Long Memory

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Abstract

We survey the continuous and descrete time long memory SV models.

1 Long memory in continuous time SV

1.1 Fractional BM

$$x_t = \int_0^t a(t-s)dW_s \tag{1.1}$$

$$y_t = \int_{-\infty}^t a(t-s)dW_s \tag{1.2}$$

where (1.2) is the stationary process and $\int_0^\infty a^2(t)dt < +\infty$. Then $x_t = y_t$ in quadratic mean.

Definition 1 *If $a(t) = t^\alpha \tilde{a}(t)/\Gamma(1+\alpha)$ for $|\alpha| < 1/2$ where $\tilde{a}$ is continuously differentiable on $[0,T]$, then (1.1) and (1.2) called fractional processes.*

Fractional processes can also written by

$$x_t = \int_0^t c(t-s)dW_{\alpha s}, \quad W_{\alpha s} = \int_0^s \frac{(s-u)^\alpha}{\Gamma(1+\alpha)}dW_u$$

where, W_α is the so-called fractional BM of order α.

- The relation between a and c is one to one.
- W_α is not semi-martingale (see e.g.[Rog97]) but stochastic integration w.r.t W_α can be defined.
- The processes are long memory if

$$\lim_{t\to\infty} t\tilde{a}(t) = a(\infty), \quad 0 < \alpha < \frac{1}{2} \text{ and } 0 < a(\infty) < +\infty.$$

Example 1

$$dx_t = -kx_t dt + \sigma dW_{\alpha t},\ W_{\alpha t} = \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} dW_s,\ x_0 = 0,\ 0 < \alpha < \frac{1}{2}$$

has a solution of

$$x_t = \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} dx_s^{(\alpha)},$$
$$x_t^{(\alpha)} = \int_0^t e^{-k(t-s)} \sigma dW_s.$$

- $x_t^{(\alpha)}$ *is the derivative of order* α *of* x_t*. See the Appendix A.*
- $x_t^{(\alpha)}$ *is a solution of the linear SDE:* $dx_t^{(\alpha)} = -kx_t^{(\alpha)} dt + \sigma dW_t$.[1]

1.2 Fractional SV

$$\frac{dS_t}{S_t} = \sigma_t dW_t \tag{1.3}$$
$$d\log\sigma_t = -k\log\sigma_t dt + \gamma dW_{\alpha t} \tag{1.4}$$

where $k > 0$ and $0 \leq \alpha < 1/2$.

- The fractional exponent α provides some degree of freedom in order of regularity of the volatility process; the greater α, the smoother the path of volatility process.
- The volatility process itself (not only its logarithm) has hyperbolic decay of the correlogram.
- The persistence of volatility shocks yields leptokurtic features for returns. It vanishes with temporal aggregation at slow hyperbolic rate of decay.[2]

1.3 Filtering and discrete time approximations

By Example 1, the solution of (1.4) is given by

$$\log\sigma_t = \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} d\log\sigma_s^{(\alpha)} \tag{1.5}$$
$$\log\sigma_s^{(\alpha)} = \int_0^t e^{-k(t-s)} \sigma dW_s \tag{1.6}$$

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[1] A linear SDE:

$$dz_t = -kz_t dt + \sigma dW_t$$

has a solution of

$$z_t = z_0 \int_0^t e^{-k(t-s)} \sigma dW_s.$$

[2] With usual CARCH or SV models, it vanish at an exponential rate (see [DN93] and [DW96] for these issues in short memory case).

where $\log\sigma_s^{(\alpha)}$ follows the O-U process: $d\log\sigma_s^{(\alpha)} = -k\log\sigma_s^{(\alpha)}dt + \gamma dW_t$. To descretize (1.5), divide $[0,t]$ at time points $j/n,\ j=0,1,...,[nt]$. First descretize (1.6) as

$$(1-\rho L)\log\sigma_{j/n}^{(\alpha)} = u_{i/n}$$

where $\rho = \exp(-k/n)$ and $u_{i/n}$ is the associated innovations process.

$$\begin{aligned}\log\sigma_{j/n} &\approx \sum_{i=1}^{j}\frac{(j-(i-1))^{\alpha}}{n^{\alpha}\Gamma(1+\alpha)}\Delta\log\sigma_{i/n}^{(\alpha)} \\ &= \left[\sum_{i=0}^{j-1}\frac{(i+1)^{\alpha}-i^{\alpha}}{n^{\alpha}\Gamma(1+\alpha)}L^i\right]\log\sigma_{i/n}^{(\alpha)} \\ &= \left[\sum_{i=0}^{j-1}\frac{(i+1)^{\alpha}-i^{\alpha}}{n^{\alpha}\Gamma(1+\alpha)}L^i\right](1-\rho L)^{-1}u_{i/n}\end{aligned}$$

long memory filter $\left[\sum_{i=0}^{j-1}\frac{(i+1)^{\alpha}-i^{\alpha}}{n^{\alpha}\Gamma(1+\alpha)}L^i\right]$

short memory filter $(1-\rho L)^{-1}$

2 Long memory in discrete time models

(1) FIGARCH (2) FIEGARCH (3) Long-memory SV

2.0.1 Long-memory SV

$$h_t = (1-L)^{-d}\eta_t,\quad \eta_t \sim NID\left(0,\sigma_\eta^2\right)$$

where $h_t \equiv \log\sigma_t^2$.

$d=0$	white noise
$-1<d<0$	stationary intermediate-memory process
$0<d<1/2$	stationary long-memory process
$d=1$	random walk

3 Stylized facts about long memory

A Fractional Calculus

Denote the nth derivative D^n and the $n-$fold integral D^{-n}. Then

$$D^{-1}f(t) = \int_0^t f(x)\,dx$$

Now if

$$D^{-n}f\left(t\right)=\frac{1}{\left(n-1\right)!}\int_{0}^{t}\left(t-x\right)^{n-1}f\left(x\right)dx \tag{A.1}$$

is true for n, then

$$\begin{aligned}D^{-(n+1)}f\left(t\right)&=D^{-1}\left[\frac{1}{\left(n-1\right)!}\int_{0}^{t}\left(t-x\right)^{n-1}f\left(x\right)dx\right]\\&=\int_{0}^{t}\left[\frac{1}{\left(n-1\right)!}\int_{0}^{s}\left(s-x\right)^{n-1}f\left(x\right)dx\right]ds\\&=\frac{1}{\left(n-1\right)!}\int_{0}^{t}\int_{0}^{s}\left(s-x\right)^{n-1}f\left(x\right)dxds\\&=\frac{1}{\left(n-1\right)!}\int_{0}^{t}\int_{0}^{x}\left(s-x\right)^{n-1}f\left(x\right)dsdx\\&=\frac{1}{n!}\int_{0}^{t}\left(t-x\right)^{n}f\left(x\right)dx\end{aligned}$$

But (A.1) is true for $n=1$, so it is also true for all n by induction. The fractional integration can be defined by

$$D^{-\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-x\right)^{\alpha-1}f\left(x\right)dx.$$

References

[DN93] Feike C. Drost and Theo E. Nijman, *Temporal aggregation of GARCH processes*, Econometrica **61** (1993), 909–927.

[DW96] Feike C. Drost and Bas J.M. Werker, *Closing the GARCH gaps: Continuous time GARCH modeling*, Journal of Econometrics **74** (1996), 31–57.

[Rog97] L. C. G. Rogers, *Arbitrage with fractional brownian motion*, Mathmatical Finance **7** (1997), 95–105.