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From block-spin expectation values to renormalized couplings

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We show that given a set of expectation values one can estimate the underlying probability distribution by use of "Schwinger-Dyson equations." The method can be applied to obtain the evolution of coupling constants in Monte Carlo renormalization-group calculations. Some results for three- and four-dimensional scalar theory are presented briefly.

This paper addresses the problem of reconstructing the probability distribution given the expectation values. It is a special case of the so-called moment problem. In many cases the probability measure is such that an infinite set of equations involving the expectation values must hold. We will refer to these equations as Schwinger-Dyson equations[1] following the name given to them in quantum field theory. The point we want to stress is how these equations allow good approximations to the probability distribution to be obtained.

In present day physics, this problem arises in the context of the renormalization group[2] (RG). Under a RG transformation the probability distribution, described by the Euclidean action (or Hamiltonian) S, changes. Since the original distribution is known, the expectation values with respect to the transformed action $\tilde{S}$ can be determined by numerical techniques. In particular, Monte Carlo methods[3] have been especially successful in this respect. Now, in order to obtain $\tilde{S}$ from these values, some form of solution of the moment problem must be applied. In some cases, the solution might not be unique. This is a disadvantage of this type of approach, which we will comment on later in the text.

Since Monte Carlo methods were originally applied in the study of the RG,[4] several authors have considered the problem and several solutions have been proposed.[5] Nevertheless, the method presented in this paper has a distinguished character of elegance, generality, and simplicity. Only one Monte Carlo simulation is required; the main equations are in most cases linear; the method can be applied to any lattice theory. These are, among others, the better advantages of our proposal.

Before entering into the description of our method we want to mention the usefulness of determining the evolution of the couplings under the RG. In fact, most analyses restrict themselves to the determination of the critical exponents, given their universal character. However, one must keep in mind that the basic assumptions of the RG concern the evolution of the couplings, from which the physical consequences, such as the universality of the critical exponents, follow. A rich structure of domains and different fixed points is, however, possible and must be studied. Furthermore, from a completely different standpoint of the lattice system as a regularized field theory, it has been shown[6] that one observes simple scaling behaviors of observables when one approaches the continuum limit (fixed point) by performing Monte Carlo simulations along the renormalized trajectories.

The action S describing the probability distribution $[\sim\exp(-S)]$ is parametrized as a linear combination of symmetric operators (translationally invariant and invariant under internal symmetries) O_i:

$$S=\sum_i \beta_i O_i \ . \tag{1}$$

The number of couplings β_i is, in general, infinite even for a finite problem. Determining S is equivalent to fixing the value of the couplings β_i. However, it is clear that any numerical method can only deal with a finite number of unknowns; thus, one must assume that all except for a finite number of couplings $(\beta_1,\ldots,\beta_N)$ are zero. This assumption is certainly wrong and, even if most of the couplings are "small," one must still be sure that an infinite number of small terms is negligible.

Our method, which we will now describe for the simplest case of a scalar field theory in a d-dimensional lattice, has the ability to give a precise mathematical meaning to these truncated actions. Later in this paper applications will be given, including the physically interesting cases of three- and four-dimensional scalar field theory.

Schwinger-Dyson equations are a consequence of the invariance properties of the integration measure. For lattice scalar fields (with no constraints) the measure is invariant under $\phi(x)\rightarrow\phi(x)+\varepsilon(x)$. Neither the action nor other operators are invariant and so their variation must compensate when averaging over the field. Consider, for example, the expectation value $\langle\phi(x_0)\rangle$ of the field at some point x_0 of the lattice. If we make a change of variables of the previous type with $\varepsilon(x_0)=\varepsilon$ and $\varepsilon(x)=0$ for $x\neq x_0$ we get

$$1=\left\langle \phi(x_0)\frac{\delta S}{\delta\phi(x_0)}\right\rangle \tag{2}$$

by equating to zero all terms linear in ε. Notice that Eq. (2) is linear in S and, thus, also in the couplings. We can sum on x_0 to obtain an equation involving translationally invariant quantities. If we choose the operators O_i to be monomials in the field as customarily done, we have

$$\dot{O}_i\equiv\sum_{x_0}\phi(x_0)\frac{\delta O_i}{\delta\phi(x_0)}=d_i O_i \ , \tag{3}$$

where we will refer to d_i as the dimension of the operator

O_i. Using (3), Eq. (2) takes the simple form

$$V=\sum_i \beta_i d_i \langle O_i \rangle \ , \tag{4}$$

where V is the lattice volume.

