Optimally Weighted Realized Volatility

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Introduction

■ Estimation of volatility (Conditional variance of asset return).
$\Rightarrow$ Option pricing, VaR, etc.

■ High-frequency data.
e.g. Hourly data, 30 minutes data, $\cdots$, 5 minutes data, Transaction data (Raw data)

DGP:

$$\underbrace{dp(t)}_{n\times 1} = \underbrace{\mu(t)}_{n\times 1} dt + \underbrace{\Sigma(t)}_{n\times n} \underbrace{dz(t)}_{n\times 1}, \quad 0 \leq t \leq T$$

Volatility matrix:

$$\Omega(t) \equiv \Sigma(t)\,\Sigma(t)'$$

Discretely observed time points:

$$0 = t_0^i < t_1^i < \cdots < t_k^i < \cdots < t_{N_i-1}^i < t_{N_i}^i = T$$

$\boxed{\text{Estimation of integrated volatility } \int_0^T \Omega(t)dt}$

$\Rightarrow$ Quadratic variation:

$$\lim_{N_i \to \infty} \underbrace{\sum_{k=1}^{N_i} (\Delta p_i(t_k^i))^2}_{\text{Realized volatility}} = \int_0^T \omega_{ii}(t)dt$$

Weighted realized volatility

$$\widehat{\omega}_{ij} = \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} w_{kl} \Delta p_i(t_k^i) \Delta p_j(t_l^j) = \underbrace{\Delta p_i'}_{1 \times N_i} \overbrace{W}^{N_i \times N_j} \underbrace{\Delta p_j}_{N_j \times 1}$$

where $\Delta p_i = (\Delta p_i(t_1^i), \cdots, \Delta p_i(t_{N_i}^i))'$.

Examples:

1. Realized volatility calculated by using interpolated data
2. Realized volatility calculated by using raw data
3. Fourier series estimator (Malliavin and Mancino, 2002)

