Improving the Availability of Mutual Exclusion Systems on Incomplete Networks

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Abstract—We model a distributed system by a graph $G = (V, E)$, where V represents the set of processes and E the set of bidirectional communication links between two processes. G may not be complete. A popular (distributed) mutual exclusion algorithm on G uses a coterie $\mathcal{C}(\subseteq 2^V)$, which is a nonempty set of nonempty subsets of V (called quorums) such that, for any two quorums $P, Q \in \mathcal{C}$, 1) $P \cap Q \neq \emptyset$ and 2) $P \not\subset Q$ hold. The availability is the probability that the algorithm tolerates process and/or link failures, given the probabilities that a process and a link, respectively, are operational. The availability depends on the coterie used in the algorithm. This paper proposes a method to improve the availability by transforming a given coterie.

Index Terms—Availability, coteries, distributed systems, G-nondominatedness, graph theory, mutual exclusion problems, quorums.

1 Introduction

Let V be a finite set. A nonempty set $\mathcal{C}$ $(\subseteq 2^V)$ of nonempty subsets of V (called **quorums**) is called a **coterie** under V if it satisfies 1) Intersection Property: For any $P, Q \in \mathcal{C}$, $P \cap Q \neq \emptyset$, and 2) Minimality: For any $P, Q \in \mathcal{C}$, $P \not\subset Q$.

The concept of a coterie was introduced by Garcia-Molina and Barbara [5] as an extension of the majority consensus [6], [16], and has been used in algorithms for many kinds of synchronization problems in distributed systems (see, e.g., [1], [3], [5], [6], [12], [15], [16]). Among them the (distributed) mutual exclusion problem is perhaps the most well-known. Suppose that a distributed system is modeled by a graph $G = (V, E)$, where V represents the set of processes and E the set of bidirectional communication links between two processes. Then, Maekawa's mutual exclusion algorithm uses a coterie under V [12]. Since its property varies depending on the coterie selected, many methods have been proposed to produce "good" coteries [7], [14], [15].

Given the probabilities that a vertex (i.e., a process) and an edge (i.e., a link), respectively, are operational, the **availability** $A_G(\mathcal{C})$ of coterie $\mathcal{C}$ is the probability that there remains a connected subgraph $G' = (V', E')$ of G consisting only of operational vertices and edges such that $Q \subseteq V'$ for some quorum $Q \in \mathcal{C}$. It is the probability that Maekawa's algorithm that uses $\mathcal{C}$ tolerates process and/or link errors and is considered to be one of the most important goodness measures [3], [15], [17]. However, the problem of computing the availability of a coterie is an extremely hard problem belonging to the class of #P-hard problems [2]. We therefore developed methods to search for an optimal coterie (in terms of the availability).

A powerful and useful concept is the nondominatedness [5]. A coterie $\mathcal{C}$ is said to **dominate** a coterie $\mathcal{D}$ if $\mathcal{C} \neq \mathcal{D}$ and, for any quorum $Q \in \mathcal{D}$, there is a quorum $P \in \mathcal{C}$ such that $P \subseteq Q$. A coterie $\mathcal{C}$ is said to be **nondominated** (ND) if no coterie dominates $\mathcal{C}$. By definition, if $\mathcal{C}$ dominates $\mathcal{D}$, then $A_G(\mathcal{C}) \geq A_G(\mathcal{D})$. Thus, searching for an ND coterie, instead of searching for an optimal one, might be a practical way of obtaining a better coterie. Unfortunately, the membership problem for ND coteries is a famous open problem which is probably co-NP-complete [9], although the ND coteries are beautifully characterizable in terms of the self-dual Boolean functions [9]. In this context, several efficient methods have been proposed for enumerating or constructing ND coteries [4], [5], [9], [13].

