

Field-strength description of non-Abelian gauge theories

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The method of dual transformation developed by Sugamoto is applied to the SU(2) pure Yang-Mills theory and the SO(3) Georgi-Glashow model. After the dual transformation, the partition functions are expressed only in terms of field strengths.

I. INTRODUCTION

In the past several years, many theorists have attempted to understand the quark-confining mechanism. On the one hand, Wilson¹ has shown that quarks are confined by electric flux in a lattice gauge model. On the other hand, Nambu² has shown that monopoles are confined by magnetic flux in the Nielsen-Olesen model,³ where the Higgs field conveys the electric charge. In the latter model, if the Higgs field were not electrically but magnetically charged and the roles of electric and magnetic fields were interchanged, electrically charged quarks would be confined by electric flux as in the Wilson model. At this time it is preferable to introduce the magnetically charged Higgs field as a topological operator rather than an elementary field. This idea is called Mandelstam-'t Hooft duality.^{4,5}

In order to embody this idea, many authors have applied the method of dual transformation to lattice gauge theories.⁶ The method of dual transformation was invented by Kramers and Wannier⁷ in the study of the two-dimensional Ising model. Though this method is easily applied to Abelian lattice gauge theories, the extensions to non-Abelian cases are not yet successful.

Recently Sugamoto⁸ has noticed that the dual transformation in lattice theories is a kind of Fourier transformation and he has developed a method of dual transformation in conventional gauge theories. The Abelian Higgs model with one magnetic vortex string of Nielsen and Olesen is dually transformed to the Kalb-Ramond⁹ and Nambu¹⁰ model of relativistic hydrodynamics with an external vorticity source. His method is easily extended to non-Abelian Higgs-Kibble models, which are dually related to the Freedman model,¹¹ a non-Abelian version of the Kalb-Ramond and Nambu model.¹² It may not be appropriate to call his method "dual transformation," in the light of its original meaning. If it becomes possible to treat the topological excitations as dynamical degrees of freedom in his framework, the program of dual transformation will be com-

plete. There have been several works¹³ along this line; in particular the work of Bardakci and Samuel is interesting.

In the present paper we apply the method of Sugamoto to non-Abelian gauge theories, where a complete or partial gauge symmetry is still retained. After the transformation we shall obtain variants of the Freedman model. In Sec. II, we consider the SU(2) pure Yang-Mills theory in the temporal gauge. Some properties of the Lagrangian obtained by the dual transformation will be examined. In Sec. III, the Georgi-Glashow model is considered. This model allows the existence of monopoles as classical solutions.¹⁴ Our concern in the present paper is directed only to the one-monopole sector. In Sec. IV, the meaning of our transformation is discussed and our result is compared to that of Halpern,¹⁵ who has formulated gauge theories only in terms of field strengths.

II. DUAL TRANSFORMATION IN THE PURE YANG-MILLS THEORY

The Lagrangian density for the SU(2) pure Yang-Mills theory is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2, \quad (2.1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc}A_\mu^b A_\nu^c.$$

For the sake of simplicity, we take the temporal gauge ($A_0^a = 0$). In this gauge the partition function is simply given by

$$Z = \int \mathcal{D}A_i^a \exp\left(i \int d^4x \mathcal{L}\right). \quad (2.2)$$

First, we perform the Fourier transformation for $F_{\mu\nu}^a$ in the integrand of (2.2) as

$$\exp\left\{i \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^a)^2\right]\right\} \propto \int \mathcal{D}W_{\mu\nu}^a \exp\left\{i \int d^4x \left[-\frac{1}{4}(W_{\mu\nu}^a)^2 + \frac{1}{2}\bar{W}_{\mu\nu}^a F^{\mu\nu,a}\right]\right\}, \quad (2.3)$$

where $W_{\mu\nu}^a$ is an antisymmetric tensor field, and $\bar{W}_{\mu\nu}^a$ is its dual tensor, namely

$$\bar{W}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} W^{\lambda\rho,a}. \quad (2.4)$$