Interpolation and realized volatility

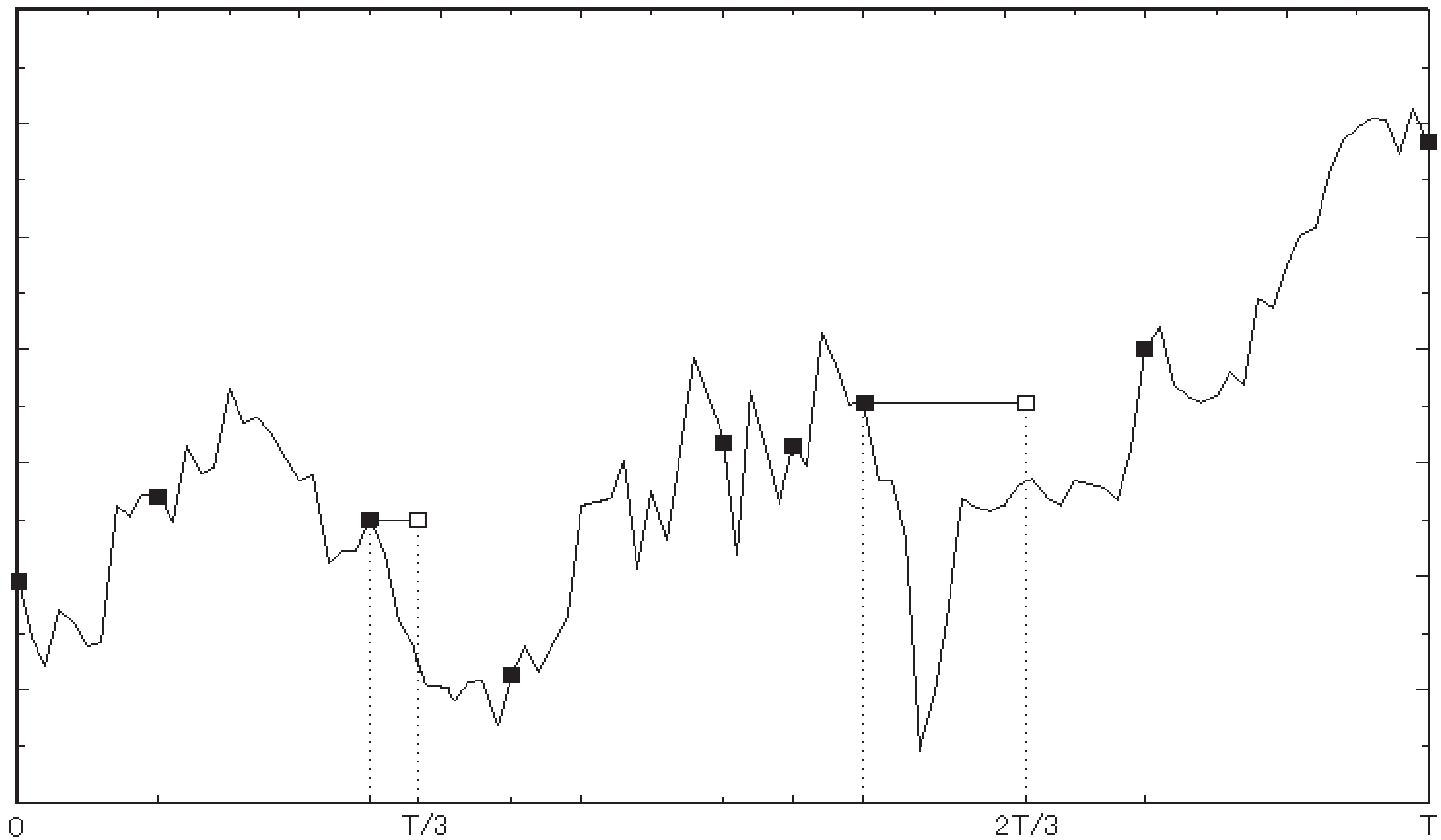
- $\overbrace{\{p_i(t_k^i)\}_{k=1}^{N_i}}^{\text{Unevenly spaced data}} \quad \rightarrow \boxed{\text{Interpolation}} \rightarrow \overbrace{\{q_i(mT/M)\}_{m=1}^{M}}^{\text{Evenly spaced data}}$

- Realized (cross) volatility

$$\widehat{\omega}_{ij} = \sum_{m=1}^{M} \left\{ q_i\left(\frac{mT}{M}\right) - q_i\left(\frac{(m-1)T}{M}\right) \right\} \left\{ q_j\left(\frac{mT}{M}\right) - q_j\left(\frac{(m-1)T}{M}\right) \right\}$$

