

## An example of convex heptagon with Heesch number one

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**Abstract:** We give an example of convex heptagon whose Heesch number is just equal to one, and among fourteen kinds of edge-to-edge coronas of this tile we present some of them, one of which admits a family of continuous deformations.

**Key words:** monohedral tiling, convex heptagon, corona, Heesch number

1. In this paper we consider edge-to-edge monohedral tilings of a region of the Euclidean plane by convex polygons. A tiling is called *edge-to-edge* if no vertex of any polygon lies on the interior of an edge of any other polygon, and is called *monohedral* if the region of the plane is tiled by congruent or reflected copies of a prototile  $T$ . For definitions of several fundamental concepts on tilings, see [18], [29].

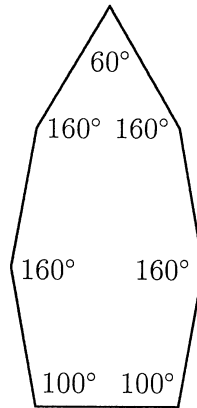
It is well known that any triangle, or any quadrangle including the non-convex one can tile the whole plane monohedrally, but any convex  $n$ -gon with  $n \geq 7$  does not admit a monohedral tiling. As for the hexagon, Reinhardt [25] showed that there are three classes of convex hexagons that can tile the plane monohedrally, and as for the pentagon, we now know 14 classes of convex pentagon admitting a monohedral tiling. But the classification for the pentagonal case is not completed yet. For the detailed results and the complicated history of the classification of monohedral tilings by convex polygons, see the references listed up at the end of this paper. The present situation is extensively summarized in the excellent book Grünbaum-Shephard [18; p.495, 517~518]. (See also [16], [17], [28], [32].) A similar problem for the spherical case is also interesting and is considered in the papers [6]~[10], [31], [33]~[35], etc.

2. On the other hand, concerning local tilings of the plane, the concept “Heesch number” was introduced in order to measure the ability of a tile to tile the plane to what extent. To explain this concept, we first consider a “corona” of a tile. A *corona* is a tiling around one tile  $T$  by congruent or reflected copies of  $T$  such that no point of  $T$  cannot be visible from the exterior in the plane, which satisfies the following minimal condition: If we drop one tile from this local tiling around  $T$ , then a part of  $T$  is visible from the exterior. Starting from one tile  $T$ , we construct a corona around  $T$ , and next we construct a second corona around the first corona, satisfying the same conditions as above, where we replace  $T$  with the first corona in this case. We continue to repeat this procedure until we cannot construct a new corona. Among several construction of

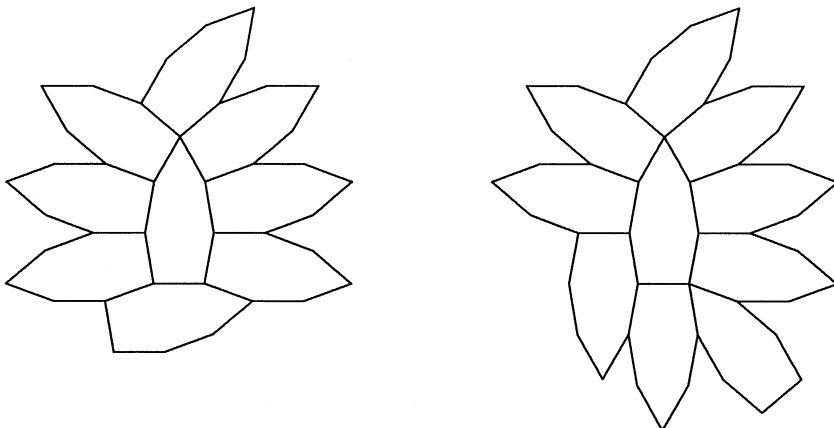
coronas the maximum number of layers is called the “Heesch number” of  $T$  and is denoted by  $H(T)$ . In case  $T$  admits a monohedral tiling of the plane we set  $H(T) = \infty$ . Of course for most  $n$ -gons with  $n \geq 5$  we have  $H(T) = 0$ . Several interesting examples and results on the Heesch number are given in Fontaine [14], Senechal [30], Raedschelders [24], Friedman [15], Mann [21], etc. Note that in considering the Heesch number, we usually ignore the conditions on a tile or a tiling such as “polygon”, “convexity”, or the property “edge-to-edge”.