Then the functional integration over A_i^a becomes the following Gaussian form:

$$\int \mathcal{D}A_i^a \exp\left[i \int d^4x \left(\frac{1}{2}g A^{i,a} K_{ij}^{ab} A^{j,b} - V^{i,a} A_i^a\right)\right], \quad (2.5)$$

where K_{ij}^{ab} and V_i^a are defined by

$$K_{ij}^{ab} = \epsilon_{abc} \bar{W}_{ij}^c \equiv \epsilon_{abc} \epsilon_{ijk} e^{k,c}, \quad (2.6)$$

$$V_i^a = \partial^\mu \bar{W}_{\mu i}^a. \quad (2.7)$$

After the integration, we obtain the final result

$$Z \propto \int \mathcal{D}W_{\mu\nu}^a (\det K_{ij}^{ab})^{-1/2} \exp\left(i \int d^4x \mathcal{L}^*\right), \quad (2.8)$$

where \mathcal{L}^* is given by

$$\mathcal{L}^* = -\frac{1}{2g} V^{i,a} M_{ij}^{ab} V^{j,b} - \frac{1}{4} (W_{\mu\nu}^a)^2. \quad (2.9)$$

In Eq. (2.9) M_{ij}^{ab} is the inverse matrix of K_{ij}^{ab} , and its explicit form and its determinant are given by

$$M_{ij}^{ab} = -\frac{1}{2 \det e} (2e_i^b e_j^a - e_i^a e_j^b), \quad (2.10)$$

$$\det M_{ij}^{ab} = (\det K_{ij}^{ab})^{-1} = [-2(\det e)^3]^{-1}, \quad (2.11)$$

with

$$\det e = \det \begin{pmatrix} e_1^1 & e_1^2 & e_1^3 \\ e_2^1 & e_2^2 & e_2^3 \\ e_3^1 & e_3^2 & e_3^3 \end{pmatrix}. \quad (2.12)$$

The formula (2.10) is proved in Appendix A.

The Lagrangian (2.9) was obtained by the same technique as in Ref. 12, and has a similar form to the one in Ref. 12. There is, however, an essential difference between them. In the previous case, the gauge symmetry is spontaneously broken and the inverse of the matrix K is well defined, meanwhile, in the present case, the gauge symmetry is not broken and the inverse of K is not well defined for the special field configurations satisfying $\det e = 0$. For such a configuration, the matrix K has some zero modes. In the functional integral (2.5), the integration for such a mode is not Gaussian, and it simply gives a δ function meaning that the component of V_i^a proportional to the zero mode should vanish. For such a special configuration, however, the expression (2.8) is formally valid, since a δ func-

tion can be rewritten as

$$\delta(x) = \lim_{\alpha \rightarrow 0} \left(\frac{i}{\pi\alpha}\right)^{1/2} \exp\left(-\frac{i}{2\alpha} x^2\right),$$

where α corresponds to an eigenvalue of K . For later convenience, we set

$$P_i^a \equiv M_{ij}^{ab} V^{j,b}. \quad (2.13)$$

As is shown in the following, this vector field P_i^a behaves like a gauge field.

In the temporal gauge, the Lagrangian (2.1) is still invariant under time-independent gauge transformations. Correspondingly, we consider the following infinitesimal transformation:

$$\delta W_{\mu\nu}^a = \epsilon_{abc} \omega^b W_{\mu\nu}^c, \quad (2.14)$$

where $\omega^a(x)$ is an infinitesimal, time-independent function. Noting that

$$\delta M_{ij}^{ab} = -M_{ik}^{ac} \delta K^{k,l,cd} M_{lj}^{db}, \quad (2.15)$$

we know the variations of the vector P_i^a and the Lagrangian density

$$\delta P_i^a = \epsilon_{abc} \omega^b P_i^c - \partial_i \omega^a, \quad (2.16)$$

$$\delta \mathcal{L}^* = \frac{1}{g} \partial^\mu (\omega^a V_\mu^a), \quad (2.17)$$

where V_μ^a is a natural extension of V_i^a and is given by

$$V_\mu^a = \partial^\nu \bar{W}_{\nu\mu}^a. \quad (2.18)$$

Thus the vector field P_i^a transforms like a gauge field and the action is invariant.