$\Rightarrow$ Interpolation bias

Previous-tick interpolation

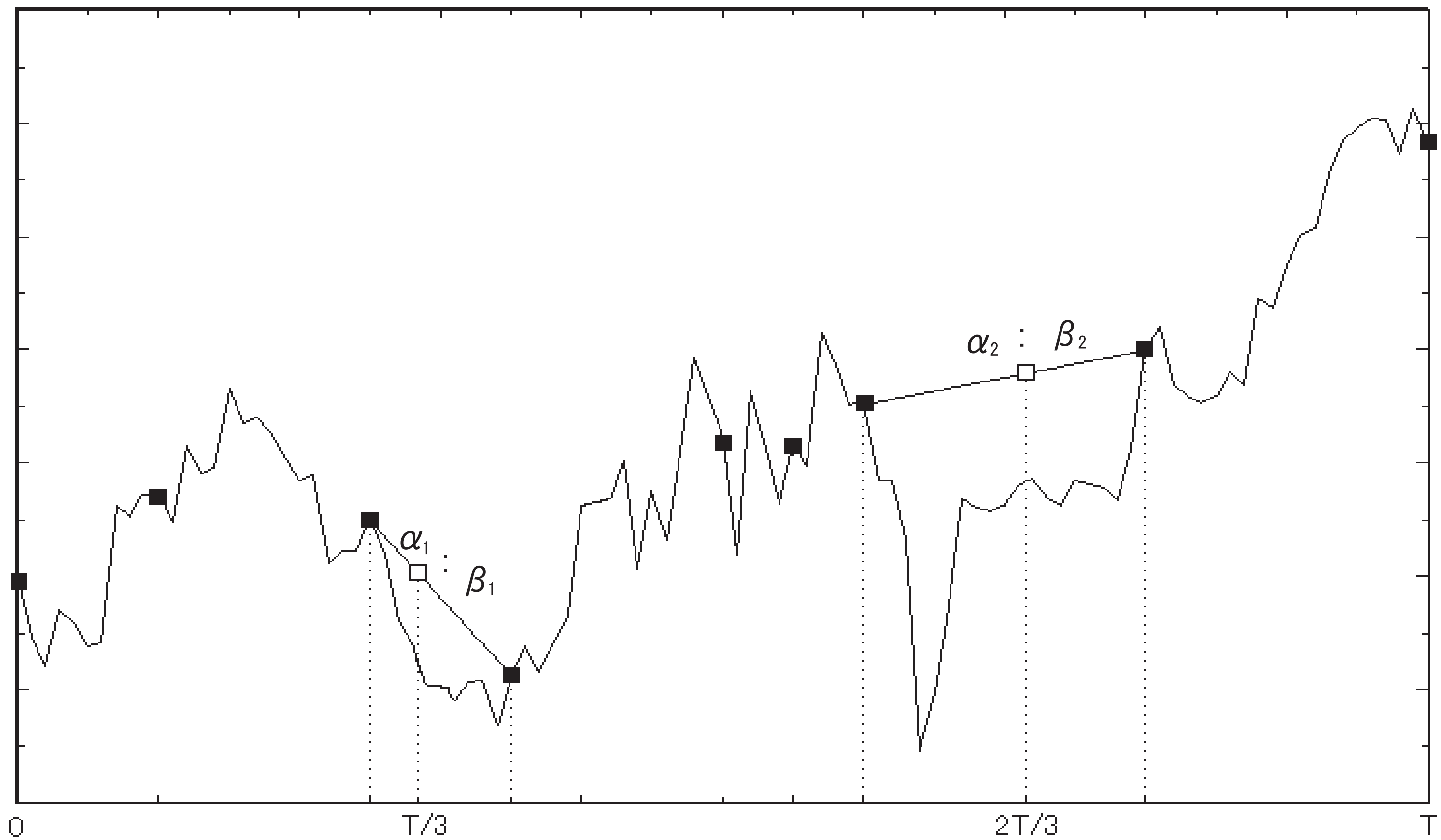


■ Transaction; □ Interpolated

Weight matrix for the previous-tick interpolation ($i = j$):

$$W = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear interpolation



■ Transaction; □ Interpolated

Weight matrix for the linear interpolation ($i = j$):

$$W = \begin{pmatrix} 1 & 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_1 & \alpha_1^2 + \beta_1^2 & \beta_1 & \beta_1 & \beta_1 & \beta_1\alpha_2 & 0 \\ 0 & 0 & \beta_1 & 1 & 1 & 1 & \alpha_2 & 0 \\ 0 & 0 & \beta_1 & 1 & 1 & 1 & \alpha_2 & 0 \\ 0 & 0 & \beta_1 & 1 & 1 & 1 & \alpha_2 & 0 \\ 0 & 0 & \beta_1\alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2^2 + \beta_2^2 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_2 & 1 \end{pmatrix}$$

When we can use all transaction data,...

