k-Coteries for Tolerating Network 2-Partition

Takashi Harada, *Member*, *IEEE Computer Society*, and
Masafumi Yamashita, *Member*, *IEEE Computer Society*

Abstract—Network partition, which makes it impossible for some pairs of precesses to communicate with each other, is one of the most serious network failures. Although the notion of *k*-coterie is introduced to design a *k*-mutual exclusion algorithm robust against network failures, the number of processes allowed to simultaneously access the critical section may fatally decrease once network partition occurs. This paper discusses how to construct a *k*-coterie such that the *k*-mutual exclusion algorithm adopting it is robust against network 2-partition. To this end, we introduce the notion of *complemental* *k*-coterie, and show that complemental *k*-coteries meet our purpose. We then give methods for constructing complemental *k*-coteries, and show a necessary and sufficient condition for a *k*-coteries to be complemental.

Index Terms—Distributed systems, complemental, *k*-coteries, *k*-semicoteries, *k*-mutual exclusion problem, network 2-partition, nondominatedness, quorums.

1 Introduction

Suppose that there is a distributed system whose processes share a resource. In order to keep the access regulation on the resource consistent, the processes are requested to access the resource only in a specified program section called *critical section*. Our concerns are protocols for entering and leaving the critical section, which of course depend on the access regulation. Many of the shared resources ask a mutually exclusive access, in the sense that exactly one process is granted to access at a time. Then, we encounter the problem of designing protocols (for the critical section) which guarantee that no more than one process is in the critical section simultaneously. This problem is called the *mutual exclusion problem*. If the resource is more generous so that at most k processes are granted to access simultaneously, then the corresponding protocol design problem is called the *k-mutual exclusion problem* (or the *(k+1)-exclusion problem*) [6], [11], [15], [16].

A typical *k*-mutual exclusion algorithm, i.e., a pair of entering and leaving protocols, uses an information structure called a *k*-coterie. Let U be the set of all processes in a distributed system. A *k-semicoterie* $\mathcal{C}$ under U is a set of nonempty subsets Q of U satisfying the following two conditions:

1. **Minimality:** For all $P, Q \in \mathcal{C}, P \not\subset Q$.
2. **Intersection Property:** There are k pairwise disjoint quorums in $\mathcal{C}$, but no more than k.

A member Q of a *k*-semicoterie $\mathcal{C}$ is referred to as a *quorum*. A *k*-semicoterie $\mathcal{C}$ is called a *k-coterie* if it satisfies the following condition [4], [11]:

3. **Nonintersection Property:** For any set $\mathcal{D}$ of h $(< k)$ pairwise disjoint quorums in $\mathcal{C}$, there is a set $\mathcal{D}'$ of k pairwise disjoint quorums in $\mathcal{C}$ such that $\mathcal{D}' \supset \mathcal{D}$.

For example, $\mathcal{C} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$ is a 2-coterie under $U = \{1,2,3,4\}$. Note that 1-semicoteries, which are by definition 1-coteries, are known as *coteries* [5].

We would like to explain an outline of the *k*-mutual exclusion algorithm that uses a *k*-coterie $\mathcal{C}$.[1] We prepare a single token $\texttt{permission}_v$ for each process $v \in U$, and initially place it in v.

1. A process u wishing to enter the critical section selects a quorum $P \in \mathcal{C}$ and requests $\texttt{permission}_v$ to each process $v \in P$.
2. Upon receiving the request from u, each v sends $\texttt{permission}_v$ to u as soon as v has it.
3. Upon receiving $\texttt{permission}_v$ from each process $v \in P$, u enters the critical section.
4. Upon leaving the critical section, u returns $\texttt{permission}_v$ to each process $v \in P$.

An obvious but important observation is the following: If a process u is in the critical section, then there is a quorum $P \in \mathcal{C}$ such that u possesses the tokens $\texttt{permission}_v$ of all processes $v \in P$. Then, the number of processes who are granted to enter the critical section is bounded by the number of pairwise disjoint quorums in $\mathcal{C}$, which is k by Intersection Property. However, there might be a case in which a choice of less than k quorums does not leave a quorum that does not intersect with each of the quorums in the choice, and only a small number of processes could enjoy the privilege of entering the critical section. Nonintersection Property guarantees that such cases never happen.

The robustness against network failures is an advantage of using the above *k*-mutual exclusion algorithm. However, the extent of fault tolerance capability of the algorithm primarily depends on the *k*-coterie that it adopts. Much

- T. Harada is with the Graduate School of Management, Hiroshima Univeristy, 1-1-89 Higashi Senda, Naka-ku, Hiroshima, 730-0053, Japan. E-mail: haradat@mgt.hiroshima-u.ac.jp.
- M. Yamashita is with the Department of Computer Science and Communication Engineering, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan. E-mail: mak@csce.kyushu-u.ac.jp.

Manuscript received 10 July 2002; revised 1 Aug. 2003; accepted 28 Dec. 2003.
For information on obtaining reprints of this article, please send e-mail to: tpds@computer.org, and reference IEEECS Log Number 116911.

1. This rough sketch of the algorithm does not include tricks to make the algorithm starvation and deadlock free. See [11] for a full implementation.

effort is hence devoted to construction of a large variety of k-coteries and pursuit of the best k-coterie in terms of the availability, i.e., the probability that the algorithm tolerates network failures and grants a process the access right [1], [3], [4], [6], [7], [9], [10], [12], [13].

Along this context, this paper investigates the robustness of k-coteries against network 2-partition. Let $\mathcal{C}$ be a k-coterie adopted in the algorithm, and suppose that the network is partitioned into two groups S and $\overline{S}$ $(= U \setminus S)$, and that no two processes $u \in S$ and $v \in \overline{S}$ can communicate with each other (but that communication inside each of the groups is complete). Let $t(S)$ (respectively, $t(\overline{S})$) be the maximum number of pairwise disjoint quorums Q in $\mathcal{C}$ such that $Q \subseteq S$ (respectively, $Q \subseteq \overline{S}$). Let $\rho(S) = t(S) + t(\overline{S})$. Then, the algorithm can grant at most $\rho(S)$ processes the right to enter the critical section and, hence, $\rho = \min_{S \subseteq U} \rho(S)$ is a natural measure to evaluate the robustness of the algorithm against network 2-partition. We say that a k-coterie is *complemental* if $\rho = k$, i.e., if $t(S) + t(\overline{S}) = k$ holds for any $S \subseteq U$. The complementalness of k-semicoterie is defined in the same way. That $\mathcal{C}$ is complemental is a necessary condition for the algorithm to achieve k-mutual exclusion no matter how the network is partitioned into two subnetworks. The objective of this paper is to characterize complemental k-coteries.

A k-coterie (respectively, k-semicoterie) $\mathcal{C}$ is said to *dominate* a k-coterie (respectively, k-semicoterie) $\mathcal{D}$, if $\mathcal{C} \neq \mathcal{D}$ and for any quorum $P \in \mathcal{D}$, there exists a quorum $Q \in \mathcal{C}$ such that $Q \subseteq P$. A k-coterie (respectively, k-semicoterie) $\mathcal{C}$ is said to be *nondominated* (ND, for short), if $\mathcal{C}$ is not dominated by any k-coterie (respectively, k-semicoterie).[2]

In [2], Barbara and Garcia-Molina showed that all ND coteries are complemental. Little effort was however made to clarify the tolerance capability of k-coteries for network 2-partition, and indeed whether or not any ND k-coterie is complemental is not known. Our contributions are summarized as follows:

1. We show that all ND 2-coteries are complemental, but that for $k \geq 3$, there is a noncomplemental ND k-coterie.
2. For each of several typical k-coterie construction methods, we derive a condition for it to produce a complemental k-coterie.
3. We give a necessary and sufficient condition for a k-coterie to be complemental.