A difficulty of searching for an optimal coterie on a graph lies in the lack of an efficient algorithm for comparing two ND coteries (because it is unlikely that there is an efficient algorithm for calculating the availability of a coterie). To attack this problem, Ibaraki et al. introduced the concept of **graph-nondominatedness** (G-NDness) and identified the optimal coteries on rings and trees, using a characterization of G-ND coteries on rings and trees [11]. Recently Harada and Yamashita [8] presented a necessary and sufficient condition for a coterie on a general graph to be G-dominated, but, as expected, its test requires exponential time [8].

The following fact is important from the view of searching for an optimal coterie: If $\mathcal{C}$ G-dominates $\mathcal{D}$, then $A_G(\mathcal{C}) \geq A_G(\mathcal{D})$. On the other hand, on some graph G, coteries that are not ND can G-dominate some ND coteries and, in fact, there are many ND coteries that are not G-ND. Hence, enumerating or constructing G-ND coteries is a better approach (than enumerating or constructing just ND coteries). Motivated by this, as an enumeration and a construction of ND coteries are possible, in this paper, we focus on how to obtain a G-ND coterie from a given ND coterie.

We first present a necessary and sufficient condition for an ND coterie on a graph G to be G-ND. Surprisingly, the condition is testable in polynomial time and the G-NDness of ND coteries is therefore efficiently decidable. Although this test procedure may not determine the G-NDness of dominated coteries, by combining it with an enumeration or a construction method for ND coteries we can construct or enumerate G-ND coteries (that are ND) for a given graph G.

We next propose a method to improve the availability by modifying a given coterie to obtain a G-ND coterie. To this end, we introduce a polynomial time function **Replace** that produces, given a G-dominated coterie, a new coterie that G-dominates it. Since Replace preserves the NDness, a naive method that repeats applying Replace while the current ND coterie is G-dominated produces a G-ND coterie.

The idea of modifying a coterie so as to satisfy a desirable property is not our original. For complete graphs, coterie transformation algorithms have been proposed 1) for constructing a large ND coterie from simpler ones, 2) for enumerating ND coteries, and 3) for obtaining a new coterie with a better performance [4], [5], [9], [13]. In particular, the coterie transformation [5] and operator ρ [4] are similar to Replace function in that they are functions on the set of coteries that preserve the NDness. However (of course), they are not designed to map a given coterie to a new coterie that G-dominates the given one.

After preparing definitions, we characterize G-ND coteries that are ND in Section 2. Section 3 introduces Replace function and discusses its properties. Section 4 concludes the paper.

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2 Characterizing G-ND Coteries That Are ND

Without loss of generality, we assume that the underlying graph $G = (V, E)$ is undirected and connected. The following definitions are from [10], [11].

Definition 1. *Let $\mathcal{C}$ be a coterie on a graph $G = (V, E)$. The set of all connected minimal subgraphs $H = (V_H, E_H)$ of G such that $Q \subseteq V_H$ for some $Q \in \mathcal{C}$ is denoted by $\mathcal{H}_G(\mathcal{C})$, where H is* minimal *in the sense*

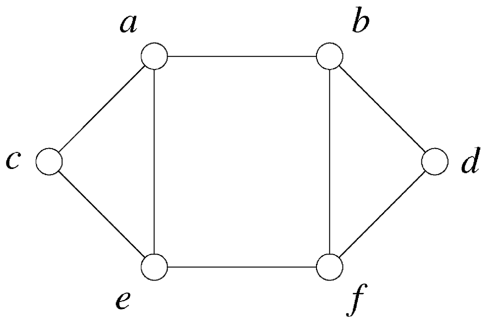

Fig. 1. A graph G with six vertices.

that no proper subgraph of H satisfies the above condition. Hence, $\mathcal{H}_G(\mathcal{C})$ is a set of trees.

Let $\mathcal{H}^*_G(\mathcal{C})$ denote the subset of $\mathcal{H}_G(\mathcal{C})$ constructed from $\mathcal{H}_G(\mathcal{C})$ by removing each tree if its proper subtree is also in $\mathcal{H}_G(\mathcal{C})$.