Next we consider the transformation introduced by Freedman,¹¹

$$\delta W_{\mu\nu}^a = (\nabla_\mu \Lambda_\nu)^a - (\nabla_\nu \Lambda_\mu)^a, \quad (2.19)$$

where Λ_μ^a is an arbitrary infinitesimal vector function, and ∇_μ^{ab} is an analog of the covariant derivative with P_μ^a ($P_0^a = 0$) regarded as a gauge field, that is,

$$\nabla_\mu^{ab} = \partial_\mu \delta_{ab} + \epsilon_{acb} P_\mu^c. \quad (2.20)$$

Under this transformation the first term of the Lagrangian (2.9) is invariant except for a total divergence

$$-\frac{1}{2g} \epsilon_{abc} \epsilon^{\mu ij\nu} \partial_\mu (P_i^a P_j^b \Lambda_\nu^c).$$

The second term of \mathcal{L}^* , however, breaks the symmetry explicitly.

Finally, the Euler-Lagrange equation is given as

$$\frac{1}{g}(\partial_\mu P_\nu^a - \partial_\nu P_\mu^a + \epsilon_{abc} P_\mu^b P_\nu^c) + \bar{W}_{\mu\nu}^a = 0. \quad (2.21)$$

This can be regarded as the definition of the field strength in terms of the gauge field. In addition we point out that the functional determinant in Eq. (2.8) is the necessary functional measure for the nonlinear Lagrangian (2.9).¹⁶

III. DUAL TRANSFORMATION IN THE GEORGI-GLASHOW MODEL

The Lagrangian density of the SO(3) Georgi-Glashow model is given by

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \frac{1}{2}(D_\mu\phi)^2 - V(\phi^2), \quad (3.1)$$

where ϕ^a is an isotriplet Higgs scalar, $(D_\mu\phi)^a$ is its covariant derivative, and $V(\phi^2)$ is a double-well potential. As is well known, this model has a classical solution representing an extended monopole.¹⁴ We pick up the one-monopole sector through a gauge-fixing condition. Any topologically trivial field configuration can be gauge transformed into such a configuration that the direction of the Higgs scalar is constant everywhere. Likewise any field configuration with unit topological quantum number is gauge equivalent to a configuration such that

$$\phi^a(x) = \frac{x^a}{|\mathbf{x}|} |\phi|. \quad (3.2)$$

Then we fix the gauge by imposing the condition (3.2) on the Higgs scalar field. There still remains a gauge degree of freedom, since the direction of the Higgs scalar is invariant under gauge rotations around the axis along the Higgs scalar. In order to get rid of this degree of freedom, we further impose the condition

$$\partial^\mu \left[\hat{\phi}^a A_\mu^a - \frac{i}{g} \text{Tr}(\partial_\mu \Omega \Omega^\dagger \tau^3) \right] = 0. \quad (3.3)$$

Here $\hat{\phi}^a$ denotes the unit vector along the Higgs scalar and Ω is a singular gauge transformation which brings the direction of the Higgs scalar into the one along the third axis everywhere, that is,

$$\Omega \hat{\phi}^a \tau^a \Omega^\dagger = \tau^3. \quad (3.4)$$

$$Z \propto \int |\phi|^2 \mathcal{D}|\phi| \mathcal{D}A_\mu^a \mathcal{D}B \mathcal{D}W_{\mu\nu}^a \exp \left\{ i \int d^4x \left[\frac{1}{2} g^2 A^{\mu,a} K_{\mu\nu}^{ab} A^{\nu,b} - \frac{1}{4} (W_{\mu\nu}^a)^2 - (V_\mu^a - g j_\mu^a) A^{\mu,a} - (V_\mu - \partial_\mu B) A^\mu + B \partial_\mu C^\mu + \frac{1}{2} (\partial_\mu \phi^a)^2 - V \right] \right\}, \quad (3.11)$$

where V_μ^a and j_μ^a are defined by

$$V_\mu^a = \partial^\nu \bar{W}_{\nu\mu}^a, \quad (3.12)$$

$$j_\mu^a = |\phi|^2 \epsilon_{abc} \hat{\phi}^b \partial_\mu \hat{\phi}^c. \quad (3.13)$$

The gauge field transformed by Ω is given by

$$A_\mu^{\prime a} \tau^a = \Omega A_\mu^a \tau^a \Omega^\dagger - \frac{2i}{g} \partial_\mu \Omega \Omega^\dagger. \quad (3.5)$$

In particular, the third component of the new gauge field takes the form

$$A_\mu^{\prime 3} = \hat{\phi}^a A_\mu^a - \frac{i}{g} \text{Tr}(\partial_\mu \Omega \Omega^\dagger \tau^3). \quad (3.6)$$

Thus the gauge condition (3.3) is nothing but the Lorentz condition for $A_\mu^{\prime 3}$.