The rest of this paper is organized as follows: Section 2 shows that every ND 2-coterie is complemental. In Section 3, we introduce the concept of r-complemental k-coteries, where k-coterie is complemental if and only if it is r-complemental for all $1 \leq r \leq k$. Based on the concept, Section 4 investigates how to construct a complemental majority k-coterie, a complemental composite k-coterie and a complemental tree k-coterie. Finally, Section 5 completely characterizes r-complemental k-coteries. Section 6 concludes the paper.

2. By definition, an ND k-semicoterie satisfying Nonintersection Property is an ND k-coterie. Note, however, that an ND k-coterie may not be an ND k-semicoterie, since there may be an ND k-coterie which is dominated by a k-semicoterie (that is not a k-coterie).

2 Network 2-Partition and ND 2-Coteries

We model a distributed system by a connected undirected graph $G = (U, E)$, where U and E represent the set of processes and the set of bidirectional communication links, respectively. If communication links in $F \subseteq E$ are down, communication between a pair (u, v) of processes belonging to different connected components of $G' = (U, E - F)$ becomes impossible. We say that the system suffers from a network ℓ-*partition* if G' consists of ℓ connected components. We assume that the processes never fail.

This paper investigates a network 2-partition. A set F of communication link failures causing a network 2-partition clearly defines a partition U_1, U_2 of U, and communication between two processes is possible if and only if both of them belong to one of the two partites. However, there may not be a set F of communication link that realizes a given partition U_1, U_2 of U, depending on G. Nevertheless, in this paper, we say that a k-mutual exclusion algorithm tolerates a network 2-partition only when it tolerates any partition U_1, U_2 of U, since the algorithm cannot select an underlying network. In other words, we assume that G is complete.

Under the above assumptions, that a k-coterie is complemental is necessary for the k-mutual exclusion algorithm to tolerate network 2-partition. In what follows, we show that any ND 2-coterie is complemental, i.e., $\rho = 2$.[3] Let $\mathcal{C}$ be a k-semicoterie under U and consider the following condition **C**: There is a set $S \subseteq U$ such that both of the following two conditions hold:

1. (C1) For any quorum $P \in \mathcal{C}$, $P \not\subseteq S$.
2. (C2) For any k pairwise disjoint quorums $P_1, P_2, \dots, P_k \in \mathcal{C}$, $P_i \cap S \neq \emptyset$ for some $1 \leq i \leq k$.

Theorem 1 [9], [13], [14].

1. *For any $k \geq 1$, a k-semicoterie is dominated if and only if it satisfies Condition **C**.*
2. *For any $k \geq 1$, if a k-coterie $\mathcal{C}$ is dominated then it satisfies Condition **C**. On the other hand, for any $k \leq 2$, a k-coterie $\mathcal{C}$ is dominated if it satisfies Condition **C**.*

Note that it is still open to decide whether or not the second claim of item 2 of Theorem 1 holds for any $k \geq 3$. Given a set $\mathcal{D}$ of nonempty subsets of U, $Min(\mathcal{D})$ denotes a subset of $\mathcal{D}$ constructed from $\mathcal{D}$ by removing each element if a proper subset of the element is in $\mathcal{D}$.

Definition 1. *Let $\mathcal{C}$ be a k-semicoterie under U, and r be an integer such that $1 \leq r \leq k$. The* r-contraction *of $\mathcal{C}$, denoted by $\mathcal{C}^r$, is defined by*

$$\mathcal{C}^r = Min(\{P \mid P = P_1 \cup P_2 \cup \cdots \cup P_r,\ P_i \in \mathcal{C} \text{ for all } 1 \leq i \leq r, \text{ and } P_i \cap P_j = \emptyset \text{ for all } 1 \leq i < j \leq r\}).$$

That is, the r-contraction of a k-semicoterie $\mathcal{C}$ is the set of all minimal subsets of U that contains as a subset the union of r pairwise disjoint quorums of $\mathcal{C}$. Note that $\mathcal{C}^1 = \mathcal{C}$ by definition. We restate Theorem 1 as Corollary 1.

3. The definition of ρ was given in Section 1.

Consider the following condition **D** for $\mathcal{C}$: For any $S \subseteq U$, either one of the following two conditions holds:

1. (D1) $P \subseteq S$ for some $P \in \mathcal{C}$.
2. (D2) $P \subseteq \overline{S}$ for some $P \in \mathcal{C}^k$.

Corollary 1.

1. *For any $k \geq 1$, a k-semicoterie $\mathcal{C}$ is ND if and only if it satisfies Condition **D**.*
2. *For any $k \geq 1$, a k-coterie $\mathcal{C}$ is ND if it satisfies Condition **D**. On the other hand, for any $k \leq 2$, if a k-coterie $\mathcal{C}$ is ND then it satisfies Condition **D**.*

Theorem 2. *Every ND 2-coterie $\mathcal{C}$ is complemental.*

Proof. Let $\mathcal{C}$ be an ND 2-coterie. Then, it satisfies Condition **D** by Corollary 1. Suppose that U is partitioned into S and $\overline{S}$. Let $\mathcal{C}(S) = \{P \in \mathcal{C} \mid P \subseteq S\}$ and $\mathcal{C}(\overline{S}) = \{P \in \mathcal{C} \mid P \subseteq \overline{S}\}$. By Intersection Property, $t(S) + t(\overline{S}) \leq 2$. If neither of $\mathcal{C}(S)$ and $\mathcal{C}(\overline{S})$ are empty, then clearly $t(S) + t(\overline{S}) \geq 2$. Hence, suppose, without loss of generality, that either $\mathcal{C}(S)$ or $\mathcal{C}(\overline{S})$ is empty. Without loss of generality, we may assume that $\mathcal{C}(S) = \emptyset$, i.e., (D1) does not hold. Then, (D2) holds and $t(\overline{S}) = 2$. □

The NDness is hence sufficient for a 2-coterie to be complemental. However, we cannot extend it to any $k \geq 3$, as the following counterexample shows.

Example 1. Let $U = \{1,2,3,4,5,6,7,8\}$ and consider a 3-coterie

$$\mathcal{C} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{1,8\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{2,7\},\{2,8\},\{3,4\},\{3,5\},\{3,6\},\{3,7\},\{3,8\},\{4,5,6\},\{4,5,7\},\{4,5,8\},\{4,6,7\},\{4,6,8\},\{4,7,8\},\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}\}$$

under U. We first show that $\mathcal{C}$ is an ND 3-coterie. In fact, $\mathcal{C}$ is a *vote assignable* k-coterie [7]. To see this, define a weight function w by $w(i) = 2$ for $1 \leq i \leq 3$ and $w(i) = 1$ for $4 \leq i \leq 8$, and take a threshold $\theta = 3$. Then,

$$\mathcal{C} = Min\left(\left\{S \subseteq U \mid \sum_{i \in S} w(i) \geq \theta\right\}\right).$$

Thus, $\mathcal{C}$ is a vote assignable 3-coterie. For all $S \subseteq U$, if $\sum_{i \in S} w(i) < 3$, then $\overline{S}$ contains three pairwise disjoint quorums of $\mathcal{C}$. $\mathcal{C}$ is hence ND, by Corollary 1.