As stated above, any convex heptagon  $T$  cannot tile the plane monohedrally, and so the Heesch number is finite for any convex heptagon. Keeping this fact in mind, Morimoto [22] offered a conjecture that  $H(T) = 0$  for any convex heptagon. Namely he conjectured that any convex heptagon  $T$  cannot be surrounded by congruent or reflected copies of  $T$ . The purpose of this paper is to give a counter-example to this conjecture. We give an example of a convex heptagon  $T$  with  $H(T) = 1$ . The results are summarized in the following form:

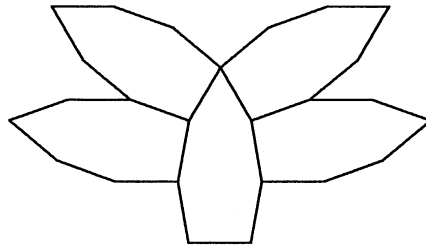
**Theorem.** *The Heesch number of the following equilateral convex heptagon (bamboo shoot) is one.*



*We can construct a corona of this tile exactly in fourteen ways up to Euclidean transformations, all of which are necessarily edge-to-edge. Among them we give two examples here:*

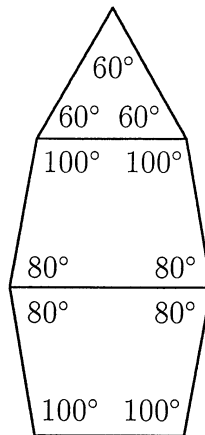


3. **Outline of the proof.** Detailed examinations of statements in Theorem are easy to verify, and we here only sketch an outline of the proof. First, it is an easy task to check that the above figures actually give coronas of the central heptagon. In addition, if a vertex of surrounding tiles lies on the interior of an edge of the central heptagon, then an exterior angle  $20^\circ$  or  $80^\circ$  necessarily appears in some place, and so we cannot construct a complete corona. Hence any first corona of this tile must be edge-to-edge. By an elementary argument we can also show that any coronas necessarily contain the following partial tiling:



And we can fill the upper exterior part of this figure essentially in a unique way up to an axial reflection. Also there are essentially eight ways to fill the lower exterior part, two of which are axially symmetric. By considering all combinations up and below, we know that there are exactly fourteen kinds of coronas up to Euclidean transformations. For all these coronas an angle  $40^\circ$  appears as an exterior angle, and so we cannot construct a second corona in any case, which shows that the Heesch number of this tile is exactly one.

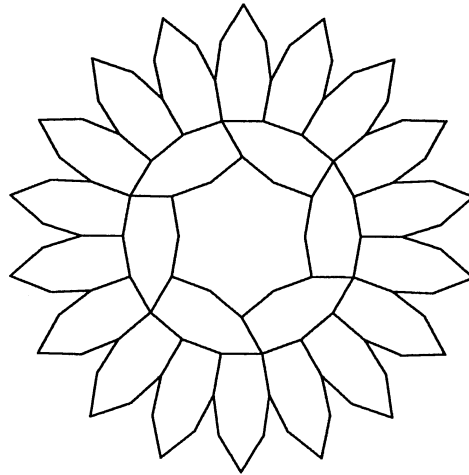
The most fundamental part of the proof is to ensure the existence of an equilateral convex heptagon satisfying the conditions on angles stated in Theorem, because angles and lengths of edges cannot be independently chosen. In general such an examination requires many calculations. But in this case we can easily see the existence by drawing two line segments in the tile as follows:



By this figure we know that the desired tile can be constructed from two congruent trapezoids and one equilateral triangle. And thus we obtain Theorem. q.e.d.