Definition 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two coteries on a graph $G = (V, E)$. A coterie $\mathcal{C}$ is said to G-*dominate* $\mathcal{D}$ if $\mathcal{H}^*_G(\mathcal{C}) \neq \mathcal{H}^*_G(\mathcal{D})$ and, for any $J \in \mathcal{H}^*_G(D)$, there is an $H \in \mathcal{H}^*_G(C)$ such that H is a subtree of J. A coterie $\mathcal{C}$ is said to be G-*nondominated* (G-ND) if no coteries G-dominate $\mathcal{C}$.*

Example 1. Consider a graph G in Fig. 1. Let $\mathcal{D}_1 = \{\{a, f\}\}$ and $\mathcal{D}_2 = \{\{a, b\}, \{a, e\}\}$ be two coteries on G. Figs. 2 and 3 illustrate all elements in $\mathcal{H}^*_G(\mathcal{D}_1)$ and $\mathcal{H}^*_G(\mathcal{D}_2)$, respectively. By definition, $\mathcal{D}_2$ G-dominates $\mathcal{D}_1$. Consider another coterie $\mathcal{D}_3 = \{\{a, b, f\}, \{a, e, f\}\}$ on G. Then, $\mathcal{D}_1$ dominates $\mathcal{D}_3$, but $\mathcal{D}_1$ does not G-dominate $\mathcal{D}_3$, since $\mathcal{H}^*_G(\mathcal{D}_1) = \mathcal{H}^*_G(\mathcal{D}_3)$.

In what follows, we use the following notations: As defined in Section 1, $A_G(\mathcal{C})$ is the availability of a coterie $\mathcal{C}$ on G, given the probabilities that a vertex and a link, respectively, are operational. For any $U \subseteq V$, $\overline{U} = V - U$ denotes the complement of U. $G_{|U}$ denotes the subgraph of G induced by U; i.e., $G_{|U} = (U, (U \times U) \cap E)$. By $F \subseteq H$ ($F \subset H$), we denote that F is a (proper) subgraph of H. By $\mathcal{T}(G)$, we denote the set of all connected acyclic (not necessarily spanning) subgraphs of a graph G. Finally, we denote $|E|$ and $|V|$ by m and n, respectively. In the rest of this section, we present a necessary and sufficient condition for an ND coterie to be G-ND. We use the following three theorems.

Theorem 1 [8]. *Let $\mathcal{C}$ be a coterie on a graph $G = (V, E)$. For any tree $F = (V_F, E_F) \in \mathcal{T}(G)$, if $Q \subseteq V_F$ for some quorum $Q \in \mathcal{C}$, then $H \subseteq F$ for some tree $H \in \mathcal{H}^*_G(\mathcal{C})$.*

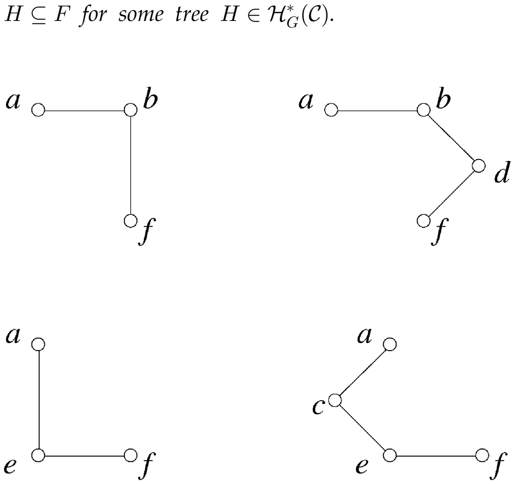

Fig. 2. The trees in $\mathcal{H}^*_G(\mathcal{D}_1)$.

