Chiral property of domain-wall fermion from eigenvalues of 4D Wilson-Dirac Operator *

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We investigate a chiral property of the domain-wall fermion (DWF) system using the four-dimensional hermitian Wilson-Dirac operator H_W. A formula expressing the Ward-Takahashi identity quark mass m_{5q} with eigenvalues of this operator is derived, which well explains the N_5 dependence of m_{5q} observed in previous numerical simulations. We further discuss the chiral property of DWF in the large volume in terms of the spectra of H_W.

1. Introduction

Recently a suitable characterization of chiral symmetry on the lattice has appeared in the form of the Ginsberg-Wilson relation [1]. A number of numerical analysis have been carried out for the domain wall fermion (DWF) [2], which is one of the simplest solutions of this relation [3,4]. A chiral property of DWF is investigated numerically in two ways; using either the chiral symmetry breaking Ward-Takahashi identity mass m_{5q} [5,6] or the zero-eigenvalue density $\rho(0)$ of the four-dimensional hermitian Wilson-Dirac operator H_W [7,8]. The latter is also related to the parity-flavor breaking order parameter [9,10]. However, conclusions are contradictory whether chiral symmetry is realized in DWF.

In this article we derive a formula which expresses m_{5q} in terms of the eigenvalues of H_W, which helps to resolve the mutual inconsistency. We further discuss the chiral property of DWF, investigating the lattice volume dependence of m_{5q} in this formula.

2. Formula

The anomalous quark mass m_{5q} is defined through a Green function of the chiral symmetry breaking term J_{5q} [2] of the Ward-Takahashi identity:

$$m_{5q} = \lim_{t \to \infty} \frac{\sum_{\vec{x}} \langle J_{5q}(t,\vec{x})P(0)\rangle}{\sum_{\vec{x}} \langle P(t,\vec{x})P(0)\rangle}, \qquad (1)$$

where $P(x)$ is the pseudo scalar density.

We try to express m_{5q} in terms of the eigenvalues of H_W. For this purpose we expand the Green function by inserting $1 = \sum |n\rangle \langle n|$ where $|n\rangle$ is the eigenstate of a deformed Hamiltonian $\widetilde{H}$ defined by

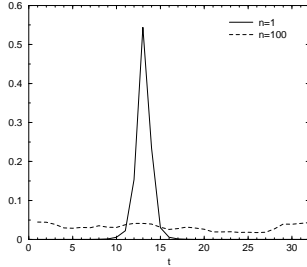$$\widetilde{H} = \log\frac{1+H'}{1-H'}, \quad H' = H_W \frac{1}{2+\gamma_5 H_W}. \quad (2)$$

Figure 1. $\sum_{\vec{x}} |\langle x|n\rangle|^2$ as a function of t.

To simplify the resulting expression, we notice that the eigenvalues of H_W or $\widetilde{H}$ are classified into isolated eigenvalues distributed in the lower region and almost continuous ones in the upper region. As is seen in Fig. 1 the eigenfunctions for the isolated eigenvalues are localized exponentially around a center and higher excited states with continuous eigenvalues are almost plane waves, rapidly oscillating before absolute square is taken.

We adopt an exponential localization for the low modes and a plane wave function for the excited modes as an approximation. The formula for m_{5q} then becomes [11]

$$m_{5q} = \frac{1}{12V}\left(\sum_{\text{local}} \tilde{h}\left(\frac{1}{2\cosh\frac{N_5}{2}\widetilde{\lambda}}\right)^2 + \sum_{\text{continuous}}\left(\frac{1}{2\cosh\frac{N_5}{2}\widetilde{\lambda}}\right)^2\right), \quad (3)$$

with V the space-time volume, $\widetilde{\lambda}$ an eigenvalue of $\widetilde{H}$ and $\tilde{h}$ being related to the exponential width δ of the localized eigenfunctions as $\tilde{h} = (4\delta)^4$.

3. Numerical results

In Figs. 2 and 3 we compare m_{5q} obtained with (3) (lines) with those directly calculated in simulations, both for the plaquette and the RG improved gauge actions, where circles and squares are from [5,6] and triangles are from this work.