To see that $\mathcal{C}$ is not complemental, let $S = \{1,2,3\}$ and $\overline{S} = \{4,5,6,7,8\}$. Since $t(S) = t(\overline{S}) = 1$, $t(S) + t(\overline{S}) = 2 < 3$.

3 *r*-Complemental *k*-Coteries

In this section, we define *r*-complemental *k*-coteries and show their basic properties.

Definition 2. *Let $\mathcal{C}$ and r, respectively, be a k-coterie under U and an integer such that $1 \leq r \leq k$. $\mathcal{C}$ is said to be r-*complemental*, if for any $S \subseteq U$, either one of the following two conditions holds:*

1. *(E1) $P \subseteq S$ for some $P \in \mathcal{C}^r$.*
2. *(E2) $P \subseteq \overline{S}$ for some $P \in \mathcal{C}^{k-r+1}$.*

Proposition 1. *A k-coterie $\mathcal{C}$ is complemental if and only if it is r-complemental for all $1 \leq r \leq k$.*

Proof. Suppose that a *k*-coterie $\mathcal{C}$ is *r*-complemental for all $1 \leq r \leq k$, and consider any partition S and $\overline{S}$ of U. Let r be the maximum number of pairwise disjoint quorums $Q \in \mathcal{C}$ such that $Q \subseteq S$, i.e., $r = t(S)$. If $r = k$, then $t(S) + t(\overline{S}) = k$, by Intersection Property of $\mathcal{C}$. So, suppose that $r < k$. Then, by definition, there are no $r+1$ pairwise disjoint quorums $Q \in \mathcal{C}$ such that $Q \subseteq S$. Hence, there are $k - r$ pairwise disjoint quorums $Q \in \mathcal{C}$ such that $Q \subseteq \overline{S}$, since $\mathcal{C}$ is $(r+1)$-complemental, which implies that $t(S) + t(\overline{S}) = k$, i.e., $\mathcal{C}$ is complemental.

On the other hand, suppose that $\mathcal{C}$ is complemental, but is not *r*-complemental for some $1 \leq r \leq k$, i.e., there is an $S \subseteq U$, such that both of (E1) and (E2) do not hold, which implies that $t(S) \leq r - 1$ and $t(\overline{S}) \leq k - r$. Hence, $t(S) + t(\overline{S}) \leq k - 1$, a contradiction. □

This motivates the investigation of *r*-complemental *k*-coteries in the following. It is worth noting that, by Definition 2 and Corollary 1, 1-complemental *k*-coteries are always ND.

Proposition 2. *A k-coterie $\mathcal{C}$ is r-complemental if and only if it is $(k-r+1)$-complemental.*

Proof. Suppose that $\mathcal{C}$ is not *r*-complemental. Then, there is an $S \subseteq U$ such that $P \not\subseteq S$ for all $P \in \mathcal{C}^r$ and $P \not\subseteq \overline{S}$ for all $P \in \mathcal{C}^{k-r+1}$, which implies that $\mathcal{C}$ is not $(k-r+1)$-complemental, since $k - (k-r+1) + 1 = r$. The other direction can be shown in the same way. □

By Proposition 2, a *k*-coterie is complemental, if it is *r*-complemental for all $1 \leq r \leq \lceil k/2 \rceil$.

Example 2. Let $U = \{1,2,3,4,5\}$, and consider a 3-coterie $\mathcal{C}$

$$\mathcal{C} = \{\{1\},\{2,3\},\{4,5\}\}$$

under U. Then, we have

$$\mathcal{C}^2 = \{\{1,2,3\},\{1,4,5\},\{2,3,4,5\}\}$$

and

$$\mathcal{C}^3 = \{\{1,2,3,4,5\}\},$$

by Definition 1. Then, $\mathcal{C}$ is neither 1 nor 3-complemental by Proposition 2, since $P \not\subseteq S$ for any $P \in \mathcal{C}$ and $P \not\subseteq \overline{S}$ for any $P \in \mathcal{C}^3$, where $S = \{2\}$. It is not 2-complemental either since $P \not\subseteq S$ and $P \not\subseteq \overline{S}$ hold for any $P \in \mathcal{C}^2$, where $S = \{1,2\}$.

Consider another 3-coterie $\mathcal{D}$

$$\mathcal{D} = \{\{1\},\{2\},\{3,4\},\{3,5\},\{4,5\}\}$$

under U. We have

$$\mathcal{D}^2 = \{\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$$

and

$$\mathcal{D}^3 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\}\}.$$

By Proposition 2, $\mathcal{D}$ is both 1 and 3-complemental since for any $S \subseteq U$, $P \subseteq S$ for some $P \in \mathcal{D}$ or $P \subseteq \overline{S}$ for some $P \in \mathcal{D}^3$. It is also 2-complemental since for any $S \subseteq U$, there is a $P \in \mathcal{D}^2$ such that either $P \subseteq S$ or $P \subseteq \overline{S}$ holds. $\mathcal{D}$ is thus complemental.

4 Classes of Complemental *k*-Coteries

The problem of determining whether a *k*-coterie is complemental is difficult, in general, because $t(S) + t(\overline{S})$ must be checked for all subsets S of U by definition. The problem, however, becomes tractable if we restrict ourselves to some classes of *k*-coteries. In this section, we drive conditions for several typical classes of *k*-coteries to be complemental, which enable us to construct complemental *k*-coteries efficiently.

4.1 Majority *k*-Coteries

Given $k \geq 1$, the set

$$k\text{-}Maj(U) = \{P \subseteq U \mid |P| = \lceil (|U|+1)/(k+1) \rceil\},$$

is called a *majority k-coterie* under U, if it is a *k*-coterie [4].[4] Let $U = \{1,2,3,4,5\}$ and $k = 2$. Then, we have

$$\begin{aligned}2\text{-}Maj(U) =&\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\\ &\{3,4\},\{3,5\},\{4,5\}\},\end{aligned}$$

which is a 2-coterie.

