_j}$ and $\{t_i\}_{i=0}^{N_k}$ respectively. Define the sequence: $\{t_i\}_{i=0}^{N_{jk}} \equiv \left\{ t : \{t_i\}_{i=0}^{N_j} \cup \{t_i\}_{i=0}^{N_k} \right\}$.

$$\hat{\Omega}_{(jk)}(m) = \sum_{i=1}^{N_{jk}} \left(p_j\left(t_i\right) - p_j\left(t_{i-1}\right)\right)\left(p_k\left(t_i\right) - p_k\left(t_{i-1}\right)\right)$$

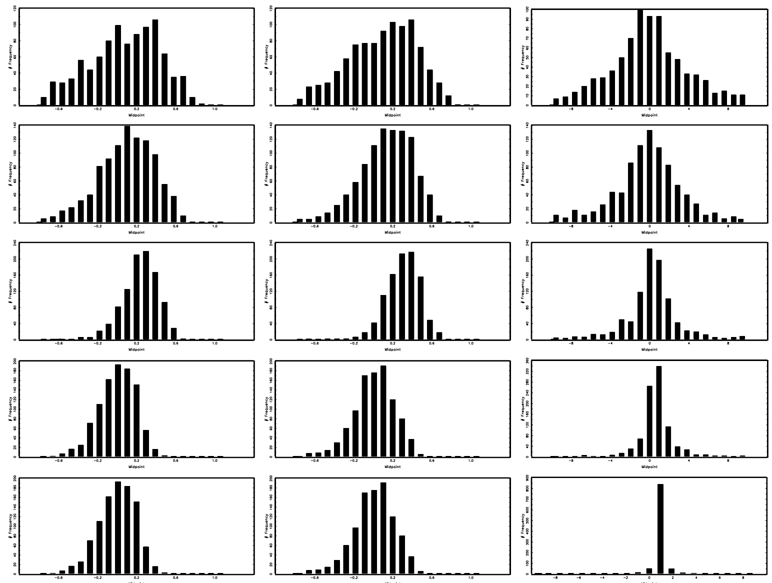

Figure 2: Volatilities and cross-volatility. The distributions are compared with 1,000 replications.

2.4 Monte Carlo simulations

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

$$d\sigma_{jk}(t) = -\kappa_{jk}\sigma_{jk}(t)\,dt + \gamma_{jk}dW_{jk}(t)\,,\ j,k = 1,2.$$

where $\kappa_{jk} = 0.99$ and $\gamma_{jk} = 0.01$ for any j, k.

Figure 2 reports the distributions of

$$1 - \frac{\hat{\Omega}_{(jk)}}{\int_0^T \Omega_{(jk)}(t)\,dt}.$$

3 Conclusion

Let us use (7) in scaler case. However, we expect that Fourier estimator is good for cross-volatility. There are many remaining works:

- Asymptotic distribution of the estimators.
- Linear interpolation bias correction.
- Long memory.

Acknowledgements

Thank you for your reading. To be continued.

A Fourier estimator of [MM02]

The method will be the following: first compute the Fourier coefficients of dp_i, the obtain a mathematical expression of the Fourier coefficients of Ω_{jk} using the Fourier coefficients of dp_i.

References

[ABDL03] Torben G. Andersen, Tim Bollerslev, Francis X. Diebold, and Paul Labys, *Modeling and forecasting realized volatility*, Econometrica **71** (2003), 579–625.

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