We realize the gauge conditions (3.2) and (3.3) by inserting the following δ functions into the functional integral:

$$\delta(\cos\Theta - \cos\theta) \delta(\Phi - \varphi) \delta(\partial^\mu (\hat{\phi}^a A_\mu^a + C_\mu)), \quad (3.7)$$

where (Θ, Φ) and (θ, φ) are the polar coordinates of $\hat{\phi}^a$ and the spatial position, respectively, and C_μ is the second term of the right-hand side of Eq. (3.6). It is easily confirmed that the Faddeev-Popov determinant for (3.7) is constant. This fact is valid so far as we take, instead of Eq. (3.3), any gauge condition linearly depending on $\hat{\phi}^a A_\mu^a$.

Now the partition function for the one-monopole sector is given by

$$Z = \int |\phi|^2 \mathcal{D}|\phi| \mathcal{D}(\cos\Theta) \mathcal{D}\Phi \mathcal{D}A_\mu^a \delta(\cos\Theta - \cos\theta) \times \delta(\Phi - \varphi) \delta(\partial^\mu (\hat{\phi}^a A_\mu^a + C_\mu)) \times \exp \left(i \int d^4x \mathcal{L} \right). \quad (3.8)$$

Integrations over Θ and Φ are trivial and we get

$$Z \propto \int |\phi|^2 \mathcal{D}|\phi| \mathcal{D}A_\mu^a \mathcal{D}B \times \exp \left\{ i \int d^4x [\mathcal{L} + \partial^\mu (\hat{\phi}^a A_\mu^a + C_\mu) B] \right\}. \quad (3.9)$$

It is convenient to decompose any isovector A^a into components parallel and perpendicular to $\hat{\phi}^a$ as follows:

$$A^a = \hat{\phi}^a A + A^\perp. \quad (3.10)$$

Again, performing the Fourier transformation (2.3), we get

The coefficient matrix of the quadratic term in A_μ^a is given by

$$K_{\mu\nu}^{ab} = \mathbf{K}_{\mu\nu}^{ab} + \frac{1}{g} \hat{\mathbf{W}}_{\mu\nu}^{ab}, \quad (3.14)$$

with

$$\mathbf{K}_{\mu\nu}^{ab} = |\phi|^2 g_{\mu\nu} (\delta_{ab} - \hat{\phi}^a \hat{\phi}^b) + \frac{1}{g} \epsilon_{abc} \hat{\phi}^c \tilde{W}_{\mu\nu}, \quad (3.15)$$

$$\hat{\mathbf{W}}_{\mu\nu}^{ab} = \epsilon_{abc} \tilde{W}_{\mu\nu}^c. \quad (3.16)$$

If we direct $\hat{\phi}^a$ along the third axis, $K_{\mu\nu}^{ab}$ takes the form

$$K_{\mu\nu}^{ab} = \begin{bmatrix} |\phi|^2 g_{\mu\nu} & \frac{1}{g} \tilde{W}_{\mu\nu}^3 & -\frac{1}{g} \tilde{W}_{\mu\nu}^2 \\ -\frac{1}{g} \tilde{W}_{\mu\nu}^3 & |\phi|^2 g_{\mu\nu} & \frac{1}{g} \tilde{W}_{\mu\nu}^1 \\ \frac{1}{g} \tilde{W}_{\mu\nu}^2 & -\frac{1}{g} \tilde{W}_{\mu\nu}^1 & 0 \end{bmatrix}. \quad (3.17)$$