Theorem 3. *A majority k-coterie $k\text{-}Maj(U)$ is complemental if $|U|+1$ is divisible by $k+1$.*

Proof. Let $\mathcal{C} = k\text{-}Maj(U)$ and $n = |U|$. Suppose, otherwise, that $\mathcal{C}$ is not *r*-complemental for some $1 \leq r \leq k$. Since $\mathcal{C}$ is not *r*-complemental, let $S \subseteq U$ be such that neither of (E1) and (E2) holds for S. By assumption, $w = \lceil (n+1)/(k+1) \rceil = (n+1)/(k+1)$, which implies that $|S| \leq rw - 1$, since, otherwise, $P \subseteq S$ would hold for some $P \in \mathcal{C}^r$. Thus, $|\overline{S}| \geq n - (rw-1) = (k-r+1)w$, which, however, implies that $P \subseteq \overline{S}$ for some $P \in \mathcal{C}^{k-r+1}$, a contradiction. □

4.2 Composite *k*-Coteries

Given an integer m $(2 \leq m \leq |U|)$, let $\{U_1, U_2, \ldots, U_m\}$ be an *m*-partition of U, k_i $(1 \leq i \leq m)$ a positive integer such that $\sum_{i=1}^{m} k_i = k$, and $\mathcal{C}_i$ a k_i-coterie under U_i. A *composite k-coterie* under U [4], [13] is simply defined by

$$k\text{-}Comp(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m) = \cup_{i=1}^{m} \mathcal{C}_i.$$

Let $\mathcal{C}_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $\mathcal{C}_2 = \{\{4,5\},\{4,6\},\{5,6\}\}$ be coteries under $U_1 = \{1,2,3\}$ and $U_2 = \{4,5,6\}$, respectively. Then,

$$2\text{-}Comp(\mathcal{C}_1, \mathcal{C}_2) = \{\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{4,6\},\{5,6\}\}$$

is a composite 2-coterie under $U = \{1,2,3,4,5,6\}$.

Theorem 4. *$k\text{-}Comp(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ is complemental if and only if each of $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ is complemental.*

4. Clearly, set $k\text{-}Maj(U)$ is not a *k*-coterie if $\lceil (|U|+1)/(k+1) \rceil = \lceil (|U|+1)/k \rceil$ [17].

Proof. We only prove the case $m = 2$, but an extension to the general case is straightforward. Let $\mathcal{D} = k\text{-}Comp(\mathcal{C}_1, \mathcal{C}_2)$, where $\mathcal{C}_i$ is a k_i-coterie under U_i for $i = 1,2$.

If part: Suppose, otherwise, that $\mathcal{D}$ is not complemental, i.e., there are an r $(1 \leq r \leq k)$ and an $S \subseteq U$ such that $Q \not\subseteq S$ for all $Q \in \mathcal{D}^r$ and $Q \not\subseteq \overline{S}$ for all $Q \in \mathcal{D}^{k-r+1}$. Let $S_i = S \cap U_i$ and r_i be the maximum number of pairwise disjoint quorums of $\mathcal{C}_i$ in S_i for $i = 1,2$. Then, $r_1 + r_2 < r$ since $Q \not\subseteq S$ for all $Q \in \mathcal{D}^r$.

On the other hand, for $i = 1,2$, $P \not\subseteq S_i$ for all $P \in \mathcal{C}_i^{r_i+1}$, which implies that $P \subseteq U_i \setminus S_i$ for some $P \in \mathcal{C}_i^{k_i - r_i}$, since $\mathcal{C}_i$ is complemental. Hence, $\overline{S}$ contains, as subsets, $(k_1 - r_1) + (k_2 - r_2)$ pairwise disjoint quorums of $\mathcal{D}$. Since $(k_1 - r_1) + (k_2 - r_2) = k - (r_1 + r_2) \geq k - r + 1, Q \subseteq \overline{S}$ for some $Q \in \mathcal{D}^{k-r+1}$, a contradiction.

Only if part: Suppose, otherwise, that $\mathcal{C}_1$ is not complemental, without loss of generality. Then, there are an r_1 $(1 \leq r_1 \leq k_1)$ and an $S_1 \subseteq U_1$ such that $P \not\subseteq S_1$ for all $P \in \mathcal{C}_1^{r_1}$ and $P \not\subseteq U_1 \setminus S_1$ for all $P \in \mathcal{C}_1^{k_1 - r_1 + 1}$. Let $S = S_1 \cup U_2$. Then, $Q \not\subseteq S$ for all $Q \in \mathcal{D}^{k_2 + r_1}$.

On the other hand, $Q \not\subseteq \overline{S}$ for all $Q \in \mathcal{D}^{k_1 - r_1 + 1}$. Since $k_1 - r_1 + 1 = k - (k_2 + r_1) + 1$, $Q \not\subseteq \overline{S}$ for all $Q \in \mathcal{D}^{k-(k_2+r_1)+1}$. Hence, $\mathcal{D}$ is not complemental, a contradiction. □

4.3 Coterie Join Operation

For a *k*-coterie $\mathcal{C}$, we denote $\cup_{P \in \mathcal{C}} P$ by $\cup\mathcal{C}$.

Definition 3. *Let $\mathcal{C}_1$ be a k-coterie under U, $\mathcal{C}_2$ a coterie under U, and u an element in $\cup\mathcal{C}_1$. Assume that $\cup\mathcal{C}_1 \cap \cup\mathcal{C}_2 \subseteq \{u\}$ holds. Then, from $\mathcal{C}_1$ and $\mathcal{C}_2$, the* coterie join operation *produces a quorum set $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ defined by*

$$\begin{aligned}\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2) =&\{R \mid R = (P \setminus \{u\}) \cup Q,\ P \in \mathcal{C}_1,\ Q \in \mathcal{C}_2\\ &\text{and } u \in P\} \cup \{R \mid R = P,\ P \in \mathcal{C}_1 \text{ and } u \notin P\}.\end{aligned}$$

Jiang and Huang [9] showed that $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ is a *k*-coterie. The coterie join operation is a powerful tool to construct a new *k*-coterie.

Theorem 5. *Let $\mathcal{C}_1$ be a k-coterie under U, $\mathcal{C}_2$ a coterie under U, and r an integer such that $1 \leq r \leq k$. Assume that $\cup\mathcal{C}_1 \cap \cup\mathcal{C}_2 \subseteq \{u\}$. Then, $\mathcal{J} = \mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ is r-complemental if and only if $\mathcal{C}_1$ is r-complemental and $\mathcal{C}_2$ is (1-)complemental. Thus, $\mathcal{J}$ is complemental if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are complemental.*

Proof. Only if part: We show that $\mathcal{J}$ is not *r*-complemental if either $\mathcal{C}_1$ is not *r*-complemental or $\mathcal{C}_2$ is not 1-complemental.

(I) Suppose that $\mathcal{C}_1$ is not *r*-complemental despite that $\mathcal{J}$ is *r*-complemental, and derive a contradiction. Since $\mathcal{C}_1$ is not *r*-complemental, for some $S \subseteq \cup\mathcal{C}_1$, $P \not\subseteq S$ for all $P \in \mathcal{C}_1^r$ and $P \not\subseteq \overline{S}$ for all $P \in \mathcal{C}_1^{k-r+1}$. Then, since $\mathcal{J}$ is *r*-complemental, either $R \subseteq S$ for some $R \in \mathcal{J}^r$ or $R \subseteq \overline{S}$ for some $R \in \mathcal{J}^{k-r+1}$. Observe that $R \not\subseteq \cup\mathcal{C}_1$ since, otherwise, if $R \subseteq \cup\mathcal{C}_1$, then R belongs either to $\mathcal{C}_1^r$ or to $\mathcal{C}_1^{k-r+1}$, a contradiction because $\mathcal{C}_1$ is not *r*-complemental. Hence, $R \subseteq \overline{S}$ for $R \in \mathcal{J}^{k-r+1}$, and $R = (P \setminus \{u\}) \cup Q$ for some $P \in \mathcal{C}_1^{k-r+1}$ and $Q \in \mathcal{C}_2$. Note that $P \not\subseteq \overline{S}$ because $\mathcal{C}_1$

is not r-complemental. If $u \notin S$, then $(P \setminus \{u\}) \not\subseteq \overline{S}$ and, hence, $R \not\subseteq \overline{S}$, a contradiction. So, we may assume $u \in S$.