The right hand side of (3) is calculated as follows. We evaluate eigenvalues of the Wilson Dirac operator H_W using the Lanczos method. The eigenvalue $\widetilde{\lambda}$ of the deformed Hamiltonian is evaluated from that of H_W perturbatively for the iso-
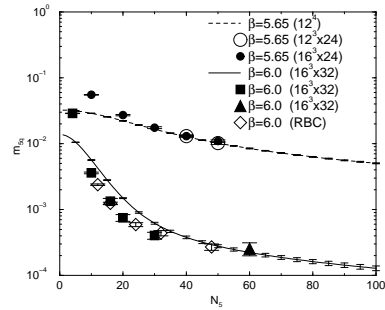

Figure 2. m_{5q} as function of N_5 for the plaquette action. Lines are our formula. RBC is a data from [6].
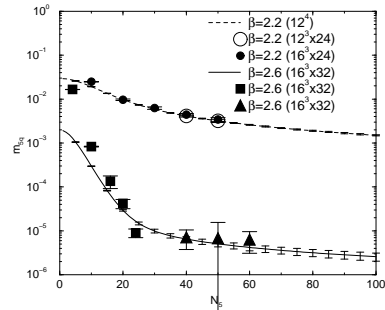

Figure 3. m_{5q} as function of N_5 for the RG action.

lated eigenvalues and directly for the continuous ones assuming the free theory relation. The separation between the isolated and the continuous eigenvalues is made by inspection of the eigenvalue distribution, and $\tilde{h}$ is treated as a free parameter to fit the data. For example, at $\beta = 2.6$ of the RG action, we adopt the number of localized eigenvectors $n_l = 20$, and $\tilde{h} = 200$. Summing up contributions from each eigenvalue we obtain m_{5q} as a function of N_5. Errors are calculated by a single elimination jackknife procedure for 100 configurations.

As is seen in these figures our formula well explains the N_5 dependence of the anomalous quark mass both in strong and weak coupling regions for the two gauge actions. An important problem in the previous numerical investigations is whether m_{5q} contains a constant term in the large N_5 limit. Now we can conclude that the exponential dumping still holds even at large N_5 but with a small decay rate, which reflects the magnitude of a few small eigenvalues. This is a reasonable
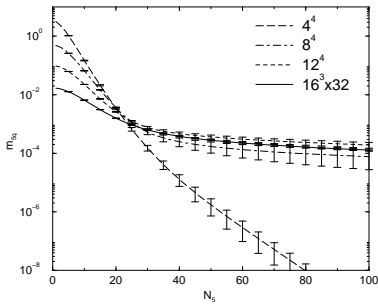
Figure 4. Volume dependence of m_{5q} for the plaquette action at $\beta = 6.0$.

Figure 5. Volume dependence of m_{5q} for the RG action at $\beta = 2.6$.

behavior for finite lattice volume where exact zero eigenvalues are absent.

A bending behavior of m_{5q} for weak coupling around $N_5 \sim 20$ can be explained with our formula as follows. For small N_5 the continuous modes, which dominate in number, mainly contribute in (3), giving a steep exponential decay. For large N_5, on the other hand, only a few of the small isolated eigenvalues contribute, so that the dumping rate of m_{5q} becomes small.

4. Volume dependence of m_{5q}

Although an exponential decay of m_{5q} in N_5 seems to hold in DWF at finite volume, there remains a crucial question whether this remains so, and hence chiral symmetry is exactly realized with DWF, in the infinite volume limit. Indeed m_{5q} decreases with some power of $1/N_5$ if the number of small eigenvalues increases linearly with the volume V [11], in which case chiral symmetry is not realized.

To examine this question we plot the volume dependence of m_{5q} at a weak coupling in Figs. 4 and 5, using the formula (3) with n_l and $\tilde{h}$ obtained in the previous section. Almost no volume dependence is observed for $V \geq 12^4$ at large N_5 for both actions, and a similar behavior holds also in the strong coupling region.

It is difficult, however, to distinguish a slow exponential decay seen in Figs. 4 and 5 from a power of $1/N_5$ at $N_5 \sim 100$. A distinction becomes possible only for $N_5 \gg 100$, which effectively corresponds to $N_5 \to \infty$ for the current lattice volumes, $V \simeq 12^4$ – 24^4, so data at much
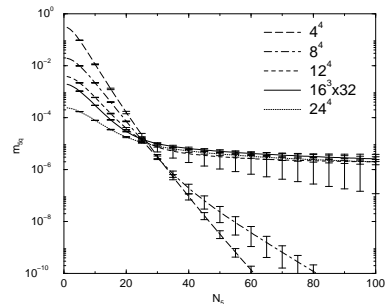
larger volumes are needed to establish the stability of an exponential decay against the increase of volume. Therefore the current data do not allow definite conclusions on the chiral property of DWF in the infinite volume limit. Further investigations will be required to answer the question.

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