Nondominated Coteries on Graphs

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Abstract—Let C and D be two distinct coteries under the vertex set V of a graph $G = (V, E)$ that models a distributed system. Coterie C is said to G-dominate D (with respect to G) if the following condition holds: For any connected subgraph H of G that contains a quorum in D (as a subset of its vertex set), there exists a connected subgraph H' of H that contains a quorum in C. A coterie C on a graph G is said to be G-nondominated (G-ND) (with respect to G) if no coterie D ($\neq C$) on G G-dominates C. Intuitively, a G-ND coterie consists of irreducible quorums.

This paper characterizes G-ND coteries in graph theoretical terms, and presents a procedure for deciding whether or not a given coterie C is G-ND with respect to a given graph G, based on this characterization. We then improve the time complexity of the decision procedure, provided that the given coterie C is nondominated in the sense of Garcia-Molina and Barbara. Finally, we characterize the class of graphs G on which the majority coterie is G-ND.

Index Terms—Availability, coteries on graphs, distributed mutual exclusion problem, G-nondominatedness, majority consensus.

1 Introduction

The (distributed) mutual exclusion problem is widely recognized as a fundamental problem in distributed computing. Let us model a distributed system as an undirected graph; the vertices represent processes and the edges represent bidirectional communication links each connecting a pair of processes. In 1985, Garcia-Molina and Barbara [1] introduced the concept of coteries, and showed its usefulness for solving the mutual exclusion problem.[1] A coterie is a set of mutually incomparable nonempty sets (called quorums) of vertices (i.e., processes) such that any two quorums intersect each other.

A coterie is used to solve the mutual exclusion problem as follows: When entering the critical section, a vertex is asked to gain permission from every vertex in a quorum and holds it until it leaves the critical section. Because of the intersection property of quorums, at most one vertex can be in the critical section, provided that a vertex never give its permission to two vertices at a time.

Suppose that the graph (i.e., distributed system) on which the mutual exclusion algorithm mentioned above is implemented is unreliable so that fail-stop failures may occur on vertices and/or edges (i.e., processes and/or communication links). Then a vertex can enter the critical section only if there is a "surviving" quorum in the sense that all vertices in the quorum are being operational and for any pair of vertices in the quorum there is a path consisting only of operating vertices and edges. Given the probabilities that a vertex and a link, respectively, are operational, the probability that there is a surviving quorum is called the *availability* of the coterie. Although the problem of finding an optimal coterie with respect to the availability is difficult and computing the availability of a given coterie on a given graph is known to be #P-hard in general [2], some studies have been done to reveal properties of optimal coteries on simple classes of graphs such as complete graphs, rings, and trees [3], [4], [5], [6], [7], [8].

In particular, Ibaraki, Nagamochi, and Kameda introduced the concept of G-domination[2] as a central concept to calculate the availability of coteries on rings and trees, and showed that if a coterie C G-dominates a coterie D, then the availability of C is not smaller than that of D in general; we can thus discard D from the candidate list for optimal coteries [7]. They also characterized G-nondominated coteries on rings and trees. However, a characterization of G-nondominated coteries on general graphs is still open. This paper characterizes G-nondominated coteries in graph theoretical terms.

We first present a necessary and sufficient condition for a coterie on a graph to be G-nondominated. In order to check the condition, however, we need to test all trees appearing in the graph as a subgraph.[3] Next, we show that if a coterie is nondominated in the sense of Garcia-Molina and Barbara [1], we can complete the test just by checking only so-called cut-trees. Finally, we discuss the majority coterie [9] on graphs. We characterize the class of graphs G on which the majority coterie is G-nondominated, and derive an easy sufficient condition on G for the majority on G to be G-nondominated.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph that models a distributed system; each vertex $v \in V$ represents a process of the

1. See [11] for a mutual exclusion algorithm using a coterie.

2. We will formally define the concept of G-domination in Section 2.

3. The number of trees is $O(2^m)$, where m is the number of edges of the graph. Hence, testing the G-nondominatedness of a coterie based on this condition requires exponential time.

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distributed system, and each edge $(u, v) \in E$ represents the bidirectional communication link between u and v. The following definitions are by Garcia-Molina and Barbara [1].