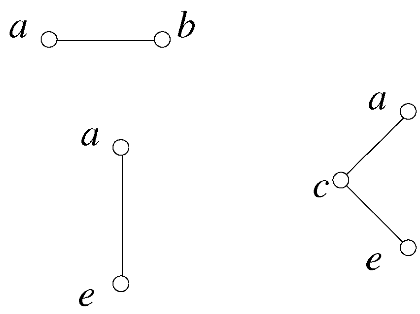

Fig. 3. The trees in $\mathcal{H}^*_G(\mathcal{D}_2)$.

Theorem 2 [8]. *Let $\mathcal{C}$ be a coterie on a graph $G = (V, E)$. $\mathcal{C}$ is G-dominated if and only if there exists a tree $F = (V_F, E_F) \in \mathcal{T}(G)$ satisfying the following formula:*

$$\text{For all } H = (V_H, E_H) \in \mathcal{H}^*_G(\mathcal{C}), H \not\subseteq F \text{ and } V_H \cap V_F \neq \emptyset. \tag{1}$$

Theorem 3 [5]. *Let $\mathcal{C}$ be a coterie on a graph $G = (V, E)$. $\mathcal{C}$ is dominated if and only if there exists a set $U \subseteq V$ such that $Q \not\subseteq U$ and $Q \cap U \neq \emptyset$, for all $Q \in \mathcal{C}$.*

Now, we are ready to prove our main theorem.

Theorem 4. *Let $\mathcal{C}$ be an ND coterie on a graph $G = (V, E)$. $\mathcal{C}$ is G-dominated if and only if there exist a quorum $Q \in \mathcal{C}$ and a connected component $N = (V_N, E_N)$ of $G_{|\overline{Q}}$ satisfying the following formula:*

$$\begin{aligned}&\text{For any connected component } N' = \\ &(V_{N'}, E_{N'}) \text{ of } G_{|\overline{V_N}} \text{ and quorum } P \in \mathcal{C},\ P \not\subseteq V_{N'}.\end{aligned} \tag{2}$$

Proof. Only if part: Suppose that $\mathcal{C}$ is G-dominated. By Theorem 2, there exists a tree $F \in \mathcal{T}(G)$ such that $H \not\subseteq F$ and $V_H \cap V_F \neq \emptyset$ for any $H \in \mathcal{H}^*_G(\mathcal{C})$. Since $\mathcal{C}$ is ND, there exists a quorum $Q \in \mathcal{C}$ such that either $Q \subseteq V_F$ or $Q \cap V_F = \emptyset$ holds by Theorem 3. However, $Q \subseteq V_F$ never hold since if $Q \subseteq V_F$, $H \subseteq F$ for some tree $H \in \mathcal{H}^*_G(\mathcal{C})$ by Theorem 1, a contradiction. Hence, $Q \cap V_F = \emptyset$ for some quorum $Q \in \mathcal{C}$, then there is a connected component $N = (V_N, E_N)$ of $G_{|\overline{Q}}$ satisfying $F \subseteq N$.

Let $N' = (V_{N'}, E_{N'})$ be any connected component of $G_{|\overline{V_N}}$. Obviously, $V_F \cap V_{N'} = \emptyset$. We now show that $P \not\subseteq V_{N'}$ for any quorum $P \in \mathcal{C}$. Assume otherwise that $P \subseteq V_{N'}$ for some $P \in \mathcal{C}$. Let $F' = (V_{F'}, E_{F'}) \in \mathcal{T}(G)$ be a spanning tree of N'. Since $P \subseteq V_{F'}$, by Theorem 1, there is a tree $H = (V_H, E_H) \in \mathcal{H}^*_G(\mathcal{C})$ such that $H \subseteq F'$. Since $V_F \cap V_{N'} = \emptyset$ and $V_H \subseteq V_{N'}$, $V_H \cap V_F = \emptyset$, a contradiction. Hence, $Q \in \mathcal{C}$ and N satisfies (2).