Raw data realized volatility: $\sum_{k=1}^{N_i} \Delta p_i(t_k^i)^2 = \Delta p_i' \Delta p_i$

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Fourier series estimator (Malliavin and Mancino, 2002):

$$\widehat{\omega}_{ij}^{F} = \frac{\pi^2}{Q} \sum_{q=1}^{Q} \left(a_q(dp_i) a_q(dp_j) + b_q(dp_i) b_q(dp_j) \right)$$

where

$$a_q(dp_i) = \frac{1}{\pi} \int_0^{2\pi} \cos(qt) dp_i(t), \;\; b_q(dp_i) = \frac{1}{\pi} \int_0^{2\pi} \sin(qt) dp_i(t).$$

$\Rightarrow$

$$w_{kl} = \begin{cases} 1 & \text{if } t_k^i = t_l^j, \\ \dfrac{\sin \frac{(Q+1)\left(t_k^i - t_l^j\right)}{2} \cos \frac{Q\left(t_k^i - t_l^j\right)}{2}}{Q \sin \frac{\left(t_k^i - t_l^j\right)}{2}} & \text{otherwise} \end{cases}.$$

- Each method is characterized by its weight matrix.

- What is the best choice of the weight matrix? $\Rightarrow$ MSE

- Solving

$$\min_{w_{kl}} E\left(\widehat{\omega}_{ij} - \int_0^T \omega_{ij}(t)dt\right)^2$$

then...

MSE-minimizing weight:

$$w^A_{kl} = \frac{\int_0^T \omega_{ij} dt \int_{I(k,l)} \omega_{ij} dt}{v_{kl}\left\{1 + \sum u_{kl}\right\}}.$$

where

$$v_{kl} = \left(\int_{I(k,l)} \omega_{ij} dt\right)^2 + \left(\int_{t_{k-1}}^{t_k} \omega_{ii} dt\right)\left(\int_{t_{l-1}}^{t_l} \omega_{jj} dt\right)$$

$$u_{kl} = \frac{\left(\int_{I(k,l)} \omega_{ij} dt\right)^2}{v_{kl}}.$$

- Densely-sampled/less volatile period $\Rightarrow$ Larger weight
- Coarsely-sampled/volatile period $\Rightarrow$ Smaller weight

A feasible estimator:

$$\int_{I(k,l)} \omega_{ij} dt \Rightarrow \Delta p_i \left(t_k^i\right) \Delta p_j \left(t_l^j\right)$$

$$\int_{t_{k-1}}^{t_k} \omega_{ii} dt \Rightarrow \left\{\Delta p_i \left(t_k^i\right)\right\}^2$$

$$\int_0^T \omega_{ij} dt \Rightarrow \sum \Delta p_i \left(t_k^i\right) \Delta p_j \left(t_l^j\right) I(A)$$

$\Rightarrow$ Naive weight

Monte Carlo study

■ DGP

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \; 0 \leq t \leq T$$

$$d\sigma_{ij}(t) = \kappa\left(\theta - \sigma_{ij}(t)\right)dt + \gamma dW_{ij}(t), \; i,j = 1,2.$$

where $\kappa = 0.01$, $\theta = 0.01$, and $\gamma = 0.001$ and $T = 60 \times 60 \times 24$ seconds.