DEFINITION 1. *Let V be a universal set of vertices. A set C of nonempty subsets of V is said to be a* coterie *under V if both of the following conditions hold*:

1) *(Intersection Property)* $\forall p, q \in C\ [p \cap q \neq \emptyset]$, *and*
2) *(Minimality Property)* $\forall p, q \in C\ [p \not\subseteq q]$.

An element of a coterie is called a quorum.

DEFINITION 2. *Let C and D be two distinct coteries under V. C is said to* dominate *D if for any quorum $p \in D$ there exists a quorum $q \in C$ such that $q \subseteq p$. A coterie C is said to be* nondominated *(ND, for short) if no coterie dominates C.*

The *reliability* of a vertex (edge) is the probability that the vertex (edge) is operational. Then the *availability* of a coterie C on G, denoted by $A_G(C)$, is defined as the probability that there is a connected subgraph $G' = (V', E')$ of G consisting only of operating vertices and edges such that $q \subseteq V'$ for some $q \in C$, given the reliabilities of a vertex and an edge. If a coterie C dominates a coterie D, then $A_G(C) \geq A_G(D)$ by definition. Thus we can substantially assume that optimal coteries (with respect to the availability) are ND. However, the nondominatedness is obviously not sufficient to pursue an optimal coterie on a graph, since the availability of the coterie depends heavily on the graph. The following concepts are introduced by Ibaraki, Nagamochi, and Kameda to analyze the availability of a coterie on a graph [7].

DEFINITION 3. *Let $G = (V, E)$ and C be a graph and a coterie under V, respectively. The set of all connected minimal subgraphs $h = (V_h, E_h)$ of G such that $q \subseteq V_h$ for some $q \in C$ is denoted by $\mathcal{H}_G(C)$, where h is "minimal" in the sense that no proper subgraph of h satisfies the above condition any more. Hence, $\mathcal{H}_G(C)$ is a set of trees.*

Let $\mathcal{H}^*_G(C)$ denote the subset of $\mathcal{H}_G(C)$ constructed from $\mathcal{H}_G(C)$ by repeatedly removing a tree whose proper subtree is in $\mathcal{H}_G(C)$. Then, for any two distinct trees, $g, h \in \mathcal{H}^*_G(C)$, $g \not\subseteq h$. This is called the minimality *property of $\mathcal{H}^*_G(C)$.*

DEFINITION 4. *Let $G = (V, E)$ be a graph, and let C and D be two coteries under V. Coterie C is said to G-*dominate *D (with respect to G) if $\mathcal{H}^*_G(C) \neq \mathcal{H}^*_G(D)$, and for any $g \in \mathcal{H}^*_G(D)$, there is an $h \in \mathcal{H}^*_G(C)$ such that h is a subtree of g. A coterie C is said to be G-*nondominated *(G-ND, for short) (with respect to G) if no coterie G-dominates C with respect to G.*

DEFINITION 5. *Let $G = (V, E)$ and C be a graph and a coterie under V, respectively. By $C_G(C)$, we denote the set of all subsets $q \subseteq V$ such that for some $h = (V_h, E_h) \in \mathcal{H}^*_G(C)$, $q = V_h$ holds.*

Let $C^*_G(C)$ be the subset of $C_G(C)$ constructed from $C_G(C)$ by repeatedly removing an element whose proper subset is in $C_G(C)$. Then, for any distinct elements $p, q \in C^*_G(C)$, $p \not\subseteq q$. This is called the minimality *property of $C^*_G(C)$.*

LEMMA 1. *$C^*_G(C)$ is a coterie.*

PROOF. $C^*_G(C)$ satisfies the minimality property by Definition 5. As for the intersection property, by Definition 5, $C^*_G(C) \subseteq C_G(C)$ implies that for each $q \in C^*_G(C)$, there exists an $h \in \mathcal{H}^*_G(C)$ such that $q = V_h$. Hence, it is sufficient to show that $V_h \cap V_f \neq \emptyset$ for any two trees $h, f \in \mathcal{H}^*_G(C)$. By Definition 3, there exist quorums p_h and p_f in C such that $p_h \subseteq V_h$ and $p_f \subseteq V_f$ hold. Since $p_h \cap p_f \neq \emptyset$, we have $V_h \cap V_f \neq \emptyset$. □

By the definition of $\mathcal{H}^*_G(C)$, we may rephrase the availability of a coterie C on G as follows: The availability is the probability that there is an $h \in \mathcal{H}^*_G(C)$ consisting only of operating vertices and edges. Hence, if C is G-dominated, then there is a G-ND coterie whose availability is not smaller than that of C.