With the help of the relations in Appendix B, we can explicitly obtain the inverse of the submatrix $\mathbf{K}_{\mu\nu}^{ab}$ as follows:

$$\begin{aligned} \mathbf{K}_{\mu\nu}^{ab} \mathbf{M}^{\nu\lambda, bc} &= \delta_\mu^\lambda (\delta_{ac} - \hat{\phi}^a \hat{\phi}^c), \\ \mathbf{M}_{\mu\nu}^{ab} &= \frac{1}{|\phi|^2 h} \left[(\delta_{ab} - \hat{\phi}^a \hat{\phi}^b) \left(f_1 g_{\mu\nu} + \frac{1}{g^2 |\phi|^4} f_2 \tilde{W}_\mu^\lambda \tilde{W}_\nu \right) + \epsilon_{abc} \hat{\phi}^c \frac{1}{g |\phi|^2} (g_1 \tilde{W}_{\mu\nu} + g_2 W_{\mu\nu}) \right], \end{aligned} \quad (3.18)$$

with

$$\begin{aligned} f_1 &= 1 + \frac{1}{2g^2 |\phi|^4} (W_{\mu\nu})^2, \\ f_2 &= g_1 = -1, \\ g_2 &= -\frac{1}{4g^2 |\phi|^4} (W_{\mu\nu} \tilde{W}^{\mu\nu}), \\ h &= 1 + \frac{1}{2g^2 |\phi|^4} (W_{\mu\nu})^2 - \frac{1}{16g^4 |\phi|^8} (W_{\mu\nu} \tilde{W}^{\mu\nu})^2. \end{aligned} \quad (3.19)$$

Further if we know the inverse $G_{\mu\nu}$ of the matrix

$$G_{\mu\nu}^{-1} = \hat{\phi}^a \tilde{W}_{\mu\lambda}^a \mathbf{M}^{\lambda\rho, bc} \hat{\mathbf{W}}_{\rho\nu}^c \hat{\phi}^a, \quad (3.20)$$

the inverse $M_{\mu\nu}^{ab}$ of the whole matrix $K_{\mu\nu}^{ab}$ is known from

$$\begin{aligned} M_{\mu\nu}^{ab} &= \mathbf{M}_{\mu\nu}^{ab} - [(\mathbf{M} \hat{\mathbf{W}} \hat{\phi}) G (\hat{\phi} \hat{\mathbf{W}} \mathbf{M})]_{\mu\nu}^{ab} + g [\mathbf{M} \hat{\mathbf{W}} \hat{\phi} G \hat{\phi}]_{\mu\nu}^{ab} \\ &\quad + g [\hat{\phi} G \hat{\phi} \hat{\mathbf{W}} \mathbf{M}]_{\mu\nu}^{ab} - g^2 \hat{\phi}^a G_{\mu\nu} \hat{\phi}^b. \end{aligned} \quad (3.21)$$

Though we have not yet obtained the explicit form of $G_{\mu\nu}$, it is characteristic of theories with an unbroken gauge symmetry that M has no well-defined limit when all of the $W_{\mu\nu}^a$'s approach zero.

Formally integrating over A_μ^a in (3.11), we arrive at the following expression for the partition function:

$$\begin{aligned} Z &\propto \int |\phi|^2 \mathcal{D}|\phi| \mathcal{D}B \mathcal{D}W_{\mu\nu}^a (\det K_{\mu\nu}^{ab})^{-1/2} \\ &\quad \times \exp \left(i \int d^4x \mathcal{L}^* \right), \end{aligned} \quad (3.22)$$

where \mathcal{L}^* is given by

$$\begin{aligned} \mathcal{L}^* &= -\frac{1}{2g^2} [(\mathbf{V}^{\mu, a} - g j^{\mu, a}) + \hat{\phi}^a (V^\mu - \partial^\mu B)] M_{\mu\nu}^{ab} \begin{pmatrix} \mu & \nu \\ a & b \end{pmatrix} \\ &\quad - \frac{1}{4} (W_{\mu\nu}^a)^2 + B \partial_\mu C^\mu + \frac{1}{2} (\partial_\mu \phi^a)^2 - V(\phi^2). \end{aligned} \quad (3.23)$$

This Lagrangian involves time derivatives of \tilde{W}_{0i} , \tilde{W}_{0i}^a , and B ; derivatives of \tilde{W}_{0i} and \tilde{W}_{0i}^a appear linearly in V_i and V_i^a , respectively. Since M is a function of fields, (3.23) is a typical nonlinear Lagrangian. For such a nonlinear Lagrangian the square root of the determinant of the coefficient matrix of the kinetic part is needed as a path-integral measure.¹⁶ As before, if we direct $\hat{\phi}^a$ along the third axis, the coefficient matrix takes the form

$$\begin{bmatrix} M_{ij}^{11} & M_{ij}^{12} & M_{i\nu}^{13} \\ M_{ij}^{21} & M_{ij}^{22} & M_{i\nu}^{23} \\ M_{\mu j}^{31} & M_{\mu j}^{32} & M_{\mu\nu}^{33} \end{bmatrix}. \quad (3.24)$$