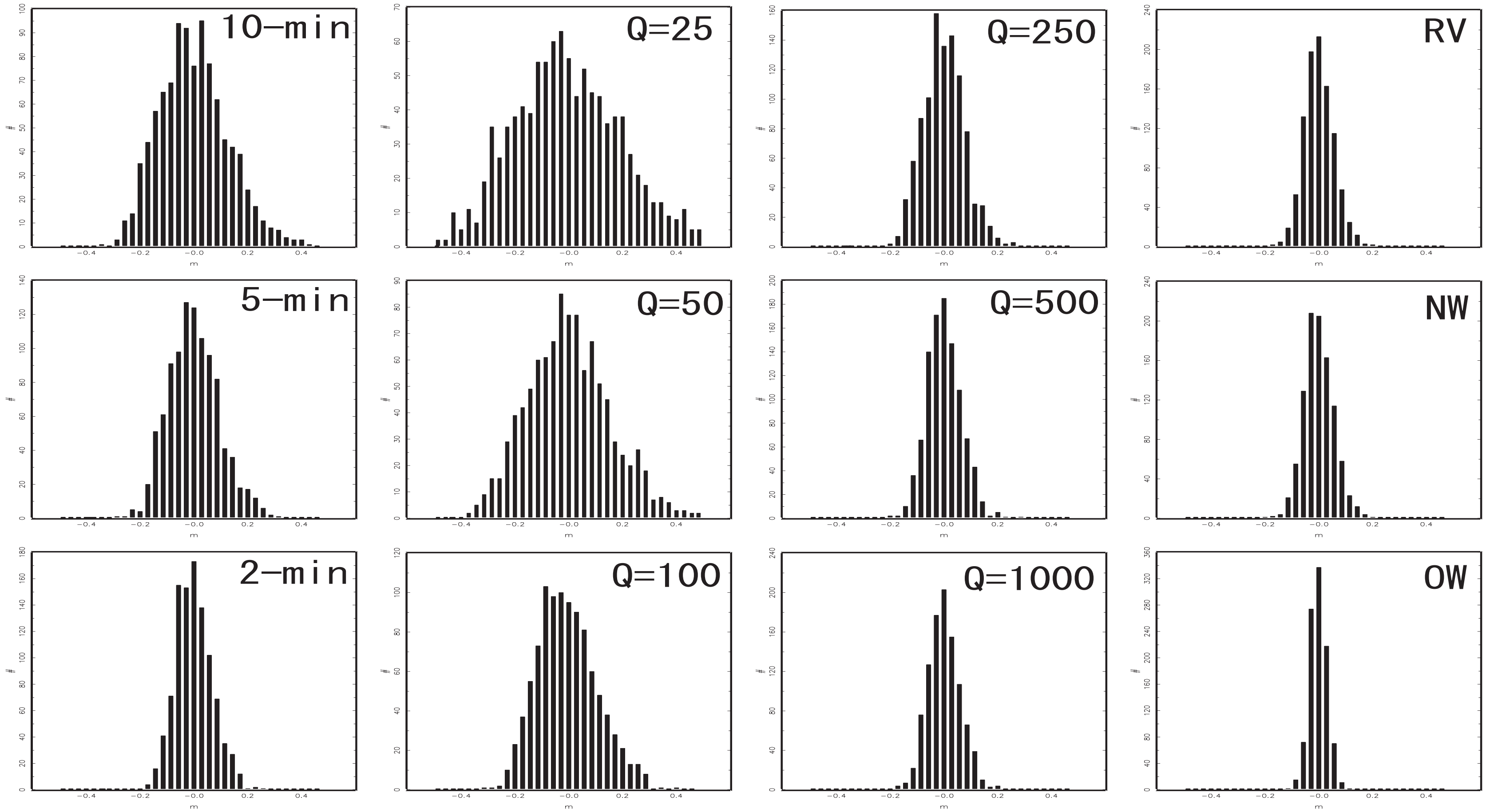
■ Time differences are drawn from an exponential distribution with mean 45 seconds for p_1 and 60 seconds for p_2:

$$F\left(t_k^i - t_{k-1}^i\right) = 1 - \exp\left\{-\lambda_i\left(t_k^i - t_{k-1}^i\right)\right\}, \; i = 1,2$$