Coterie C is said to be *closed* under G, if $C = C^*_G(C)$. If C is not closed, by definition $C^*_G(C)$ is dominated by C. They have the same availability though. As you will see in the proof of Theorem 1, $C^*_G(C)$ plays an important role in finding coteries that G-dominate C.

EXAMPLE 1. Consider a graph $G = (V, E)$ in Fig. 1, and let C be a coterie under V defined as follows:

$$C = \{\{a, b\}, \{a, d\}, \{b, d\}\}.$$

Fig. 2 illustrates all the elements in $\mathcal{H}^*_G(C)$. Note that a subgraph $(\{a, b, c\}, \{(a, b), (b, c)\})$ of G, for instance, contains quorum $\{a, b\}$ but is not an element of $\mathcal{H}^*_G(C)$, since it is not minimal.

We construct from $\mathcal{H}^*_G(C)$,

$$C_G(C) = \{\{a, b\}, \{a, b, e\}, \{a, d, e\}, \{b, c, d\}, \{b, d, e\}\}.$$

$C_G(C)$ is not a coterie, since it does not satisfy the minimality property; $\{a, b, e\}$ is a superset of $\{a, b\}$. We construct a coterie $C^*_G(C)$ from $C_G(C)$ by removing $\{a, b, e\}$:

$$C^*_G(C) = \{\{a, b\}, \{a, d, e\}, \{b, c, d\}, \{b, d, e\}\}.$$

Observe that $\{b, d\}$ in C is not an element of $C^*_G(C)$, since the subgraph of G induced by $\{b, d\}$ is not connected. It is replaced by minimal supersets, $\{b, c, d\}$ and $\{b, d, e\}$, that leave the induced subgraph connected.

3 CHARACTERIZING G-NONDOMINATED COTERIES

In this section, we characterize G-ND coteries. In what follows, notation $f \subseteq g$ ($f \subset g$) denotes that f is a subgraph (proper subgraph) of g. Also, by $\mathcal{T}(G)$, we denote the set of all connected acyclic (not necessarily spanning) subgraphs of a graph G.

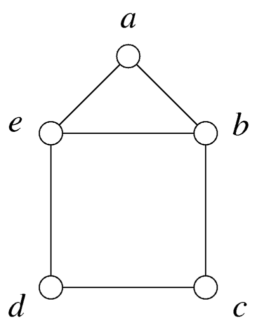

Fig. 1. A graph G with five vertices.

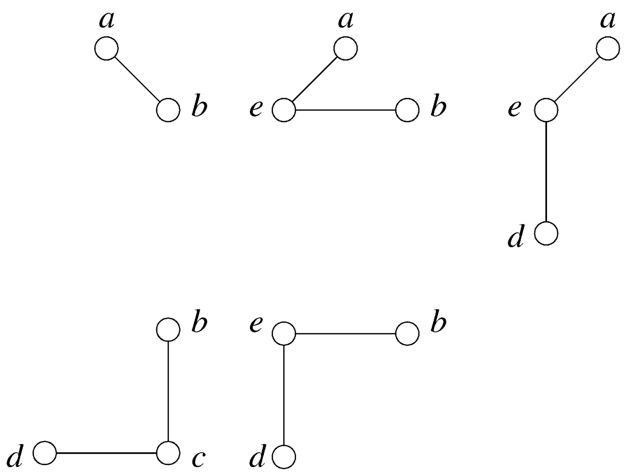

Fig. 2. Trees in $\mathcal{H}_G^*(C)$.