Let $S^+ = S \cup Q$ for some $Q \in \mathcal{C}_2$. Since $\mathcal{J}$ is r-complemental, either $R' \subseteq S^+$ for some $R' \in \mathcal{J}^r$ or $R' \subseteq \overline{S^+}$ for some $R' \in \mathcal{J}^{k-r+1}$. Note that, if $R' \subseteq \cup\mathcal{C}_1$ then $R' \subseteq S$ or $R' \subseteq \overline{S}$. Then, by the same argument above, $R' \not\subseteq \cup\mathcal{C}_1$ holds and, hence, $R' \notin \mathcal{C}^r$ and $R' \notin \mathcal{C}^{k-r+1}$. Suppose that $R' \subseteq S^+$ for $R' \in \mathcal{J}^r$. By Minimality of $\mathcal{C}_2$, $R' = (P' \setminus \{u\}) \cup Q$ for some $P' \in \mathcal{C}^r$. Since $u \in S$, it follows $P' \subseteq S$, a contradiction because $\mathcal{C}_1$ is not r-complemental. Hence, $R' \subseteq \overline{S^+}$ for $R' \in \mathcal{J}^{k-r+1}$. Since $R' \notin \mathcal{C}^{k-r+1}$, $R' = (P' \setminus \{u\}) \cup Q'$ for some $P' \in \mathcal{C}^{k-r+1}$ and $Q' \in \mathcal{C}_2$. However, by Intersection Property of $\mathcal{C}_2$, $R' \not\subseteq \overline{S^+}$ holds, a contradiction.

(II) Next, suppose that $\mathcal{C}_2$ is not 1-complemental. Then, by definition, for some $S \subseteq \cup\mathcal{C}_2$, both of $Q \not\subseteq S$ and $Q \not\subseteq \overline{S}$ hold for all $Q \in \mathcal{C}_2$. Let $A \in \mathcal{C}_1^r$ be such that $u \in A$. Note that there is such an A since $u \in \cup\mathcal{C}_1$. Put $S^+ = (A \setminus \{u\}) \cup S$. In the following, we show that $\mathcal{J}$ is not r-complemental for S^+, i.e., 1) $R \not\subseteq S^+$ for all $R \in \mathcal{J}^r$ and 2) $R \not\subseteq \overline{S^+}$ for all $R \in \mathcal{J}^{k-r+1}$ hold.

To claim 1), let $R \in \mathcal{J}^r$. If $R \subseteq \cup\mathcal{C}_1$, then $R \in \mathcal{C}_1^r$. Then, we may assume that $u \notin R$ since, otherwise, $\mathcal{C}_2 = \{\{u\}\}$, which is 1-complemental, a contradiction. Since $u \in A$, $R \not\subseteq A$ by Minimality of $\mathcal{C}_1^r$ and, hence, $R \not\subseteq S^+$. Suppose $R \not\subseteq \cup\mathcal{C}_1$. Then, $R = (P \setminus \{u\}) \cup Q$ for some $P \in \mathcal{C}_1^r$ and $Q \in \mathcal{C}_2$. By definition, $Q \not\subseteq S$. Then, $R \not\subseteq S^+$ since there is a $v \in (Q \setminus S)$ such that $v \notin S^+$.

To claim 2), let $R \in \mathcal{J}^{k-r+1}$. If $R \subseteq \cup\mathcal{C}_1$, then $R \in \mathcal{C}_1^{k-r+1}$. By the same argument as above, we may assume $u \notin R$. Since A contains r quorums of $\mathcal{C}_1$ and R contains $k-r+1$ quorums of $\mathcal{C}_1$, $R \not\subseteq \overline{A}$ by Intersection Property of $\mathcal{C}_1$. Since $u \notin R$, $R \not\subseteq \overline{A \setminus \{u\}}$ and, hence, $R \not\subseteq \overline{S^+}$. Suppose that $R \not\subseteq \cup\mathcal{C}_1$. Then, $R = (P \setminus \{u\}) \cup Q$ for some $P \in \mathcal{C}_1^{k-r+1}$ and $Q \in \mathcal{C}_2$. By definition, $Q \not\subseteq \overline{S}$. Hence, $R \not\subseteq \overline{S^+}$.

If part: Suppose that $\mathcal{J}$ is not r-complemental, and derive a contradiction. That is, there is an $S \subseteq U$ such that $R \not\subseteq S$ for all $R \in \mathcal{J}^r$ and $R \not\subseteq \overline{S}$ for all $R \in \mathcal{J}^{k-r+1}$.

Let $S^+ = S \cup \{u\}$. Since $\mathcal{C}_1$ is r-complemental, either $P \subseteq S^+$ for some $P \in \mathcal{C}_1^r$ or $P \subseteq \overline{S^+}$ for some $P \in \mathcal{C}_1^{k-r+1}$. If $P \subseteq \overline{S^+}$, then $u \notin P$ and, hence, $P \in \mathcal{J}^{k-r+1}$, a contradiction since $P \subseteq (\overline{S^+} \setminus \{u\}) \subseteq \overline{S}$. Hence, there is a $P \in \mathcal{C}_1^r$ such that $P \subseteq S^+$. We may assume that $u \in P$ since, otherwise, if $u \notin P$, then $P \in \mathcal{J}^r$, a contradiction because $P \subseteq (S^+ \setminus \{u\}) \subseteq S$. Since $\mathcal{C}_2$ is 1-complemental, either $Q \subseteq S$ or $Q \subseteq \overline{S}$ for some $Q \in \mathcal{C}_2$. Suppose that $Q \subseteq S$. Since $u \in P$, there is a $W \in \mathcal{J}^r$ such that $W = (P \setminus \{u\}) \cup Q$. However, since $(P \setminus \{u\}) \subseteq S$ and $Q \subseteq S$, $W \subseteq S$ holds, a contradiction because $\mathcal{J}$ is not r-complemental. So, we may assume that $Q \subseteq \overline{S}$ for $Q \in \mathcal{C}_2$.

Next, let $S^- = S \setminus \{u\}$. Since $\mathcal{C}_1$ is r-complemental, by arguing as above, we have a $P' \in \mathcal{C}_1^{k-r+1}$ such that $P' \subseteq \overline{S^-}$. If $u \notin P'$, then $P' \in \mathcal{J}^{k-r+1}$, a contradiction since $P' \subseteq (\overline{S^-} \setminus \{u\}) \subseteq \overline{S}$. If $u \in P'$, then $W = (P' \setminus \{u\}) \cup Q$ and $W \in \mathcal{J}^{k-r+1}$. However, since $(P' \setminus \{u\}) \subseteq (\overline{S^-} \setminus \{u\}) \subseteq \overline{S}$ and $Q \subseteq \overline{S}$, $W \subseteq \overline{S}$ holds, a contradiction. □

4.4 Tree *k*-Coteries

Let H be a subset of U such that $|H| = km + 1$ for some integer m ($m \geq 2$), and v be an element in H. A *basic tree k-coterie* [7], denoted by $k\text{-}Tree(U, H, v, m)$, is defined as

$$k\text{-}Tree(U,H,v,m) = \{Q \subseteq H \mid \{v\} \cap Q \neq \emptyset \text{ and } |Q| = 2\} \cup \{Q \subseteq H \mid \{v\} \cap Q = \emptyset \text{ and } |Q| = m\}.$$

The rooted tree T associated with $k\text{-}Tree(U, H, v, m)$ has the root v. The other elements in H are children of v and form leaves of T.

