

Integrated Volatility Measuring from Unevenly Sampled Observations

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Introduction

■ DGP:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t), \quad 0 \leq t \leq T$$

■ unevenly sampled: $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$

■ unevenly sampled observations $\{p(t_i)\}_{i=0}^N \Rightarrow$ integrated volatility

$$\int_0^T \sigma^2(t)dt$$

■ $\max_i \{t_i - t_{i-1}\} \rightarrow 0$ as $N \rightarrow \infty$

Linear Interpolation

$$\blacksquare \{p(t_i)\}_{i=0}^N \rightarrow \boxed{\text{Linear interpolation}} \rightarrow \{q(iT/M)\}_{i=0}^M$$

$$q\left(\frac{iT}{M}\right) = (1 - \rho_i)p(t_i^-) + \rho_i p(t_i^+)$$

where $\rho_i = ((iT/M) - t_i^-)/(t_i^+ - t_i^-)$.

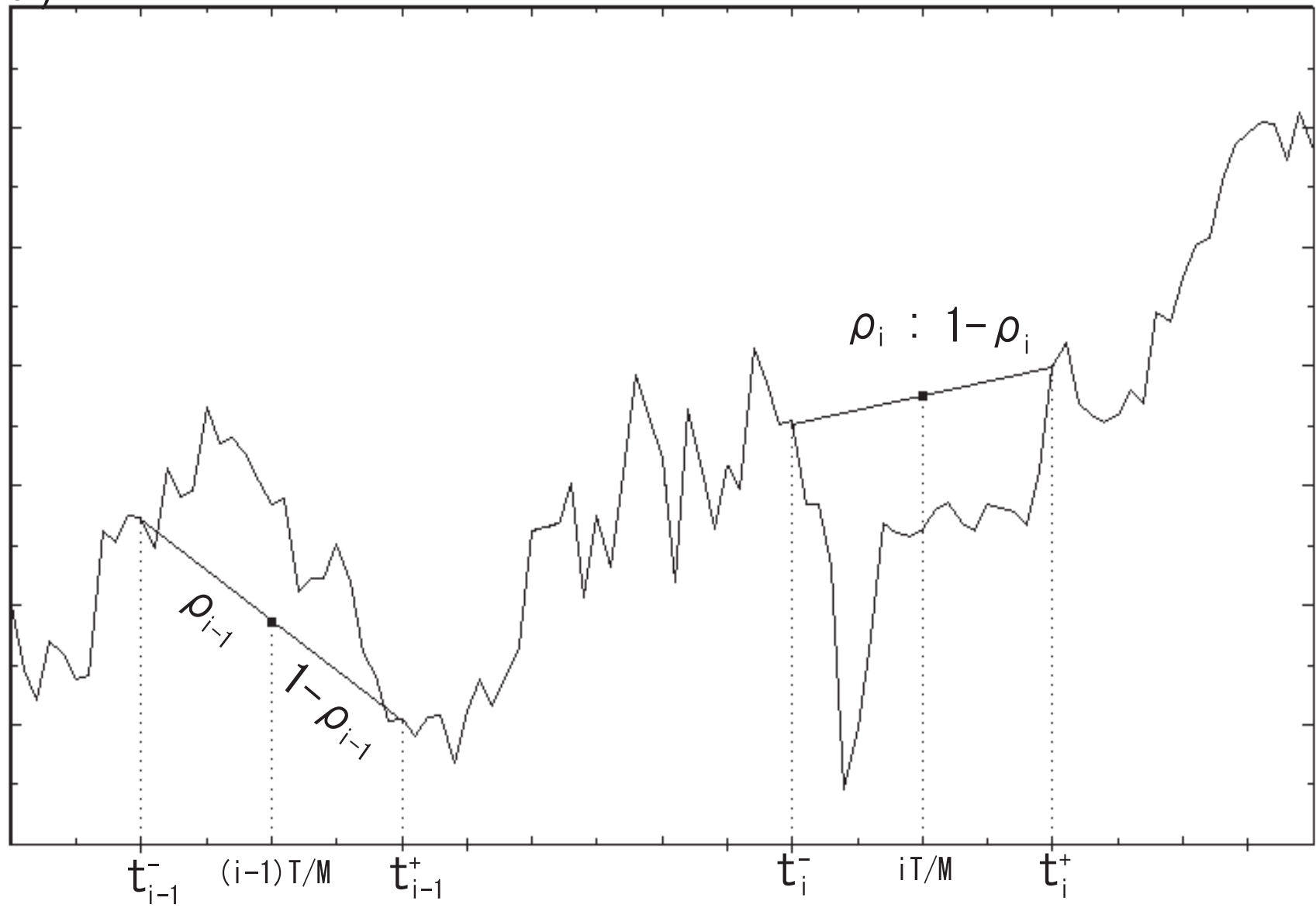
Linear interpolation Realized Volatility