LEMMA 2. *Let $G=(V, E)$ and C be a graph and a coterie under V, respectively. Let $f = (V_f, E_f)$ be any tree in $\mathcal{T}(G)$. Then $h \subseteq f$ for some $h \in \mathcal{H}_G^*(C)$, if $q \subseteq V_f$ for some $q \in C$.*

PROOF. Let $f = (V_f, E_f) \in \mathcal{T}(G)$ be any tree such that $q \subseteq V_f$ for some $q \in C$. By the definition of $\mathcal{H}_G(C)$, there exists a tree $g \in \mathcal{H}_G(C)$ such that $g \subseteq f$. Then by the definition of $\mathcal{H}_G^*(C)$, there exists a tree $h \in \mathcal{H}_G^*(C)$ such that $h \subseteq g \subseteq f$. □

Note that $h \subseteq f$ implies both $V_h \subseteq V_f$ and $E_h \subseteq E_f$.

THEOREM 1. *Let $G = (V, E)$ and C be a graph and a coterie under V, respectively. C is G-dominated if and only if there exists an $f = (V_f, E_f) \in \mathcal{T}(G)$ satisfying the following formula:*

For any $h = (V_h, E_h) \in \mathcal{H}_G^*(C)$, $h \not\subseteq f$ and $V_h \cap V_f \neq \emptyset$ hold. (1)

PROOF. **If part:** Let $f \in \mathcal{T}(G)$ be any tree satisfying (1). Fix an $h \in \mathcal{H}_G^*(C)$. Since h is connected, $V_h \not\subseteq V_f$ implies $E_h \not\subseteq E_f$. Thus $h \not\subseteq f$ if and only if $E_h \not\subseteq E_f$.

We first show $V_h \not\subseteq V_f$. Suppose otherwise that $V_h \subseteq V_f$. Since $h \in \mathcal{H}_G^*(C)$, there is a $q \in C$ such that $q \subseteq V_h \subseteq V_f$. By Lemma 2, there exists an $h' \in \mathcal{H}_G^*(C)$ such that $h' \subseteq f$, a contradiction. Hence, we have $V_h \not\subseteq V_f$, and therefore, $E_h \not\subseteq E_f$, for any $h \in \mathcal{H}_G^*(C)$.

Recall that for any $q \in C_G^*(C)$ there exists an $h \in \mathcal{H}_G^*(C)$ satisfying $q = V_h$. Since $V_h \not\subseteq V_f$ for any $h \in \mathcal{H}_G^*(C)$, $q \not\subseteq V_f$ for any $q \in C_G^*(C)$. Now, we define a new coterie D as the set constructed from $C_G^*(C) \cup \{V_f\}$ by repeatedly removing a quorum that is a superset of V_f so that the resulting set D satisfies the minimality property of coterie. In the rest, we show that D G-dominates C.

By construction, D satisfies the minimality property. Since $V_h \cap V_f \neq \emptyset$ for any $h \in \mathcal{H}_G^*(C)$, the intersection property also holds. Thus D is certainly a coterie under V. That D G-dominates C follows from the fact that D G-dominates $C_G^*(C)$.

Only if part: Let D be any coterie under V that G-dominates C. There are two cases to consider: the case $\mathcal{H}_G^*(C) \subset \mathcal{H}_G^*(D)$ and the case $\mathcal{H}_G^*(C) \not\subset \mathcal{H}_G^*(D)$.

1) Suppose $\mathcal{H}_G^*(C) \subset \mathcal{H}_G^*(D)$. Then there exists an $f \in \mathcal{H}_G^*(D) - \mathcal{H}_G^*(C)$. Now, we show that (1) holds for this f. Suppose otherwise that (1) does not hold for this f, i.e., there is an $h \in \mathcal{H}_G^*(C)$ such that either $h \subseteq f$ or $V_h \cap V_f = \emptyset$ hold. If $h \subseteq f$, $\mathcal{H}_G^*(D)$ contains both h and f, which contradicts the minimality of $\mathcal{H}_G^*(D)$. If $V_h \cap V_f = \emptyset$, h and f do not intersect each other, which contradicts the intersection property of $\mathcal{H}_G^*(D)$.
2) Suppose $\mathcal{H}_G^*(C) \not\subset \mathcal{H}_G^*(D)$. Then there exists a $g \in \mathcal{H}_G^*(C) - \mathcal{H}_G^*(D)$. Since D G-dominates C, there exists an $f \in \mathcal{H}_G^*(D)$ satisfying $f \subseteq g$. f must be in $\mathcal{H}_G^*(D) - \mathcal{H}_G^*(C)$, since otherwise C contains both f and g, which contradicts the minimality of $\mathcal{H}_G^*(C)$. Now we show that (1) holds for this f. Suppose otherwise that the formula does not hold for this f, i.e., there is an $h \in \mathcal{H}_G^*(C)$ such that either $h \subseteq f$ or $V_h \cap V_f = \emptyset$ holds. If $h \subseteq f$, $\mathcal{H}_G^*(C)$ contains both h and g, which contradicts the minimality of $\mathcal{H}_G^*(C)$. Finally, if $V_h \cap V_f = \emptyset$, then there is an $f' \in \mathcal{H}_G^*(D)$ such that $f' \subseteq h$ (because D G-dominates C), a contradiction since f' and f do not intersect each other. (It contradicts the intersection property of $\mathcal{H}_G^*(D)$.) □