A *tree k-coterie* is recursively constructed by using the coterie join operation. In the following construction, we assume that $u \in \cup\mathcal{C}_1$ and $\cup\mathcal{C}_1 \cap \cup\mathcal{C}_2 \subseteq \{u\}$, in order for $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ to be defined. We also associate a rooted tree T for each tree k-coterie $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$:

1. Any basic tree k-coterie $\mathcal{C}_1$ is a tree k-coteries. The rooted tree $T_{\mathcal{C}_1}$ associated with $\mathcal{C}_1$ was already defined.
2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, be a tree k-coterie and a basic tree (1-)coterie, and assume that $T_{\mathcal{C}_1}$ and $T_{\mathcal{C}_2}$ are the rooted trees associated with them.

   If $\cup\mathcal{C}_1 \cap \cup\mathcal{C}_2 = \{u\}$ and u is a leaf of $T_{\mathcal{C}_1}$, then $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ is a tree k-coterie. If $\cup\mathcal{C}_1 \cap \cup\mathcal{C}_2 = \emptyset$, then $\mathcal{J}_u(\mathcal{C}_1, \mathcal{C}_2)$ is a tree k-coterie for any leaf u of $T_{\mathcal{C}_1}$. The associated rooted tree T is constructed from $T_{\mathcal{C}_1}$ by replacing leaf u with tree $T_{\mathcal{C}_2}$, i.e., we remove leaf u and place the root of $T_{\mathcal{C}_2}$ instead of u. All leaves of $T_{\mathcal{C}_1}$ except u, and all leaves of $T_{\mathcal{C}_2}$ are now leaves of T.

No other quorum sets are tree k-coterie.

Theorem 6. *Let m be an integer such that $m \geq 2$, H a subset of U, and v an element in H. Assume that $k\text{-}Tree(U, H, v, m)$ is definable, i.e., $|H| = km + 1$. Then, $k\text{-}Tree(U, H, v, m)$ is complemental.*

Proof. We denote $k\text{-}Tree(U, H, v, m)$ by $\mathcal{C}$, and let T be the rooted tree associated to $\mathcal{C}$. Suppose that $\mathcal{C}$ is not r-complemental and derive a contradiction. By definition, for some $S \subseteq U$, $P \not\subseteq S$ for all $P \in \mathcal{C}^r$ and $P \not\subseteq \overline{S}$ for all $P \in \mathcal{C}^{k-r+1}$.

Suppose first that $v \in S$. If S contains $(r-1)m + 1$ leaves of T, $P \subseteq S$ for some $P \in \mathcal{C}^r$ by the definition of basic tree k-coterie. We hence assume that $\overline{S}$ contains at least $(k-r+1)m$ leaves, which however, implies that $P \subseteq \overline{S}$ for some $P \in \mathcal{C}^{k-r+1}$, a contradiction.

Suppose next that $v \notin S$. If S contains rm leaves of T, then $P \subseteq S$ for some $P \in \mathcal{C}^r$. We hence assume that $\overline{S}$ contains $(k-r)m + 1$ leaves. However, since $v \in \overline{S}$, $P \subseteq \overline{S}$ for some $P \in \mathcal{C}^{k-r+1}$, a contradiction. □

The next corollary follows.

Corollary 2. *Tree k-coteries are complemental.*

5 Characterizing *r*-Complemental *k*-Coteries

The objective of this paper is to understand complemental k-coteries. To this end, in Section 3, we introduced the concept of r-complemental k-coteries and showed that a k-coterie is complemental if and only if it is r-complemental for all $1 \leq r \leq k$. In Section 4, we showed that the k-coterie in some classes are complemental by using the concept of r-complemental k-coterie, but we did not give their complete characterization. The aim of this section is to completely characterize the r-complemental k-coteries, with the hope that it gives us a new viewpoint to understand the complemental k-coteries, although currently, it seems to be useless to reduce the time complexity of the membership problem. We begin this section with several lemmas.

Lemma 1. *Let $\mathcal{C}$ be a k-coterie under U. For any $1 \le r \le k$ and $S \subseteq U$, $|S| \ge r$ if Condition (E2) does not hold.*

Proof. Suppose, otherwise, that $|S| = m < r$ and let $P_1, P_2, \ldots, P_k$ be k pairwise disjoint quorums in $\mathcal{C}$. Since $|S| = m$, $P_i \cap S \ne \emptyset$ for at most m P_is. There are hence, at least $k - m$ P_is such that $P_i \subseteq \overline{S}$. Let P be the union of $k - r + 1$ out of those P_is. Then, $P \in \mathcal{C}^{k-r+1}$, a contradiction. □

Let S be a subset of U. We denote a set of r-partition of S by $\mathcal{S}_r = \{S_1, S_2, \ldots, S_r\}$, where $S_i \ne \emptyset$ for all $1 \le i \le r$, $\cup_{i=1}^{r} S_i = S$, and $S_i \cap S_j = \emptyset$ for all $1 \le i < j \le r$.

Lemma 2. *Let $\mathcal{C}$ be a k-coterie under U. If neither of Conditions (E1) and (E2) hold for some $S \subseteq U$ and $1 \le r \le k$, then there is an r-partition $\mathcal{S}_r$ of S such that $P \not\subseteq S'$ for all $P \in \mathcal{C}$ and $S' \in \mathcal{S}_r$.*

Proof. Suppose that (E2) does not hold for S and r. By Lemma 1, $|S| \ge r$. Since (E1) does not hold either, S contains at most ℓ pairwise disjoint quorums $P_1, P_2, \ldots, P_\ell$ of $\mathcal{C}$, where $0 \le \ell \le r - 1$. By definition, $P_0 = S \setminus \cup_{i=1}^{\ell} P_i \not\supset P$ for any $P \in \mathcal{C}$ and, hence, any subset Q of P_i $(0 \le i \le \ell)$ is not a proper superset of a quorum P of $\mathcal{C}$. Since $|S| \ge r$, we can obviously construct a desired $\mathcal{S}_r$ by partitioning some P_i $(0 \le i \le \ell)$ into several disjoint subsets. □

Lemma 3. *Let $\mathcal{C}$ be a k-coterie. Suppose that Condition (E2) does not hold for some $S \subseteq U$ and $1 \le r \le k$, and let $\mathcal{S}_r$ be an r-partition of S. Then, $\mathcal{D} = Min(\mathcal{C} \cup \mathcal{S}_r)$ is an ℓ-semicoterie for some $k \le \ell \le k + r - 1$.*

Proof. Suppose that (E2) does not hold for S and r. Since $|S| \ge r$ by Lemma 1, let $\mathcal{S}_r$ be any r-partition of S, and define $\mathcal{D} = Min(\mathcal{C} \cup \mathcal{S}_r)$. Then, $\mathcal{D}$ satisfies Minimality by definition. We hence examine Intersection Property in the following.