An infinite set of equations can be obtained by differentiating (4) with respect to β_i:

$$d_i \langle O_i \rangle = \sum_j \langle O_i O_j \rangle^c \beta_j d_j \ . \tag{5}$$

The superscript c stands for "connected part." The couplings are given by the solution of (5). This solution is unique if there exists no operator O such that $\langle O^2 \rangle^c = 0$.[7] In practice, however, one can consider only a finite number of couplings $(\beta_1, \ldots, \beta_N)$ and, correspondingly, only the first N equations of (5) are used. Let $\beta_1', \ldots, \beta_N'$ be the solution of these equations, and let

$$S' = \sum_{i=1}^{N} \beta_i' O_i \tag{6}$$

be the corresponding truncated action. In what sense is S' an approximation to the true action S? The answer is that S' is out of all truncated actions (with the first N operators) the one "closest" to the true action S. The meaning of "closest" is determined by the distance operation associated to the norm

$$||A||^2 \equiv \langle (\dot{A})^2 \rangle^c \ , \tag{7}$$

where $\dot{A}$ is obtained from A as in Eq. (3). In fact, minimizing $||S-S'||$ we obtain again Eq. (5), but restricted to N couplings and N equations. The value of $||S-S'||^2$ at the minimum is $\langle \dot{S} - \dot{S}' \rangle$. Notice that Eq. (4) can be rewritten as

$$\langle \dot{S} - \dot{S}' \rangle = 0 \ , \tag{8}$$

so that in the case that $S-S'$ has some definite dimension D, its value measures how far we sit from $||S-S'||^2 = D\langle \dot{S} - \dot{S}' \rangle = 0$.

It is important to notice that many more truncated approximations can be obtained from the Schwinger-Dyson equations. They are associated with the family of norms

$$||A||_M^2 = \left\langle \sum_x \sum_y M(x,y) \left(\frac{\delta A}{\delta \phi(x)} \right) \left(\frac{\delta A}{\delta \phi(y)} \right) \right\rangle \ , \tag{9}$$

where M is any positive symmetric functional of the field. In many cases these equations contain averages rather than connected parts and can be useful.

Before presenting physical results obtained using Eqs. (4) and (5) for three and four space-time dimensions, we want to present two simple cases requiring very little computer time. The first case is that of a single random variable $-\infty < x < +\infty$ (zero-dimensional field theory) and the second case is the one-dimensional Gaussian model.

For the first case, defining $\mu_n \equiv \langle O_n \rangle = \langle x^n \rangle$ we have

$$1 = V = \sum_n 2n \mu_{2n} \beta_{2n} \tag{10}$$

and

$$(2n+1)\mu_{2n} = \sum_m \mu_{2n+2m} 2m \beta_{2m} \ . \tag{11}$$

The first equation is the equivalent of Eq. (4) and the second one a combination of (4) and (5). The action is then approximated by a polynomial in x^2 of degree N. We have considered three different functions: (a) $S = \cosh x - 1$, (b) $S = x^4/(1+x^2)$, and (c) $S = x^{9/4}$, the first two being analytic at the origin with radius of convergence ∞ and 1, respectively. In the first case (a) the first coefficients of the Taylor expansion of the function are nicely reproduced. For $N=4$ the relative error in these coefficients is $\frac{1}{2500}$, $\frac{1}{300}$, $\frac{1}{25}$, and $\frac{1}{5}$, respectively. For the more singular cases (b) and (c) the results are presented in Fig. 1, showing good agreement in the region where the probability is non-negligible. These results are obtained by using only Eq. (11), but Eq. (10) is satisfied to within $2/10^5$, 3/100, and 3/1000 in case (a), (b), and (c), respectively, and $N=4$.

Now we turn to the case of the Gaussian model in one dimension. The partition function of this model is given by

$$Z = \int \prod_n d\phi_n \exp\left(-\sum_l \beta_l O_l \right) \ , \qquad O_l = \sum_n \phi_n \phi_{n+l} \ . \tag{12}$$