EXAMPLE 2. Consider graph G in Fig. 1 and coterie C given in Example 1. A subtree $(\{b, e\}, \{(b, e)\})$ of G satisfies (1). As in the proof of Theorem 1, we can construct a new coterie D from $C_G^*(C) \cup \{\{b, e\}\}$ by removing $\{b, d, e\}$ that is a superset of $\{b, e\}$:

$$D = \{\{a, b\}, \{b, e\}, \{a, d, e\}, \{b, c, d\}\}.$$

Fig. 3 illustrates all the elements in $\mathcal{H}_G^*(D)$. Comparing Fig. 2 with Fig. 3, coterie D G-dominates coterie C.

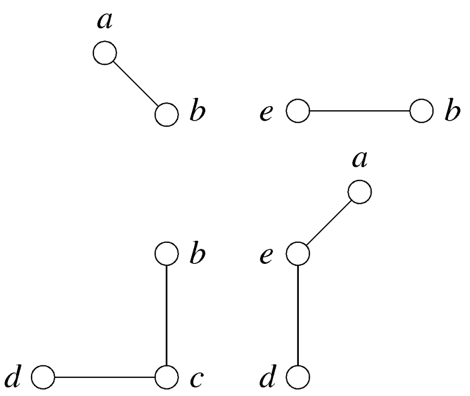

Fig. 3. Trees in $\mathcal{H}_G^*(D)$.

We need to check all the trees in $\mathcal{T}(G)$ to test the G-nondominatedness of a given coterie C based on Theorem 1. If C is ND, we can test its G-nondominatedness by checking a smaller number of trees as we will see next. A tree $f \in \mathcal{T}(G)$ is called a *cut-tree* if the removal of f from G disconnects G, or more formally:

DEFINITION 6. *A tree $f = (V_f, E_f) \in \mathcal{T}(G)$ is called a* cut-tree *of G if there is no tree in $\mathcal{T}(G)$ with vertex set $\overline{V_f} = V - V_f$.*

The following lemma is by Ibaraki and Kameda [10].

LEMMA 3 [10]. *Let C be a coterie under V. Then C is ND if and only if for any $x \subseteq V$, there exists a quorum $q \in C$ such that*

$$(q \subseteq x) \oplus (q \subseteq \overline{x}),$$

where $\oplus$ denotes the exclusive OR, and $\overline{x}$ is the complement of x (i.e., $\overline{x} = V - x$).

THEOREM 2. *Let $G = (V, E)$ and C be a graph and an ND coterie under V, respectively. Let $f = (V_f, E_f)$ be any tree in $\mathcal{T}(G)$. If f satisfies* (1), *then f is a cut-tree of G.*

PROOF. Suppose that f is not a cut-tree and derive a contradiction. Since f is not a cut-tree, there is a $g = (V_g, E_g) \in \mathcal{T}(G)$ such that $V_g = \overline{V_f}$. Since C is ND, there exists a $q \in C$ such that exactly one of $q \subseteq V_f$ or $q \subseteq V_g$ holds by Lemma 3.