$$\hat{\sigma}^2(M) = \sum_{i=1}^M \left\{ q\left(\frac{iT}{M}\right) - q\left(\frac{(i-1)T}{M}\right) \right\}^2$$

$$\blacksquare \text{bias} = E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t) dt:$$

$$\sum_{i=0}^M E \left[\left\{ q\left(\frac{iT}{M}\right) - q\left(\frac{(i-1)T}{M}\right) \right\}^2 \right] - \int_{(i-1)T/M}^{iT/M} \sigma^2(t) dt$$

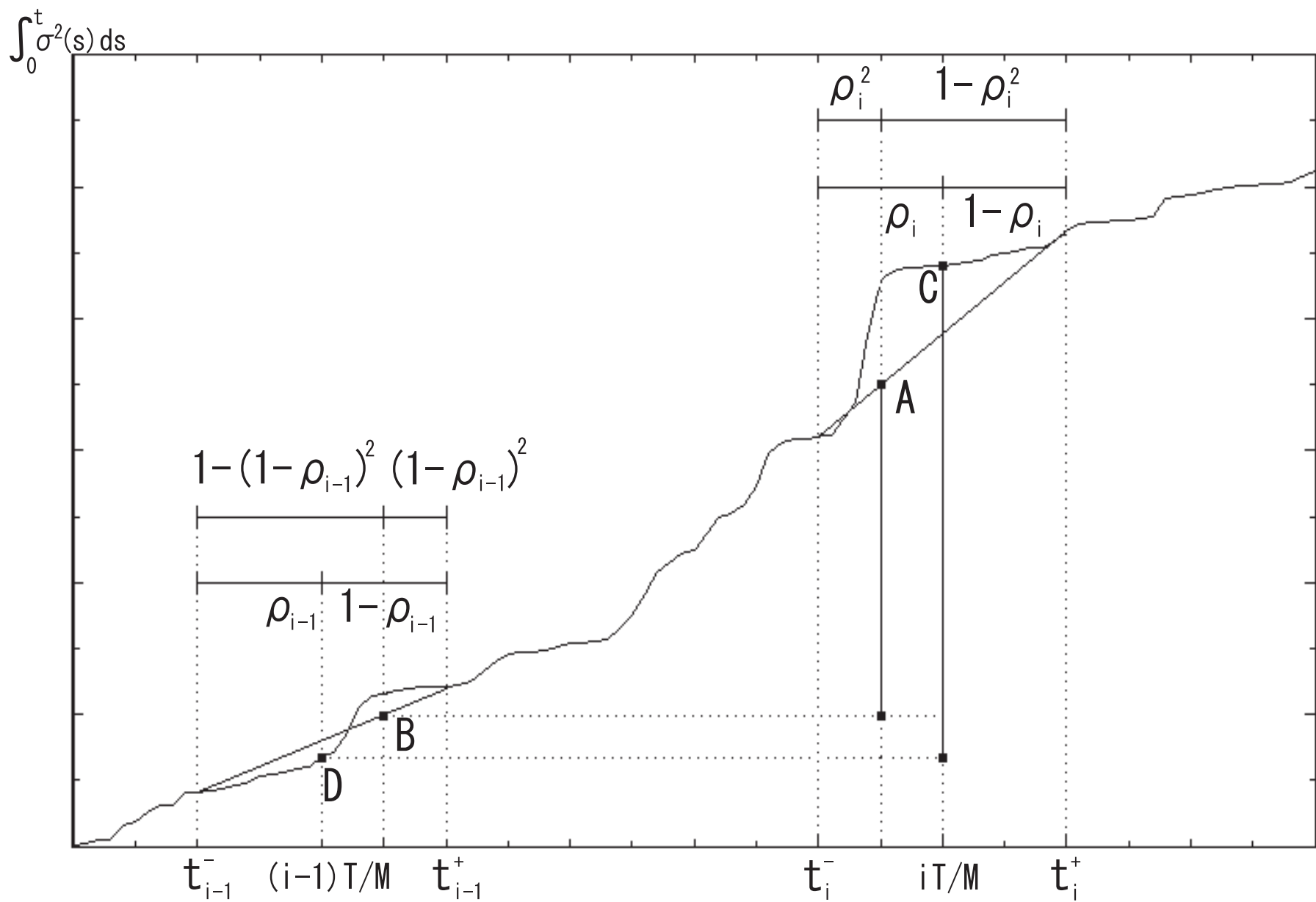
$p(t)$



Downward Bias

■ i th term of the bias $E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t)dt$