By Fourier analyzing Eqs. (4) and (5) we get

$$2\int \frac{d\theta}{2\pi} P(\theta) \hat{\beta}(\theta) = 1 \ , \tag{13a}$$

$$P(\theta) = 2P^2(\theta)\hat{\beta}(\theta), \tag{13b}$$

where $P(\theta)$ is the Fourier transform of the propagator $\langle \phi_0 \phi_l \rangle$, and $\hat{\beta}(\theta)$ the transform of β_l. A given truncation of the problem is

$$\hat{\beta}'(\theta) = \sum_{i=0}^{N-1} \cos(i\theta)\beta_i' \tag{14}$$

and, correspondingly, only the first N Fourier components of Eq. (13b) are used. It is easy to see, however, that the only simultaneous solution of (13a) and (13b) is the true solution $\hat{\beta}(\theta) = \sum_{i=0}^{\infty} \cos(i\theta)\beta_i'$ since all operators O_l have

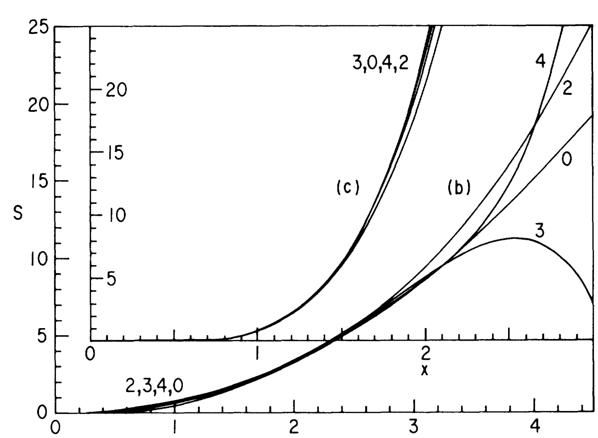

FIG. 1. Results of zero-dimensional field theory with the action (b) $S = x^4/(1+x^2)$, (c) $S = x^{9/4}$. In Figs. 1 and 2 the exact curves are indicated by the index 0, and the indices 2, 3, and 4 refer to the degree N of the approximation.

the definite dimension $D=2$ [see the discussion below Eq. (8)]. In Fig. 2 we show $\hat{\beta}'(\theta)$ for $N=2$, 3, 4 and $\hat{\beta}(\theta)=\theta^2+1$.

The last part of this paper is devoted to the application of our method to determine the evolution of coupling constants under the RG. We chose the following RG transformation:

$$\phi(\text{block})=\frac{\lambda}{2^d}\sum_{x\in\text{block}}\phi(x) \tag{15}$$

with blocks of 2^d lattice points. Only for one particular value of λ can one have a fixed point with finite expectation values.[8]

If we start with an action S, the transformation (15) defines a renormalized action $\tilde{S}$ by

$$\langle F(\phi(\text{block}))\rangle_S=\langle F(\phi)\rangle_{\tilde{S}} , \tag{16}$$

where F is any functional of the field and $\langle\,\rangle_S$ and $\langle\,\rangle_{\tilde{S}}$ are expectation values in the probability measure defined by S and $\tilde{S}$, respectively. Thus, the renormalized expectation values can be computed through the identity (16) and our inversion formulas (4) and (5) can be used to obtain approximations to $\tilde{S}$.

We applied the previous procedure to the cases of three- and four-dimensional scalar theory. The operators used in the analysis are displayed in Table I. In fact, in the three-dimensional case β_7 and β_8 were set equal to zero, and in the four-dimensional run β_5 and β_6 were set to zero. The original couplings were

$$\beta_1=-1,\ \beta_2=2.756,\ \beta_3=0.18 ,$$
$$\beta_1=-0.4276,\ \beta_2=-1,\ \beta_3=2 , \tag{17}$$

for three and four dimensions, respectively. In both cases they correspond to the vicinity of the critical surfaces. Then we follow the RG flows of the renormalized couplings within the critical surfaces. These flows should approach the fixed points of the corresponding theories.

Our results can be summarized concisely in terms of the

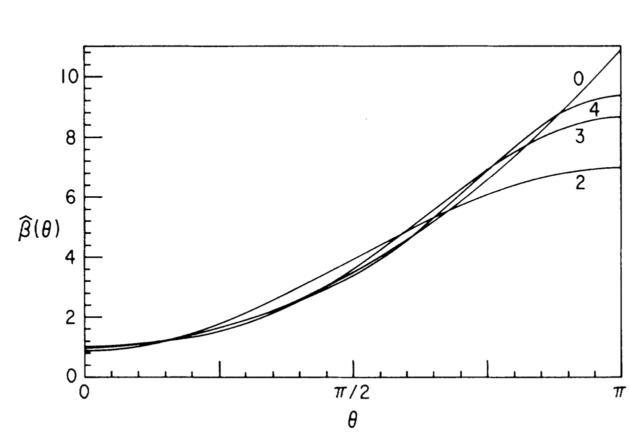

FIG. 2. Result for one-dimensional Gaussian model.