Suppose that $q \subseteq V_f$ and $q \not\subseteq V_g$ hold. By Lemma 2, $h \subseteq f$ for some $h \in \mathcal{H}_G^*(C)$, a contradiction. Hence, $q \not\subseteq V_f$ and $q \subseteq V_g$ hold. Again by Lemma 2, there exists an $h \in \mathcal{H}_G^*(C)$ such that $h \subseteq g$ holds, which implies $V_h \cap V_f = \emptyset$ (because $V_g = \overline{V_f}$), a contradiction. □

By Theorem 2, checking all the cut-trees is sufficient to test whether or not a given ND coterie C is G-ND; we do not need to check all the trees in $\mathcal{T}(G)$. This definitely improves the time complexity of the decision procedure, but the determination of its time complexity is still open.

EXAMPLE 3. Consider graph G in Fig. 1 and coterie C given in Example 1. G has three cut-trees $(\{b, e\}, \{(b, e)\})$, $(\{b, c, e\}, \{(b, c), (b, e)\})$ and $(\{b, d, e\}, \{(b, e), (d, e)\})$. The first two satisfy (1), and the last one belongs to $\mathcal{H}_G^*(C)$ (see Fig. 2). Since C is nondominated but is G-dominated, there are cut-trees satisfying (1).

THEOREM 3. *Let $G = (V, E)$ and C be a graph and a coterie under V, respectively. C is G-dominated if there exists a cut-tree $f = (V_f, E_f)$ of G satisfying the following formula*:

$$\text{For some } q \in C,\ V_f = \overline{q} \text{ holds.} \tag{2}$$

PROOF. We show that every cut-tree f satisfying (2) also satisfies (1). Let f be a cut-tree of G satisfying $V_f = \overline{q}$ for some $q \in C$.

We first show $V_h \cap V_f \neq \emptyset$ for any $h \in \mathcal{H}_G^*(C)$. For any quorum $p \in C - \{q\}$, $p \cap \overline{q} \neq \emptyset$ by the minimality property of C, which implies that $V_h \cap V_f \neq \emptyset$ for any tree $h = (V_h, E_h) \in \mathcal{H}_G^*(C - \{q\})$, since V_h contains a quorum, $p \in C$, as a subset. Since f is a cut-tree, there is no tree in $\mathcal{T}(G)$ with vertex set $\overline{V_f} = q$. Thus, $q \subset V_h$ for any $h \in \mathcal{H}_G^*(\{q\})$, which implies $V_h \cap V_f \neq \emptyset$. Hence, $V_h \cap V_f \neq \emptyset$ for any $h \in \mathcal{H}_G^*(C)$, since $\mathcal{H}_G^*(C) \subseteq \mathcal{H}_G^*(C - \{q\}) \cup \mathcal{H}_G^*(\{q\})$.

Next, we show $h \not\subseteq f$ for any $h \in \mathcal{H}_G^*(C)$. Suppose otherwise that there exists an $h \in \mathcal{H}_G^*(C)$ satisfying $V_h \subseteq V_f$. This implies that there is a $p \in C$ satisfying $p \subseteq \overline{q}$. However, it is a contradiction since $p \cap q = \emptyset$ contradicts the intersection property of C. Hence, $V_h \not\subseteq V_f$ for any $h \in \mathcal{H}_G^*(C)$, and $h \not\subseteq f$. □

As a final remark in this section, since the connectivity of a given graph can be tested in time $O(m + n)$, the sufficient condition of Theorem 3 can be tested in time $O((m + n)|C|)$, where m and n are the sizes of vertex and edge sets, respectively.

4 THE MAJORITY COTERIE ON GRAPHS

The majority coterie is one of the most well-studied coteries. This section discusses on which graphs the majority coterie becomes G-ND.

DEFINITION 7. *The majority coterie C under V is defined as follows:*

1) *When $|V|$ is odd, C is the set of all subsets of V whose cardinality is exactly $(|V| + 1)/2$.*
2) *When $|V|$ is even, let v be an arbitrary fixed vertex in V. Then $C = C_1 \cup C_2$, where C_1 is the set of all subsets of V, containing v, whose cardinality is exactly $|V|/2$, and C_2 is the set of all subsets of V, not containing v, whose cardinality is exactly $|V|/2 + 1$.*

We call v the semiprimary vertex.

We start with two simple lemmas.