$$\begin{aligned}
 & E \left[\underbrace{\left\{ q \left(\frac{iT}{M} \right) - q \left(\frac{(i-1)T}{M} \right) \right\}^2}_A - \int_{(i-1)T/M}^{iT/M} \sigma^2(t)dt \right] \\
 &= \underbrace{\left[\rho_i^2 \int_0^{t_i^+} \sigma^2(t)dt + (1 - \rho_i^2) \int_0^{t_i^-} \sigma^2(t)dt \right]}_B \\
 & - \underbrace{\left[(1 - (1 - \rho_{i-1})^2) \int_0^{t_{i-1}^+} \sigma^2(t)dt + (1 - \rho_{i-1})^2 \int_0^{t_{i-1}^-} \sigma^2(t)dt \right]}_C \\
 & - \underbrace{\int_0^{iT/M} \sigma^2(t)dt}_D + \underbrace{\int_0^{(i-1)T/M} \sigma^2(t)dt}_D
 \end{aligned}$$



Downward Bias

■ $E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t)dt$:

$$(1) \quad - \sum_{i=1}^M \{ \rho_{i-1}(1 - \rho_{i-1})(t_{i-1}^+ - t_{i-1}^-) + \rho_i(1 - \rho_i)(t_i^+ - t_i^-) \} \\ \times \frac{M}{T} \int_{(i-1)T/M}^{iT/M} \sigma^2(t)dt$$

Fourier Series Estimator of Malliavin and Mancino (2002)

■ Fourier representation of $\sigma^2(t)$:

$$\sigma^2(t) = \sum_{k=0}^{\infty} a_k(\sigma^2) \cos(kt) + b_k(\sigma^2) \sin(kt)$$

where $a_k(\sigma^2)$ and $b_k(\sigma^2)$ are Fourier coefficients of $\sigma^2(t)$.

■ $\int_0^{2\pi} \sigma^2(t) dt = 2\pi a_0(\sigma^2)$

■ Malliavin and Mancino (2002)

$$a_0(\sigma^2) = \lim_{K \rightarrow \infty} \frac{\pi}{2(K+1-k_0)} \sum_{k=k_0}^K a_k^2(dp) + b_k^2(dp)$$

where $a_k(dp)$ and $b_k(dp)$ are Fourier coefficients of $dp(t)$.

■ $\hat{\sigma}_F^2 \equiv \{\pi^2 / (K+1-k_0)\} \sum_{k=k_0}^K a_k^2(dp) + b_k^2(dp)$.

Realized Volatility

$$\hat{\sigma}^2 = \sum_{i=1}^N (p(t_i) - p(t_{i-1}))^2$$

Conjecture 1.

$$\hat{\sigma}_F^2 - \hat{\sigma}^2 = O_p(N^{-1})$$

Conjecture 2.

$$V(\hat{\sigma}_F^2) = O_p(N^{-1}), \quad V(\hat{\sigma}^2) = O_p(N^{-1})$$

$$\blacksquare \hat{\sigma}_F^2 - \int \sigma^2(t)dt = O_p(N^{-1/2}), \quad \hat{\sigma}^2 - \int \sigma^2(t)dt = O_p(N^{-1/2})$$

$$1. \hat{\sigma}_F^2 - \hat{\sigma}^2 \rightarrow 0$$

$$2. \hat{\sigma}_F^2 \rightarrow \int \sigma^2(t)dt, \quad \hat{\sigma}^2 \rightarrow \int \sigma^2(t)dt$$

Monte Carlo Study

■ DGP:

$$d \log \sigma^2(t) = -\kappa \log \sigma^2(t) dt + \gamma dW(t)$$

$$dp(t) = \sigma(t) dW(t), \quad 0 \leq t \leq 60 \times 60 \times 24 \text{ seconds}$$

where $\kappa = 0.01, \gamma = 0.1$.