Let ℓ be the maximum number of pairwise disjoint elements of $\mathcal{D}$ and show $k \le \ell < k + r$. Let $P_1, P_2, \ldots P_k$ be k pairwise disjoint quorums in $\mathcal{C}$. Since either P_i or its proper subset is in $\mathcal{D}$ by definition, $k \le \ell$.

To show $\ell < k + r$, suppose, otherwise, that there are $k + r$ pairwise disjoint quorums $Q_1, Q_2, \ldots, Q_{k+r}$ in $\mathcal{D}$. Since $\mathcal{D}$ is a subset of $\mathcal{C} \cup \mathcal{S}_r$ and since $\mathcal{C}$ is a k-semicoterie, without loss of generality, we may assume that k quorums $Q_1, Q_2, \ldots, Q_k$ belong to $\mathcal{C}$, while the other r quorums $Q_{k+1}, Q_{k+2}, \ldots, Q_{k+r}$ to $\mathcal{S}_r$, i.e., $\mathcal{S}_r = \{Q_{k+1}, Q_{k+2}, \ldots, Q_{k+r}\}$. Since (E2) does not hold, there are at most $k - r$ pairwise disjoint quorums $Q \in \mathcal{C}$ such that $Q \subseteq \overline{S}$, a contradiction, since $Q_i \subseteq \overline{S}$ for all $1 \le i \le k$. □

Lemma 4. *Let $\mathcal{C}$ and $\mathcal{D}$, respectively, be a k and an ℓ-semicoteries, and suppose that for any $P \in \mathcal{C}$, there is a $Q \in \mathcal{D}$ such that $Q \subseteq P$. Then, for all r $(1 \le r \le k)$ and $P \in \mathcal{C}^r$, there is a $Q \in \mathcal{D}^r$ such that $Q \subseteq P$.*

Proof. Let $P = \cup_{i=1}^{r} P_i \in \mathcal{C}^r$, where $P_1, P_2, \ldots, P_r \in \mathcal{C}$ are r pairwise disjoint quorums in $\mathcal{C}$. By assumption, for each P_i $(1 \le i \le r)$, there is a $Q_i \in \mathcal{D}$ such that $Q_i \subseteq P_i$. Since P_is are pairwise disjoint, so are Q_is. Let $Q = \cup_{i=1}^{r} Q_i \in \mathcal{D}^r$. Then, $Q \subseteq P$. □

We are now ready to characterize the r-complemental k-coteries.

Theorem 7. *Let $\mathcal{C}$ be a k-coterie under U. $\mathcal{C}$ is not r-complemental if and only if there is an ℓ-semicoterie $\mathcal{D}$ under U satisfying both of the following two conditions, where $k \le \ell \le k + r - 1$:*

1. *(F1) For any $P \in \mathcal{C}$, $Q \subseteq P$ for some $Q \in \mathcal{D}$.*
2. *(F2) For some $S \in \mathcal{D}^r \setminus \mathcal{C}^r$, $Q \not\subseteq \overline{S}$ for all $Q \in \mathcal{D}^{k-r+1}$.*

Proof. Only if part: Since $\mathcal{C}$ is not r-complemental, neither of (E1) and (E2) holds for some $S \subseteq U$. By Lemma 2, there is an r-partition $\mathcal{S}_r$ of S such that $P \not\subseteq S'$ for all $P \in \mathcal{C}$ and $S' \in \mathcal{S}_r$. Define $\mathcal{D} = Min(\mathcal{C} \cup \mathcal{S}_r)$. Then, by Lemma 3, $\mathcal{D}$ is an ℓ-semicoterie under U, where $k \le \ell \le k + r - 1$. In what follows, we show that $\mathcal{D}$ satisfies both (F1) and (F2).

The fact that (F1) holds is obvious from the definition of $\mathcal{D}$. As for (F2), since $P \not\subseteq S'$ for all $P \in \mathcal{C}$ and $S' \in \mathcal{S}_r$, $\mathcal{S}_r \subseteq \mathcal{D}$ and, hence, $S \in \mathcal{D}^r$. Since (E1) does not hold for S, $S \notin \mathcal{C}^r$, i.e., $S \in \mathcal{D}^r \setminus \mathcal{C}^r$. To see that (F2) holds for S, if $Q \subseteq \overline{S}$ for some $Q \in \mathcal{D}^{k-r+1}$, then there would be $k - r + 1$ pairwise disjoint quorums $P_1, P_2, \ldots, P_{k-r+1} \in \mathcal{C}$ such that $P_i \subseteq \overline{S}$ for all $1 \le i \le k - r + 1$, a contradiction, since (E2) does not hold for S.

If part: Let $S \subseteq U$ and assume that (F1) and (F2) hold for S. We show that neither of (E1) and (E2) holds for S. Assume first, otherwise, that (E1) holds for S, i.e., $P \subseteq S$ for some $P \in \mathcal{C}^r$. Since (F2) holds for S, $S \notin \mathcal{C}^r$ and, hence, $P \subset S$. On the other hand, $Q \subseteq P$ for some $Q \in \mathcal{D}^r$ by (F1) and Lemma 4. Thus, $Q \subset S \in \mathcal{D}^r$, which contradicts to Minimality of $\mathcal{D}^r$.

As for (E2), again by (F1) and Lemma 4, there is a $Q \in \mathcal{D}^{k-r+1}$ such that $Q \subseteq P$, for any $P \in \mathcal{C}^{k-r+1}$. Since $Q \not\subseteq \overline{S}$ for any $Q \in \mathcal{D}^{k-r+1}$, $P \not\subseteq \overline{S}$ for any $P \in \mathcal{C}^{k-r+1}$, i.e., (E2) does not hold for S. □

Example 3. Consider again 3-coterie $\mathcal{C}$ in Example 1 and let us observe that $\mathcal{C}$ is not 2-complemental by using Theorem 7. Based on the procedure in the proof of Theorem 7, we illustrate how to construct ℓ-semicoterie $\mathcal{D}$ satisfying (F1) and (F2) from $\mathcal{C}$.

Let $S = \{4, 5, 6, 7, 8\}$ and $r = 2$. Clearly, neither of (E1) and (E2) hold for them. Lemma 2 then guarantees that there is a 2-partition $\mathcal{S}_2$ of S such that $P \not\subseteq S'$ holds for all $P \in \mathcal{C}$ and $S' \in \mathcal{S}_2$. For instance, let us select $\mathcal{S}_2 = \{\{4, 5\}, \{6, 7, 8\}\}$. Then,

$$\begin{aligned}\mathcal{D} &= Min(\mathcal{C} \cup \mathcal{S}_2) \\ &= \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{1,8\},\{2,3\}, \\ &\quad \{2,4\},\{2,5\},\{2,6\},\{2,7\},\{2,8\},\{3,4\},\{3,5\},\{3,6\}, \\ &\quad \{3,7\},\{3,8\},\{4,5\},\{4,6,7\},\{4,6,8\},\{4,7,8\}, \\ &\quad \{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}\},\end{aligned}$$

which is a 4-semicoterie. Clearly, (F1) holds. Moreover, (F2) holds since $S \in \mathcal{D}^2 \setminus \mathcal{C}^2$ and $\overline{S} = \{1, 2, 3\}$ contains no two quorums of $\mathcal{D}$ disjoint each other.