LEMMA 4. *Let $G = (V, E)$ and $f = (V_f, E_f)$ be a connected graph and a tree in $\mathcal{T}(G)$, respectively. For any nonnegative integer $k \leq |V| - |V_f|$, there exists a tree $g \in \mathcal{T}(G)$ such that $V_f \subseteq V_g$ and $|V_g| = |V_f| + k$.*

PROOF. Since G is connected, there exists a spanning tree h of G such that $f \subseteq h$. The existence of a tree g satisfying the condition of this lemma is clear from this fact. □

LEMMA 5. *Let $G = (V, E)$ and $f = (V_f, E_f)$ be a biconnected graph and a tree in $\mathcal{T}(G)$, respectively. For any nonnegative integer $k \le |V| - |V_f| - 1$ and vertex $v \in V - V_f$, there exists a tree $g \in \mathcal{T}(G)$ such that $v \notin V_g$, $V_f \subseteq V_g$ and $|V_g| = |V_f| + k$ hold.*

PROOF. Let $v \in V - V_f$ be any vertex. Since G is biconnected, $H = G - \{v\}$, i.e., the subgraph of G induced by vertex set $V - \{v\}$, is connected. By applying Lemma 4 to H, the proof completes. □

In the last section, we showed that the existence of a cut-tree f satisfying (2) is a sufficient condition for a coterie on a graph to be G-dominated. The condition, however, is not necessary. Here, we show that the condition is necessary and sufficient as long as the majority coterie is concerned. In [7], it is shown that a necessary condition for coterie C on graph G to be G-ND is that each quorum of C is included in a biconnected component of G, i.e., there is a biconnected component of G such that all quorums of C are included in it. Hence, without loss of generality, we assume that G is biconnected in the next theorem.

THEOREM 4. *Let $G = (V, E)$ and C be a biconnected graph and the majority coterie under V, respectively. C is G-dominated if and only if there exists a cut-tree $f = (V_f, E_f)$ of G satisfying (2).*

PROOF. By Theorem 3, it suffices to show the necessity. Suppose that the majority coterie C on a graph G is G-dominated. Since C is ND, by Theorems 1 and 2, there exists a cut-tree $f \in \mathcal{T}(G)$ satisfying (1). We consider the following two cases: the case where $|V|$ is odd and the case where $|V|$ is even.

1) Suppose that $|V|$ is odd. Since $h \not\subseteq f$ for any $h = (V_h, E_h) \in \mathcal{H}^*_G(C)$, $|V_f| \le (|V| - 1)/2$ by Definition 7. Then, by Lemma 4, there exists an $f' = (V_{f'}, E_{f'}) \in \mathcal{T}(G)$ such that $V_f \subseteq V_{f'}$ and $|V_{f'}| = (|V| - 1)/2$, which implies that $\overline{V_{f'}} = \overline{q}$ for some $q \in C$, since $\left|\overline{V_{f'}}\right| = (|V| + 1)/2$. It is sufficient to show that f' is a cut-tree of G. Suppose otherwise that f' is not a cut-tree. Then there is a $g \in \mathcal{T}(G)$ such that $V_g = \overline{V_{f'}} \in C$. By Lemma 2, there is a $g' = (V_{g'}, E_{g'}) \in \mathcal{H}^*_G(C)$ such that $g' \subseteq g$. Since $V_g \cap V_{f'} = \varnothing$, $V_{g'} \subseteq V_g$, and $V_f \subseteq V_{f'}$, we have $V_{g'} \cap V_f = \varnothing$, which contradicts (1).

2) Suppose that $|V|$ is even. Let v be the semi-primary vertex of C. Since $h \not\subseteq f$ for each $h \in \mathcal{H}^*_G(C)$, by Definition 7, either $v \in V_f$ and $|V_f| \le |V|/2 - 1$, or $v \notin V_f$ and $|V_f| \le |V|/2$ holds. It is sufficient to show that f' is a cut-tree in each case. First, suppose $v \in V_f$ and $|V_f| \le |V|/2 - 1$. By Lemma 4, there exists an $f' \in \mathcal{T}(G)$ such that $V_f \subseteq V_{f'}$ and $|V_{f'}| = |V|/2 - 1$. Since $v \notin \overline{V_{f'}}$ and $\left|\overline{V_{f'}}\right| = |V|/2 + 1$, $\overline{V_{f'}} = \overline{q}$ for some $q \in C$. Using the same argument as in case 1, f' can be shown to be a cut-tree. Next, suppose that $v \notin V_f$ and $|V_f| \le |V|/2$. By Lemma 5, there exists an $f' \in \mathcal{T}(G)$ such that $v \notin V_{f'}$, $V_f \subseteq V_{f'}$ and $|V_{f'}| = |V|/2$. Since $v \in \overline{V_{f'}}$ and $\left|\overline{V_{f'}}\right| = |V|/2$, $\overline{V_{f'}} = \overline{q}$, for some $q \in C$. Using the argument in case 1 again, we can show that f' is a cut-tree. □