If part: Suppose that for some quorum $Q \in \mathcal{C}$ and component N of $G_{|\overline{Q}}$, (2) holds. Obviously, $Q \cap V_N = \emptyset$. Let F be any spanning tree of N. First, we show that $H \not\subseteq F$ for any $H \in \mathcal{H}^*_G(\mathcal{C})$. Assume otherwise that there is a tree $H \in \mathcal{H}^*_G(\mathcal{C})$ such that $H \subseteq F$. By Definition 1, there exists a quorum $Q' \in \mathcal{C}$ such that $Q' \subseteq V_H \subseteq V_F$. Since $Q \cap V_N = \emptyset$ and $V_F = V_N$, $Q \cap Q' = \emptyset$, a contradiction to Intersection Property.

Next, we show that $V_H \cap V_F \neq \emptyset$ for any $H \in \mathcal{H}^*_G(\mathcal{C})$. Fix any $H \in \mathcal{H}_G(\mathcal{C})$. By definition, V_H contains a quorum $Q' \in \mathcal{C}$. Since $Q' \not\subseteq V_{N'}$ for any connected component N' of $G_{|\overline{V_N}}$, $H \in \mathcal{H}^*_G(\mathcal{C})$ is not contained in any connected component of $G_{|\overline{V_N}}$ as a subgraph. Since H is connected, $V_H \cap V_N \neq \emptyset$ and, therefore, $V_H \cap V_F \neq \emptyset$. By Theorem 2, $\mathcal{C}$ is G-dominated. □

Although Theorem 4 assumes ND coteries, the proof of if part does not use this assumption. Hence, the existence of a pair of a quorum $Q \in \mathcal{C}$ and a connected component N of $G_{|\overline{Q}}$ satisfying (2) is sufficient for *any* (not only ND) coterie $\mathcal{C}$ to be G-dominated.

Theorem 5. *The G-NDness of a given ND coterie $\mathcal{C}$ is testable in $O(n^3|\mathcal{C}|^2 \log n)$ time.*

Proof. Let us estimate the time complexity of testing the condition of Theorem 4. Formula (2) is tested at most $n|\mathcal{C}|$ times, since the number of connected components in $G_{|\overline{Q}}$ is bounded by n. In each test for (2), we test $P \not\subseteq V_{N'}$ at most $n|\mathcal{C}|$ times. Since connected components can be identified in $O(n+m)$ time and the containment problem requires $O(n \log n)$ time, testing the condition of Theorem 4 requires $O(n^3|\mathcal{C}|^2 \log n)$ time. □

3 Improving the Availability

Suppose that we obtain an evidence N that a given coterie $\mathcal{D}$ is G-dominated by testing (2). How can we use the evidence to improve the availability? The idea here is to construct another coterie $\mathcal{C}$ that G-dominates $\mathcal{D}$. The proof of Theorem 1 of [8] includes a procedure for constructing a new coterie that G-dominates $\mathcal{C}$. Unfortunately, this procedure requires exponential time since it includes, as a subproblem, the determination of $\mathcal{H}^*_G(\mathcal{C})$, which in general requires exponential time. To see this, one can check the case in which $\mathcal{C} = \{V\}$ and G is complete. In order to avoid this time consuming procedure, we introduce a polynomial time function **Replace**. It creates a new coterie that G-dominates $\mathcal{C}$ using the evidence.

Let $\mathcal{D}$ be a set of nonempty subsets of V. $MinSet(\mathcal{D})$ and $MaxSet(\mathcal{D})$, respectively, denote the subset of $\mathcal{D}$ constructed from $\mathcal{D}$ by removing each element if its proper subset is in $\mathcal{D}$ and the set of all subsets U of V such that $Q \subseteq U$ for some $Q \in \mathcal{D}$.