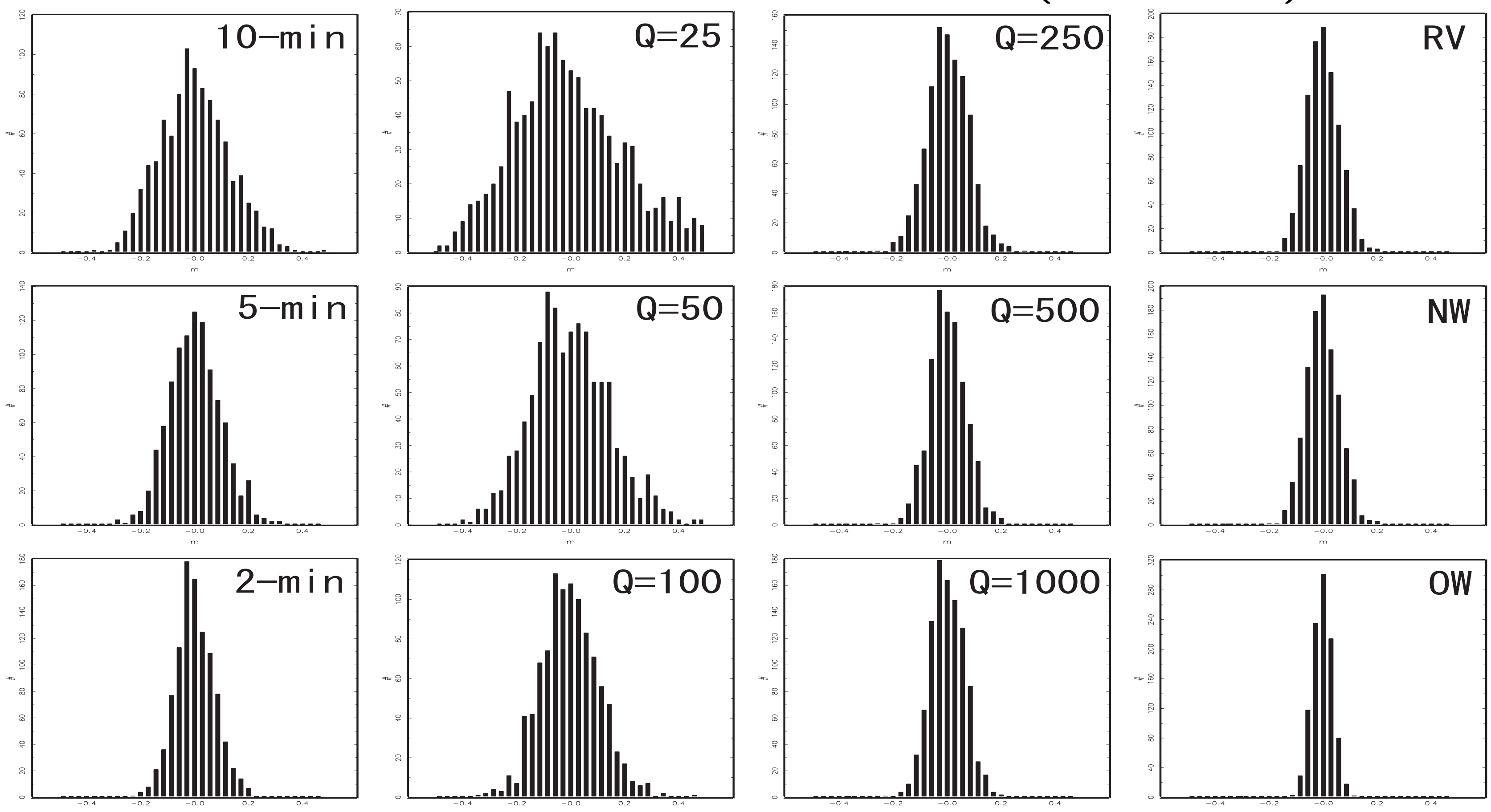
where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_1 = 1/45$ and $\lambda_2 = 1/60$.