TABLE I. List of the operators used in the analysis of three- and four-dimensional scalar theory. $\hat{\mu}$ is the unit vector in μ direction and $\hat{\mu}\neq\hat{\nu}\neq\hat{\lambda}$.

O_1	$\sum_{x,\mu}\phi(x)\phi(x+\hat{\mu})$
O_2	$\sum_{x}\phi^2(x)$
O_3	$\sum_{x}\phi^4(x)$
O_4	$\sum_{x}\phi^6(x)$
O_5	$\sum_{x,\mu,\nu}\phi(x)\phi(x+\hat{\mu}+\hat{\nu})$
O_6	$\sum_{x,\mu,\nu,\lambda}\phi(x)\phi(x+\hat{\mu}+\hat{\nu}+\hat{\lambda})$
O_7	$\sum_{x,\mu}\phi(x)\phi(x+2\hat{\mu})$
O_8	$\sum_{x}\phi^8(x)$

ratios X_i:

$$X_2=M^2/\beta_1=\frac{d\beta_1+\beta_2+2\binom{d}{2}\beta_5+4\binom{d}{3}\beta_6+d\beta_7}{\beta_1} ;$$
$$X_i=\frac{\beta_i}{\beta_1^{(d_i/2)}},\ i=3\sim 8 , \tag{18}$$

where M^2 is the lattice squared mass and d is the dimensionality of the space time. These ratios X_i are independent of λ [formula (15)] and are thus free from the ambi-

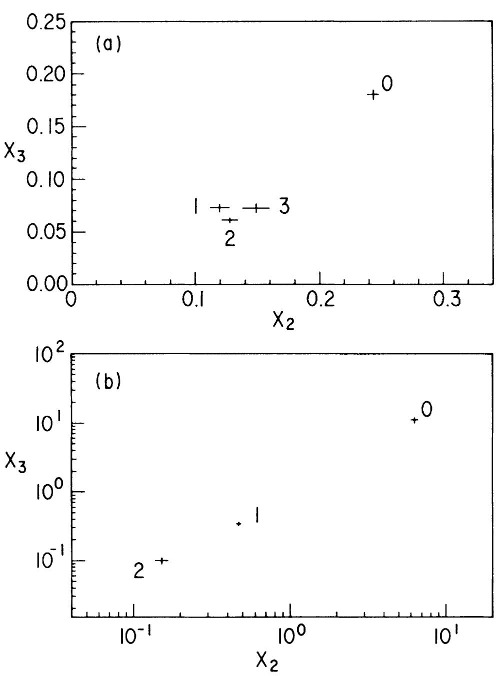

FIG. 3. Renormalized couplings (18) in (a) three-dimensional and (b) four-dimensional scalar theory. The indices 0–3 refer to the blocking level and the original lattice size is 32^3 and 16^4 in three and four dimensions, respectively.

guity of how to normalize the renormalized coupling constants. The locations of the fixed points are determined by following the RG flows of X_i. For example, in Fig. 3 results are shown for X_2 and X_3.

A more thorough analysis including several initial couplings, truncations, and lattice sizes, and a detailed study of errors and approximations will be presented elsewhere.[9] The characteristics of our Monte Carlo simulation and the renormalization of other couplings are to be found there as well. This more elaborate analysis, however, does not alter the main features shown in Fig. 3. In fact, our detailed study shows that in the three-dimensional case the RG flow approaches the nontrivial fixed point X_i^* given by

$$\begin{aligned} &X_2^* = 0.136 \pm 0.015,\; X_3^* = 0.064 \pm 0.006\;, \\ &X_4^* = -0.0004 \pm 0.0007,\; X_5^* = 0.082 \pm 0.006\;, \\ &X_6^* = 0.067 \pm 0.006\;, \end{aligned} \tag{19}$$

and in the four-dimensional case the flow approaches the trivial Gaussian fixed point $X_2^* = X_3^* = X_4^* = X_8^* = 0$.[10] Once the locations of fixed points are determined we can perform precise measurements of critical indices by running Monte Carlo simulations near the fixed points. This will also be studied in Ref. 9.[11]

To conclude, we mention that our procedure can be carried over for the case of gauge theories using the corresponding Schwinger-Dyson equations.[12]

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