Definition 3. *Let $\mathcal{C}$ and U be a coterie on a graph $G = (V, E)$ and a subset of V, respectively. Then, Replace function $\mathrm{Replace}(\mathcal{C}, U)$ returns set $MinSet(\mathcal{C}' \cup \{\overline{U}\})$, where*

$$\mathcal{C}' = MinSet(\{Q \mid Q \in MaxSet(\mathcal{C}) \text{ and } Q \not\subseteq U\}).$$

Example 2. Let $V = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{a, c\}\}$. Let us calculate $\mathrm{Replace}(\mathcal{C}, U)$ for $U = \{a, b\}$. First,

$$MaxSet(\mathcal{C}) = \{\{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

Then,

$$\begin{aligned} \mathcal{C}' &= MinSet(\{\{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}) \\ &= \{\{a, c\}, \{a, b, d\}\}, \text{ and} \\ \mathrm{Replace}(\mathcal{C}, U) &= MinSet(\{\{a, c\}, \{a, b, d\}\} \cup \{\overline{\{a, b\}}\}) \\ &= \{\{a, c\}, \{a, b, d\}, \{c, d\}\}. \end{aligned}$$

Theorem 6. *If $\mathcal{C}$ is a coterie, so is $\mathrm{Replace}(\mathcal{C}, U)$ for any proper subset U of V. Furthermore, if $\mathcal{C}$ is ND, so is $\mathrm{Replace}(\mathcal{C}, U)$.*

Proof. Arbitrarily select $U \subset V$ and fix it. By definition $\mathrm{Replace}(\mathcal{C}, U)$ satisfies Minimality. We show that $\mathrm{Replace}(\mathcal{C}, U)$ satisfies Intersection Property. Let

$$\mathcal{C}' = MinSet(\{Q \mid Q \in MaxSet(\mathcal{C}) \text{ and } Q \not\subseteq U\}).$$

Since $MaxSet(\mathcal{C}') \subseteq MaxSet(\mathcal{C})$, any two elements in $\mathcal{C}'$ intersect each other. For any $Q \in \mathcal{C}'$, $Q \cap \overline{U} \neq \emptyset$, because $Q \not\subseteq U$. Hence, $\mathrm{Replace}(\mathcal{C}, U)$ satisfies Intersection Property.

We further assume that $\mathcal{C}$ is ND. Let $W \subseteq V$ be any set. We show that either $Q \subseteq W$ or $Q \subseteq \overline{W}$ holds for some $Q \in \mathrm{Replace}(\mathcal{C}, U)$ to complete the proof by Theorem 3. Since $\mathcal{C}$ is ND, there exists a quorum $P \in \mathcal{C}$ such that $P \subseteq W$ or $P \subseteq \overline{W}$. Suppose first that $P \subseteq W$. If $W \subseteq U$, then $\overline{U} \in \mathrm{Replace}(\mathcal{C}, U)$. It follows that $\overline{U} \subseteq \overline{W}$ for $\overline{U} \in \mathrm{Replace}(\mathcal{C}, U)$. If $W \not\subseteq U$, then $Q \subseteq W$ for some $Q \in \mathrm{Replace}(\mathcal{C}, U)$, since

$$W \in \{Q \mid Q \in MaxSet(\mathcal{C}) \text{ and } Q \not\subseteq U\}.$$

Assume next that $P \subseteq \overline{W}$. If $\overline{W} \subseteq U$, then $\overline{U} \in \mathrm{Replace}(\mathcal{C}, U)$. Hence, $\overline{U} \subseteq W$ for $\overline{U} \in \mathrm{Replace}(\mathcal{C}, U)$. If $\overline{W} \not\subseteq U$, then $Q \subseteq \overline{W}$ for some $Q \in \mathrm{Replace}(\mathcal{C}, U)$, since

$$\overline{W} \in \{Q \mid Q \in MaxSet(\mathcal{C}) \text{ and } Q \not\subseteq U\}.$$