■ 1000 ‘daily’ replications

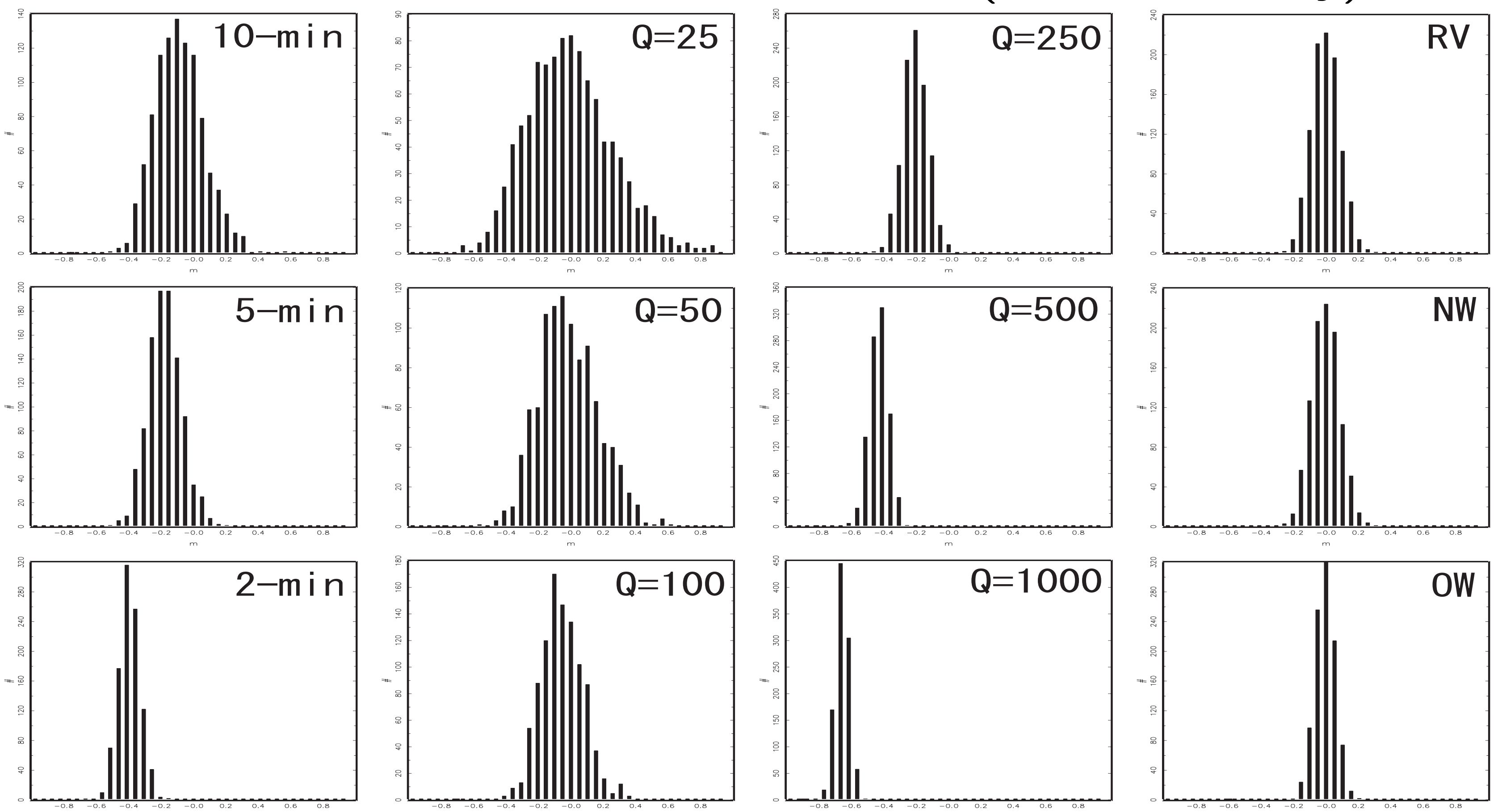
Distribution of measurement error (1st asset)



Distribution of measurement error (2nd asset)



Distribution of measurement error (cross volatility)



Summary

Contributions:

- To unify and generalize different methods
- To derive theoretically optimal weight matrix

Remaining works:

- To improve feasible estimator
- Asymptotic distribution

The latest version of this paper is available at
http://home.hiroshima-u.ac.jp/kanatani/