6 Conclusion

In this paper, we investigated the robustness of *k*-coteries against network 2-partition. We first introduced the concept of complemental *k*-coterie. Intuitively the *k*-mutual exclusion algorithm, an outline of which we illustrated in Section 1, allows k processes to enter the critical section even if network 2-partition occurs, when it adopts a complemental *k*-coterie. We then showed that the *k*-coteries of some classes are complemental, and completely characterized the complemental *k*-coterie.

As final remarks, we would like to touch several open questions. First, when network 2-partition occurs, some of the quorums of a complemental *k*-coterie may be partitioned and become useless. As a result, Nonintersection Property can be invalidated, although Intersection Property remains to hold. That is, the network 2-partition can weaken the complemental *k*-coterie to a complemental *k*-semicoterie. We leave as a future work the problem of constructing a *k*-coterie such that both Nonintersection and Intersection Properties hold, in spite of network 2-partition. Second, as mentioned in Section 2, this paper has assumed that the underlying communication network is complete, with a justification that the *k*-mutual exclusion algorithm cannot select its execution environment, i.e., the underlying communication network. It is, however, also true that if we are allowed to design a *k*-mutual exclusion algorithm for a fixed communication network, then we may be able to come up with a new property of *k*-coteries as a generalization of the complementalness. Finally, an obvious open problem is an investigation of a network ℓ-partition for $\ell \geq 3$.

Acknowledgments

The authors are grateful to the anonymous referees for their careful reviews of the paper and many helpful comments and suggestions. An extended abstract of this paper appears in the *Proceedings of the Sixth International Conference on Principles of Distributed Systems* [8]. The authors gratefully acknowledge the partial support of the Scientific Grant in Aid by the Ministry of Education, Science, Sports, Culture, and Technology of Japan.

References

[1] D. Agrawal, O. Egecioglu, and A. El Abbadi, "Analysis of Quorum-Based Protocols for Distributed (k+1)- Exclusion," *IEEE Trans. Parallel and Distributed Systems,* vol. 8, no. 5, pp. 533-537, May 1997.

[2] D. Barbara and H. Garcia-Molina, "The Vulnerability of Vote Assignments," *ACM Trans. Computer Systems,* vol. 4, no. 3, pp. 187-213, Aug. 1986.

[3] Y.I. Chang and B.H. Chen, "A Generalized Grid Quorum Strategy for *k*-Mutual Exclusion in Distributed Systems," *Information Processing Letters,* vol. 80, no. 4, pp. 205-212, Nov. 2001.

[4] S. Fujita, M. Yamashita, and T. Ae, "Distributed *k*-Mutual Exclusion Problem and *k*-Coteries," *Lecture Notes in Computer Science,* vol. 557, pp. 22-31, 1991.

[5] H. Garcia-Molina and D. Barbara, "How to Assign Votes in a Distributed Systems," *J. ACM,* vol. 32, no. 4, pp. 841-860, Oct. 1985.

[6] S.T. Huang, J.R. Jiang, and Y.C. Kuo, "K-Coteries for Fault-Tolerant k Entries to a Critical Section," *Proc. 13th Int'l Conf. Distributed Computing Systems,* pp. 74-81, May 1993.

[7] T. Harada and M. Yamashita, "Coterie Join Operation and Tree Structured *k*-Coteries," *IEEE Trans. Parallel and Distributed Systems,* vol. 12, no. 9, pp. 865-874, Sept. 2001.

[8] T. Harada and M. Yamashita, "*k*-Coteries for Tolerating Network 2-Partition," *Proc. Sixth Int'l Conf. Principles of Distributed Systems,* pp. 119-126, Dec. 2002.

[9] J.R. Jiang and S.T. Huang, "Obtaining Nondominated K-Coteries for Fault-Tolerant Distributed K-Mutual Exclusion," *Proc. Int'l Conf. Parallel and Distributed Systems,* 1994.

[10] H. Kakugawa, S. Fujita, M. Yamashita, T. Ae, "Availability of *k*-Coterie," *IEEE Trans. Computers,* vol. 42, no. 5, pp. 553-558, May 1993.

[11] H. Kakugawa, S. Fujita, M. Yamashita, and T. Ae, "A Distributed *k*-Mutual Exclusion Algorithm Using *k*-Coterie," *Information Processing Letters,* vol. 49, no. 4, pp. 213-218, Feb. 1994.

[12] Y.C. Kuo and S.T. Huang, "A Geometric Approach for Constructing Coteries and *k*-Coteries," *IEEE Trans. Parallel and Distributed Systems,* vol. 8, no. 4, pp. 402-411, Apr. 1997.

[13] M.L. Neilsen and M. Mizuno, "Nondominated *k*-Coteries for Multiple Mutual Exclusion," *Information Processing Letters,* vol. 50, no. 5, pp. 247-252, June 1994.

[14] M.L. Neilsen and M. Mizuno, "Erratum to 'Nondominated *k*-Coteries for Multiple Mutual Exclusion'," *Information Processing Letters,* vol. 60, no. 6, p. 319, Dec. 1996.

[15] K. Raymond, "A Distributed Algorithm for Multiple Entries to a Critical Section," *Information Processing Letters,* vol. 30, no. 4, pp. 189-193, Feb. 1989.

[16] P.K. Srimani and R.L.N. Reddy, "Another Distributed Algorithm for Multiple Entries to a Critical Section," *Information Processing Letters,* vol. 41, no. 1, pp. 51-57, Jan. 1992.

[17] S.M. Yuan and H.M. Chang, "Comments on 'Availability of *k*-Coterie'," *IEEE Trans. Computers,* vol. 43, no. 12, p. 1457, Dec. 1993.

Takashi Harada received the BEng and MEng degrees from Hiroshima University, Japan, in 1983 and 1985, respectively, and the DrEng degree from Kyushu University, Japan, in 2000. From 1986 to 2000, he was a research associate in the Information Processing Center at Hiroshima University. He is currently an associate professor in the Graduate School of Management at Hiroshima University. His research interests include distributed computing, fault-tolerant computing, and communication networks. He is a member of the ACM, the IEEE Computer Society, the IEICE of Japan, and the IPS of Japan.

Masafumi Yamashita received the BE and ME degrees from Kyoto University, Kyoto, Japan, in 1974 and 1977, respectively. He received the DrEng degree from Nagoya University, Nagoya, Japan, in 1981. From 1980 to 1985, he was with the Department of Information and Computer Sciences, Toyohashi University of Technology, Aichi, Japan. In 1985, he joined the faculty of engineering at Hiroshima University as an associate professor, and was a professor from 1992 to 1998. Since 1998, he has been a professor in the Graduate School of Information Science and Electrical Engineering, Kyushu University, Fukuoka, Japan. He has held visiting appointments several times with Simon Fraser University, B.C., Canada, and the University of Wisconsin-Milwaukee, WI. His research interest includes distributed algorithms/systems, parallel algorithms, and cooperative systems. He is a member of the Institute of Electronics, the Information and Communication Engineers of Japan, the Information Processing Society of Japan, the SIAM Japan, the IEEE Computer Society, and the ACM.