EXAMPLE 4. Consider graph G in Fig. 1 and the majority coterie C under the vertex set $\{a, b, c, d, e\}$. That is,

$$C = \{\{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}\}.$$

Since there is a cut-tree $f = (\{b, e\}, \{\{b, e\}\})$ such that $\overline{\{b, e\}} = \{a, c, d\} \in C$, C on G is G-dominated.

Next, on another graph G' in Fig. 4, consider C. Since there is no cut-tree of size 2 in G', there is no cut-tree f satisfying (2). Hence, C on G' is G'-ND.

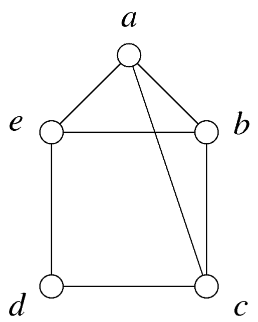

Fig. 4. A graph G' with five vertices.

Given a graph G we can decide whether or not the majority coterie is G-ND based on Theorem 4. However, its time complexity is not polynomial in $n = |V|$, although it is polynomial in $|C|$. We present an easy sufficient condition on G for the majority on G to be G-ND.

THEOREM 5. *Let $G = (V, E)$ be a biconnected graph with the minimum degree $\delta(G) \ge 3|V|/4$. Then the majority coterie C on G is G-ND.*

PROOF. Let $\kappa(G)$ and V_c be the connectivity of G and a minimal cutset of G, respectively. That is, $|V_c| = \kappa(G)$. Let graph $G_1 = (V_1, E_1)$ be a connected component of $G - \{V_c\}$, i.e., the subgraph of G induced by $V - \{V_c\}$, and let $G_2 = (V_2, E_2)$ be the graph consisting of the other connected components of $G - \{V_c\}$.

We first show $\kappa(G) > |V|/2$. To this end, assume $\kappa(G) \le |V|/2$ and derive a contradiction by showing that there is a vertex $u \in V$ such that its degree $deg(u) < 3|V|/4$. If $|V_1| \ge |V|/4$, then $|V_2 \cup V_c| \le 3|V|/4$, which implies that every vertex in V_2 has a degree at most $3|V|/4 - 1$, a contradiction. If $|V_1| < |V|/4$, then, since $|V_1 \cup V_c| < 3|V|/4$, every vertex in V_1 has a degree less than $3|V|/4$, a contradiction.

Suppose that C on G is G-dominated. Then there is a cut-tree $f = (V_f, E_f)$ such that $V_f = \overline{q}$ for some $q \in C$. Since $\kappa(G) > |V|/2$, $|V_f| \geq \kappa(G) > |V|/2$. On the other hand, $\left|V_f\right| = \left|\overline{q}\right| \leq \left|V\right|/2$, a contradiction. Hence, C on G is G-ND. □

5 Conclusion

The concept of G-domination is introduced to search for a coterie that maximizes the availability on a given graph. In this paper, we presented a necessary and sufficient condition for a coterie on a graph to be G-nondominated. We also presented a sufficient condition for a nondominated coterie on a graph to be G-nondominated. We then discussed the majority coterie, and derived a necessary and sufficient condition for the majority coterie on a graph to be G-nondominated. Finally, we derived an easy sufficient condition for the majority coterie on a graph to be G-nondominated.

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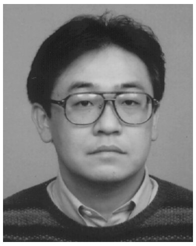

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