**Remark.** (1) We can construct several types of dihedral tilings of the whole plane by using the above heptagon and the rhombus whose adjacent angles are  $40^\circ$  and  $140^\circ$ .

(2) By patching six heptagons, we obtain a regular 18-gon in the plane, containing a star-shaped 12-gon in the interior. And by attaching 18 heptagons outside, we obtain a design of sunflower:



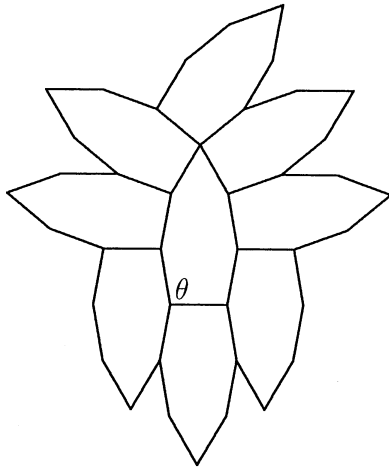
There may be other local tilings of the plane based on our heptagon.

4. There remain several important problems on the Heesch number of convex polygons. In the following we present some of them for further investigations:

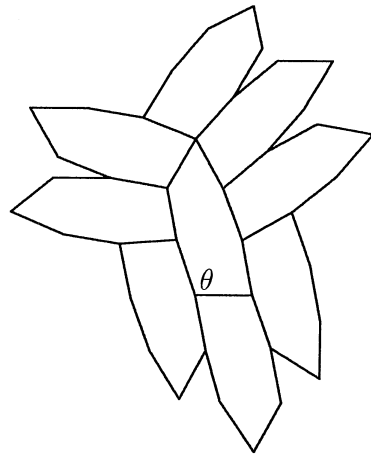
- Is there a convex heptagon  $T$  with  $H(T) \geq 2$ ?
- Classify convex heptagons  $T$  with  $H(T) \geq 1$ . Also classify their coronas, including non-edge-to-edge cases if they exist.
- Is there a convex octagon  $T$  with  $H(T) \geq 1$ ?
- Prove the finiteness of  $H(T)$  for convex  $n$ -gons  $T$  with  $n \geq 7$  directly, by using only local arguments. See the comments of Berger in [3; p.666~667].
- Is there a convex pentagon  $T$  with  $1 \leq H(T) < \infty$  which admits an edge-to-edge corona? (Note that the famous corona by Heesch's pentagon ([18; p.156]) is not edge-to-edge. And it is easy to show that this pentagon does not admit an edge-to-edge corona.)

• Consider similar problems not only in the Euclidean, but also in the affine, conformal or projective geometry, as Bowers-Stephenson [5] and Benoist [2] have done.

Note that, concerning the second problem, there are infinitely many such heptagons. In fact the corona exhibited left below can be continuously deformed to the right one:



$\theta = 100^\circ$



$\theta = 110^\circ$

The angles of this deformed equilateral heptagon are given by  $60^\circ$ ,  $360^\circ - 2\theta$ ,  $\theta + 60^\circ$ ,  $\theta$ ,  $300^\circ - 2\theta$ ,  $\theta + 60^\circ$ ,  $\theta + 60^\circ$  ( $90^\circ < \theta < 120^\circ$ ) counterclockwise from the top. The existence of such a heptagon is ensured by the same method as in Theorem. In this case the heptagon is constructed from one equilateral triangle and two trapezoids with interior angles  $300^\circ - 2\theta$ ,  $2\theta - 120^\circ$ ,  $180^\circ - \theta$ ,  $\theta$ . In case  $\theta = 120^\circ$  this heptagon degenerates to a trapezoid, and in case  $\theta = 90^\circ$  it reduces to a hexagon of Type 1, Type 2 listed in [18; p.494]. The case  $\theta = 110^\circ$  is interesting among this family of heptagons because a corona of special type appears in this case.

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