□

Theorem 7. *Let $\mathcal{C}$ be a coterie on a graph $G = (V, E)$. If there exist a quorum $Q \in \mathcal{C}$ and a connected component $N = (V_N, E_N)$ of $G_{|\overline{Q}}$ satisfying (2), then $\mathrm{Replace}(\mathcal{C}, \overline{V_N})$ G-dominates $\mathcal{C}$.*

Proof. Suppose that (2) holds for a quorum $Q \in \mathcal{C}$ and a connected component $N = (V_N, E_N)$ of $G_{|\overline{Q}}$. Let

$$\mathcal{C}' = MinSet(\{P \mid P \in MaxSet(\mathcal{C}) \text{ and } P \not\subseteq \overline{V_N}\})$$

and $\mathcal{D} = MinSet(\mathcal{C}' \cup \{V_N\}) = \mathrm{Replace}(\mathcal{C}, \overline{V_N})$. We know that $\mathcal{D}$ is a coterie. In order to show that $\mathcal{D}$ G-dominates $\mathcal{C}$, we show 1) $\mathcal{H}^*_G(\mathcal{C}') = \mathcal{H}^*_G(\mathcal{C})$ and 2) $\mathcal{D}$ G-dominates $\mathcal{C}'$.

Let $F = (V_F, E_F)$ be any spanning tree of N. Then, from the proof of if part of Theorem 4, $V_H \not\subseteq \overline{V_N}$ $(= \overline{V_F})$ for any $H \in \mathcal{H}^*_G(\mathcal{C})$. Thus, by the definition of $\mathcal{C}'$, it follows that $\mathcal{H}^*_G(\mathcal{C}') = \mathcal{H}^*_G(\mathcal{C})$, since $P \subseteq \overline{V_N}$ for any

$$P \in MaxSet(\mathcal{C}) - MaxSet(\mathcal{C}').$$

To show 2), we first observe that $V_N \in \mathcal{D}$. If $V_N \notin \mathcal{D}$, there would exist $P \in \mathcal{C}$ such that $P \subseteq V_N$, a contradiction. As shown in the proof of Theorem 4, $H \not\subseteq F$ holds for any $H \in \mathcal{H}^*_G(\mathcal{C}')$ $(= \mathcal{H}^*_G(\mathcal{C}))$. Thus, F belongs to $\mathcal{H}^*_G(\mathcal{D})$. Let $\mathcal{T}$ be the set of trees constructed from $\mathcal{H}^*_G(\mathcal{C}') \cup \{F\}$ by removing each tree if its proper subgraph is also in $\mathcal{H}^*_G(\mathcal{C}') \cup \{F\}$. Clearly, for any $H \in \mathcal{H}^*_G(\mathcal{C}')$, there exists $H' \in \mathcal{T}$ such that $H' \subseteq H$. Although the set $\mathcal{H}^*_G(\mathcal{D})$ may contain another tree $F' = (V_{F'}, E_{F'})$, except F, such that $F' \notin \mathcal{T}$ and $V_N \subseteq V_{F'}$, it follows that, for any $H \in \mathcal{T}$, $H' \subseteq H$ for some $H' \in \mathcal{H}^*_G(\mathcal{D})$. Thus, $\mathcal{D}$ G-dominates $\mathcal{C}'$. □

Theorem 8. *For any coterie $\mathcal{C}$ on a graph $G = (V, E)$ and a subset U of V, $\mathrm{Replace}(\mathcal{C}, U)$ terminates in polynomial time.*

Proof. Clearly, the most time consuming part in $\mathrm{Replace}(\mathcal{C}, U)$ is the calculation of

$$\mathcal{C}' = MinSet(\{Q \mid Q \in MaxSet(\mathcal{C}) \text{ and } Q \not\subseteq U\}).$$