The determinant of this matrix is related to that of the whole matrix $M_{\mu\nu}^{ab}$ through Jacobi's formula, that is,

$$\begin{aligned} \det \text{ of the matrix (3.24)} &= \det M_{\mu\nu}^{ab} \det \begin{bmatrix} K_{00}^{11} & K_{00}^{12} \\ K_{00}^{21} & K_{00}^{22} \end{bmatrix} \\ &= |\phi|^4 \det M_{\mu\nu}^{ab}. \end{aligned} \quad (3.25)$$

Thus, the functional measure in Eq. (3.22) coincides with the square root of (3.25), which is necessary for the nonlinear Lagrangian (3.23).

IV. DISCUSSION

Using the technique developed by Sugamoto, we have rewritten the partition functions described by gauge fields and have expressed them in terms of antisymmetric tensor fields. In Ref. 8, Sugamoto has suggested that this transformation is analogous to the one in quantum mechanics from

the coordinate representation to the momentum representation. What we have done is in fact the integration over the coordinate q instead of the momentum p in the canonical form of the partition function

$$Z \propto \int \mathcal{D}q \mathcal{D}p \exp \left\{ i \int dt [p\dot{q} - H(p, q)] \right\}. \quad (4.1)$$

Let us examine the Fourier transformation (2.3) more closely. If we introduce only \bar{W}_{oi}^a at first, the partition function (2.2) takes the form

$$Z \propto \int \mathcal{D}A_i^a \mathcal{D}\bar{W}_{oi}^a \exp \left\{ i \int d^4x [F^{oi, a} \bar{W}_{oi}^a + \frac{1}{2} \bar{W}_{oi}^a - \frac{1}{4} (F_{ij}^a)^2] \right\} = \int \mathcal{D}A_i^a \mathcal{D}\bar{W}_{oi}^a \exp \left\{ i \int d^4x [A_i^a \bar{W}_{oi}^a - \frac{1}{2} (\bar{W}_{oi}^a)^2 - \frac{1}{4} (F_{ij}^a)^2] \right\}. \quad (4.2)$$

Thus \bar{W}_{oi}^a is nothing but the electric field canonically conjugate to A_i^a . Meanwhile, \bar{W}_{ij}^a is introduced as an auxiliary field Fourier conjugate to the magnetic field F_{ij}^a ; expectation values of them coincide with each other.^{8,12} Therefore the new expression (2.8) for the partition function can be considered as the description of a gauge theory in terms of field strengths only. It is, however, interesting to notice that the Lagrangian thus obtained at least in the broken-symmetry case takes the usual form consisting of kinetic energy and potential energy, and that it has been considered by Kalb, Ramond, Nambu, and Freedman in the study of interstring interactions.

Recently Halpern¹⁵ has proposed a field-strength formulation of gauge theories. By fixing the gauge completely, gauge fields are uniquely solved in terms of field strengths. Changing the variables from gauge fields to field strengths, he has obtained the partition function described by field strengths. In his formulation the integration over field strengths is restricted to the configurations satisfying the Bianchi identities. In our formulation, $W_{\mu\nu}^a$ is introduced only through a Fourier transformation and is independent of A_i^a . Therefore, $W_{\mu\nu}^a$'s as integration variables do not satisfy the Bianchi identities, but the equation of motion for the $W_{\mu\nu}^a$'s is the usual definition of field strength in terms of gauge fields. Thus the Bianchi identities for the $W_{\mu\nu}^a$'s are valid only in the sense of expectation values.

In the previous and present works, we have considered models involving topological excitations. Our consideration, however, has been restricted to the sector with one fixed topological excitation. In Ref. 13, Bardakci and Samuel have attempted to extract the motion of topological excitations as that of singular points of a gauge function, which

yields a singular gauge transformation from a topologically nontrivial configuration to a trivial one. Further, they have tried to rewrite the Feynman path integral of these point particles into the usual field-theoretic functional integral. For this aim our transformation will offer a useful aid. In fact the Lagrangian (4.20b) of Ref. 13 is nothing but the Lagrangian (2.15) of Ref. 8 presented by Sugamoto, except that the latter involves no soliton fields and its spatial dimension is not two but three.