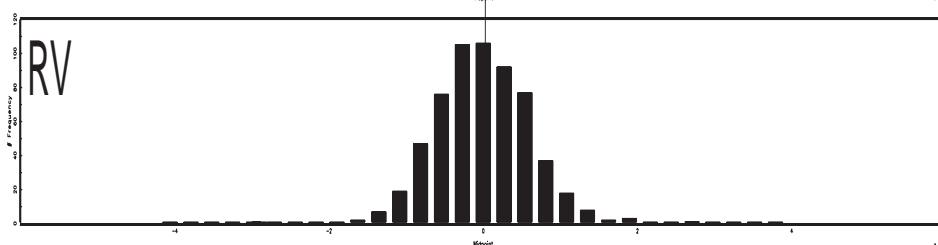
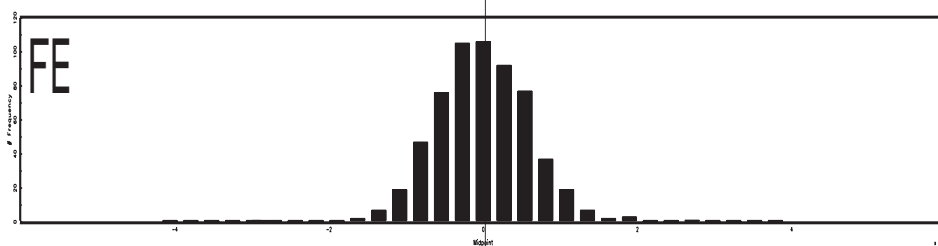
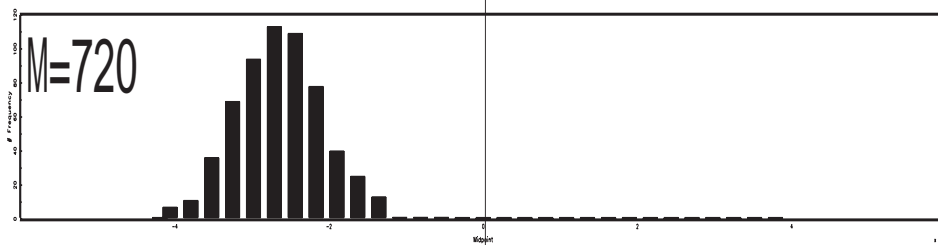
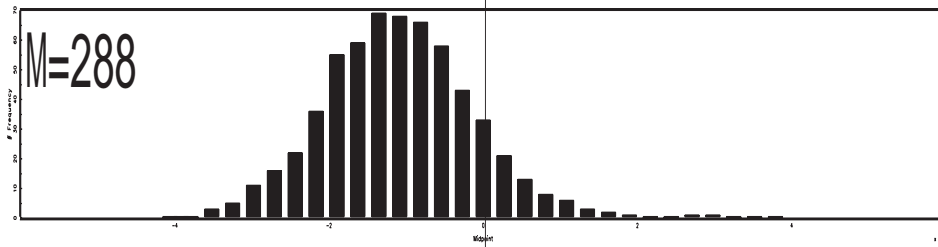
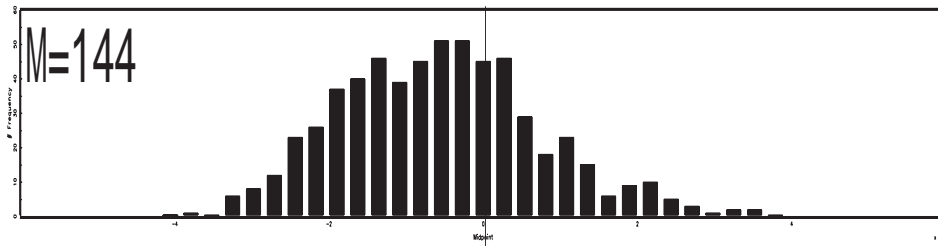
■ Time difference $t_i - t_{i-1}$ are drawn from an exponential distribution with mean equal to 45 seconds:^{*1}

$$(2) \quad F(t_i - t_{i-1}) = 1 - e^{-\lambda(t_i - t_{i-1})}$$

where $\lambda = 1/45$.

■ Distributions of $\hat{\sigma}^2(M) - \int \sigma^2(t) dt$ ($M = 144, 288, 720$), $\hat{\sigma}_F^2 - \int \sigma^2(t) dt$, and $\hat{\sigma}^2 - \int \sigma^2(t) dt$

^{*1} Barucci and Renò (2002); Aït-Sahalia and Mykland (2003); Engle and Russell (1998)



	-error	-bias (1)
$\hat{\sigma}^2(144)$	5684	5420
	(13000)	(502.0)
$\hat{\sigma}^2(288)$	10960	10900
	(9628)	(766.2)
$\hat{\sigma}^2(720)$	26480	27320
	(5762)	(1529)
$\hat{\sigma}_F^2$	2.830	
	(5954)	
$\hat{\sigma}^2$	2.864	
	(5954)	

computational time	
FE	1116.42 seconds
RV	0.25 seconds

Conclusion

- Linear interpolation bias of LRV
- Use RV rather than FE in the scalar case

Remaining Works

- Conjectures
- multivariate case

Acknowledgements

- Thank you very much for your attention.

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