In the following, we first observe that, for any $Q' \in \mathcal{C}' - \mathcal{C}$, there exists a $Q \in \mathcal{C}$ such that $Q \subset Q'$ and $|Q'| = |Q| + 1$ hold. Assume otherwise that there exists a $Q' \in \mathcal{C}' - \mathcal{C}$ such that $Q \not\subset Q'$ or $|Q'| \neq |Q| + 1$ holds for any $Q \in \mathcal{C}$. If $Q \not\subset Q'$ holds for any $Q \in \mathcal{C}$, then $Q' \notin MaxSet(\mathcal{C})$, a contradiction. So, there exists a $Q' \in \mathcal{C}' - \mathcal{C}$ such that $|Q'| \neq |Q| + 1$ for any $Q \in \mathcal{C}$. Suppose that $|Q'| = |Q| + j$ for some $j \geq 2$. By the definition of $MinSet$, there are vertices u, v $(u \neq v) \in Q' - Q$ such that $Q' - \{u\} \subseteq U$ and $Q' - \{v\} \subseteq U$ hold. This, however, implies $Q' \subseteq U$, a contradiction.

By this observation, $\mathcal{C}' = MinSet(\mathcal{D})$, where

$$\begin{aligned} \mathcal{D} = &\{Q \mid Q \in \mathcal{C} \text{ and } Q \not\subseteq U\} \cup \{Q' \mid Q' = Q \cup \{u\} \\ &\text{for some } Q \in \mathcal{C} \text{ and } u \in V \text{ and } Q' \not\subseteq U\}. \end{aligned}$$

Since $|\mathcal{D}| \leq n|\mathcal{C}|$, clearly $\mathcal{C}'$ is calculated in polynomial time. □

An outline of a procedure to improve the availability by using Replace function is therefore the following. It always terminates and produces a G-ND coterie, given an ND coterie. Let $\mathcal{C}$ be a given ND coterie:

Step 1: Search for an evidence N that $\mathcal{C}$ is G-dominated by testing (2) in Theorem 4. If the search fails, terminate the procedure (since $\mathcal{C}$ is G-ND).

Step 2: Replace $\mathcal{C}$ with Replace$(\mathcal{C}, \overline{V_N})$ and go to Step 1.

As final comments of this section, although Replace runs in polynomial time, this procedure may not run in polynomial time in the worst case since the number of repetitions may not be bounded by a polynomial. However, it always terminates since the G-domination relation over coteries forms a partial order and the number of coteries (on a fixed graph G) is finite. The fact that it produces a G-ND coterie is clear from Theorems 4, 6, and 7.

Next, this procedure cannot always guarantee proper increase of the availability. This is because the availability depends on the probabilities that vertices and edges are operational. As an extremal example, if they are all 1 (0), then the availability of every coterie is 1 (0) (and, hence, the availability cannot be improved). However, we can conclude the following: Let $\mathcal{C}$ and $\mathcal{D}$ be any ND coterie that is not G-ND and the output of this procedure for $\mathcal{C}$, respectively. Then, $A_G(\mathcal{C}) < A_G(\mathcal{D})$ holds, provided that all the operation probabilities are neither 0 nor 1.

4 Conclusions

In this paper, we discussed how to improve the availability of a coterie. We first characterized the G-ND coteries that are ND, by which the G-NDness of an ND coterie is polynomially testable. We then proposed a procedure to increase the availability by repeatedly modifying a given coterie by function Replace. Taking an ND coterie $\mathcal{C}$ that is not G-ND, Replace outputs a new ND coterie $\mathcal{D}$ that G-dominates the input, i.e., $A_G(\mathcal{D}) > A_G(\mathcal{C})$, provided that all the operational probabilities are neither 0 nor 1. Hence, the procedure can increase the availability. Although Replace terminates in polynomial time, we cannot bound the time complexity of the procedure by a polynomial in the worst case since the number of repetitions may not be bounded by a polynomial. Its analysis is left as a future work. Since we conjecture that there is an ND coterie that requires exponential time repetitions, the problem of choosing an input for which the procedure quickly terminates is interesting future work. Proposing a good polynomial time heuristic procedure based on another approach is another interesting open problem.

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