Note added. After completing this work we learned that a similar formulation was developed in the earlier works of Halpern.¹⁷ In the case of the pure Yang-Mills theory, we have chosen the temporal gauge, so that the inverse matrix M is explicitly obtained. This result is new, and the relevance of the transformation (2.19) to the Lagrangian (2.9) is first pointed out in the present paper. We thank Professor M. B. Halpern for calling our attention to his works.

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APPENDIX A: THE INVERSE OF K_{ij}^{Φ}

Equation (2.6) can be solved for e_i^a :

$$e_i^a = \frac{1}{4} \epsilon_{ijk} \epsilon_{abc} K_{jk}^{bc}. \quad (A1)$$

The determinant (2.12) is transformed as

$$\begin{aligned}
\text{dete} &= \frac{1}{3!} \epsilon_{ijk} \epsilon^{abc} e_i^a e_j^b e_k^c \\
&= \frac{1}{3!} \epsilon_{ijk} \epsilon^{abc} \frac{1}{4} \epsilon_{ilm} \epsilon^{ade} K_{ln}^{de} e_j^b e_k^c \\
&= \frac{1}{3!} K_{ij}^{ab} e_i^a e_j^b, \tag{A2}
\end{aligned}$$

where use has been made of the formula

$$\begin{aligned}
\epsilon_{abc} \epsilon_{def} &= \delta_{ad} (\delta_{be} \delta_{cf} - \delta_{bf} \delta_{ce}) \\
&\quad + \text{cyclic permutations of } (d, e, f). \tag{A3}
\end{aligned}$$

Using formula (A3) repeatedly, we can prove

$$e_i^a K_{jk}^{ab} e_k^b = 2\delta_{ij} \text{dete}. \tag{A4}$$

Further use of (A3) yields the relation

$$K_{ij}^{ab} e_j^c e_k^b = -\delta_{ac} e_k^b K_{ii}^{bd} e_i^d - K_{ij}^{ab} e_j^c e_k^b + K_{ij}^{ab} e_j^c e_k^b. \tag{A5}$$

Using (A4), we get

$$K_{ij}^{ab} (2e_j^c e_k^b - e_j^b e_k^c) = -2\delta_{ac} \delta_{ij} \text{dete}. \tag{A6}$$

Thus Eq. (2.10) has been proved. As for Eq. (2.11)

we have no formal proof. We have calculated the determinant explicitly in a special frame of the isospace, where e_3^a is along the third axis and e_2^a lies in the 1-3 plane of the isospace.¹⁸

APPENDIX B: SEVERAL FORMULAS ABOUT $W_{\mu\nu}$

With the help of the formula

$$\begin{aligned}
\epsilon_{\mu\nu\lambda\rho} \epsilon^{\mu\sigma\tau\epsilon} &= -[\delta_\nu^\sigma (\delta_\lambda^\tau \delta_\rho^\epsilon - \delta_\lambda^\epsilon \delta_\rho^\tau) \\
&\quad + \text{cyclic permutations of } (\sigma\tau\epsilon)], \tag{B1}
\end{aligned}$$

we can prove the following relations:

$$W_{\mu\nu} \bar{W}^{\nu\lambda} = -\frac{1}{4} \delta_\mu^\lambda W_{\alpha\beta} \bar{W}^{\alpha\beta}, \tag{B2}$$

$$\text{det} W_{\mu\nu} = \frac{1}{16} (W_{\mu\nu} \bar{W}^{\mu\nu})^2, \tag{B3}$$

$$W_{\mu\nu} W^{\nu\lambda} - \bar{W}_{\mu\nu} \bar{W}^{\nu\lambda} = \frac{1}{2} \delta_\mu^\lambda \bar{W}_{\alpha\beta} \bar{W}^{\alpha\beta}. \tag{B4}$$

The relation (B4) together with (B2) allows only three independent tensors of rank 2 constructed from $W_{\mu\nu}$, namely $\bar{W}_{\mu\nu}$, $W_{\mu\lambda} W^\lambda_\nu$ (or $\bar{W}_{\mu\lambda} \bar{W}^\lambda_\nu$), and itself.

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