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TOROIDAL SURGERIES ON HYPERBOLIC KNOTS, II *

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Abstract. For a hyperbolic knot K in S^3, a toroidal surgery on K is integral or half-integral. In the previous paper, we proved that all integers occur among the toroidal slopes of hyperbolic knots. Hence there is no universal upper bound for toroidal slopes, generally. We propose an upper bound in terms of genera of knots, and we show that this is the case for two special but important classes, i.e., alternating knots and genus one knots.

1. Introduction. Let K be a knot in the 3-sphere S^3, and let $E(K) = S^3 - \mathrm{Int}N(K)$ be its exterior. A *slope* on $\partial E(K)$ is the isotopy class of an essential unoriented simple loop. As usual [20], the set of slopes on $\partial E(K)$ is parameterized by $\mathbb{Q} \cup \{\infty\}$. For a slope r on $\partial E(K)$, $K(r)$ denotes the closed orientable 3-manifold obtained by *r-Dehn surgery* on K.

For a hyperbolic knot K, $K(r)$ is hyperbolic for all but finitely many r [24], which are referred to as *exceptional slopes.* A closed 3-manifold is *toroidal* if it contains an incompressible torus. If $K(r)$ is toroidal, the surgery (or the slope) is said to be *toroidal.* An exceptional slope is conjectured to yield either a toroidal manifold or a Seifert fibered manifold.

Gordon and Luecke [14] showed that if $K(m/n)$ is toroidal then $|n| \leq 2$. In other words, a toroidal slope on a hyperbolic knot is either an integer or a half-integer. In the previous paper [23], we showed that every integer can be a toroidal slope for some hyperbolic knot. Thus there is no universal upper bound for toroidal slopes, but Ichihara [18] recently showed that a toroidal slope r on a hyperbolic knot K satisfies the inequality $|r| < 3 \cdot 2^{7/4} g(K)$, where $g(K)$ denotes the genus of K.

By an inspection of the known examples, we propose the following conjecture:

Conjecture 1.1. *If a hyperbolic knot K in S^3 has a toroidal slope r, then $|r| \leq 4g(K)$.*

In this paper, we prove this conjecture for two special but important classes of hyperbolic knots, i.e., alternating knots and genus one knots. In fact, we prove slightly stronger conclusions in both cases.

Theorem 1.2. *Let K be a genus one hyperbolic knot. If r is toroidal, then $|r| = 0, 1, 2$ or 4. Furthermore, if $|r| = 2$ or 4, then $K(r)$ contains an incompressible torus meeting the attached solid torus in two meridian disks. Also, if $|r| = 4$, then K is a twist knot and it bounds a once-punctured Klein bottle whose boundary slope is r.*

In fact, the slopes $0, 2$ and 4 can be realized. For example, 0 and 4 are toroidal slopes for the figure-eight knot, and 2 is such one for 9_{46} in the knot table [6, 17]. But we do not know whether 1 can be realized or not.

Theorem 1.3. *Let K be an alternating hyperbolic knot. If r is toroidal, then $|r| \leq 4g(K)$. Furthermore, if the equality holds, then K bounds a once-punctured Klein bottle whose boundary slope is r.*

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In fact, there are examples of alternating hyperbolic knots showing this bound is best possible for each genus. The simplest one is the figure-eight knot again. It bounds a once-punctured Klein bottle whose boundary slope is ± 4. See Section 3.

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2. Genus one case. In this section, we prove Theorem 1.2.

Let K be a hyperbolic knot of genus one in S^3, and let S be a minimal genus Seifert surface of K properly embedded in $E(K)$. Suppose that $K(r)$ is toroidal for a slope r.

LEMMA 2.1. *r is an integer.*

Proof. If not, then r is a half-integer by [14]. Then K is one of Eudave-Muñoz's knots [16], and in particular, K has tunnel number one (see [13]). Scharlemann [21] showed that a genus one, tunnel number one, hyperbolic knot is 2-bridge. Therefore K is 2-bridge. This contradicts the fact that 2-bridge knots admit only integral toroidal surgeries [1, 3]. □

It is known that 0 is toroidal [8]. For our purpose, we may assume that $r > 0$.

LEMMA 2.2. $r \leq 5$.

Proof. By [12], the distance between two toroidal slopes is at most 5 except four specific manifolds. Among those four manifolds, the figure-eight knot exterior is the only case of knot exteriors in S^3. But it has exactly three toroidal slopes $-4, 0, 4$. Thus we have $r \leq 5$. □

Let $\widehat{T}$ be an incompressible torus in $K(r)$. Since $r > 0$, $\widehat{T}$ is separating. We can assume that $\widehat{T}$ intersects the attached solid torus V in a disjoint union of meridian disks. Let $|\widehat{T} \cap V| = t$. Since K is hyperbolic, $t > 0$, and also t is even. We choose $\widehat{T}$ so that t is minimal among all incompressible tori in $K(r)$. Let $T = \widehat{T} \cap E(K)$. Then it is a punctured torus properly embedded in $E(K)$ with t boundary components, each having slope r on $\partial E(K)$. Also T is incompressible in $E(K)$ by the minimality of t.

We can isotope T so that $S \cap T$ consists of loops and arcs. Since the cabling conjecture is true for genus one knots [2], $K(r)$ is irreducible. Then we can assume that no loop component of $S \cap T$ bounds a disk in S or T by the incompressibility of S and T. Furthermore, we can assume that ∂S intersects each boundary component of T in r points.

As usual ([4, 14]), the arc components of $S \cap T$ define a pair of graphs G_S and G_T on $\widehat{S}$, the capped-off surface of S, and $\widehat{T}$, respectively. Then G_S has only one (fat) vertex with valency rt and the arcs of $S \cap T$ give the edges. After labelling the boundary components of T by $1, 2, \dots, t$ in the order in which they appear on $\partial E(K)$, the endpoints of edges in G_S are labelled by $1, 2, \dots, t$, and this sequence is repeated r times around the vertex of G_S. Similarly G_T has t vertices with valency r, but there is no label. We denote by $V_{i,i+1}$ the part of V running from the component i to $i+1$ of ∂T. (When $t = 2$, $V = V_{1,2} \cup V_{2,1}$.)

Since S and T are boundary-incompressible, neither graph has trivial loops. The graph pair satisfies the parity rule [4]. It can be stated as follows in our setting: any edge of G_S has the labels with distinct parities at its endpoints. In other words, any edge of G_T connects two vertices with distinct parities.

In G_S, we can define a Scharlemann cycle, an extended Scharlemann cycle as in [14]. A (extended) Scharlemann cycle of length 2 is called an (extended) S-cycle for short. Also, a Scharlemann cycle with label pair $\{x,y\}$ is called a (x,y)-Scharlemann cycle. Finally, $\widehat{T}$ divides $K(r)$ into black and white sides, since $r>0$. Thus the disk faces of G_S are divided into black faces and white faces, according to whether the faces lie in the black or white side of $\widehat{T}$. (Note that G_S may have an annular face, which may contain essential loops of $S\cap T$.)

2.1. Case $t \geq 4$.

LEMMA 2.3.
1. *G_S cannot contain an extended S-cycle.*
2. *G_S cannot have more than $t/2+2$ mutually parallel edges. If there is $t/2+2$ mutually parallel edges, then the family contains two S-cycles with disjoint label pairs and $t \equiv 0 \pmod 4$.*
3. *G_S cannot contain three S-cycles with mutually disjoint label pairs.*

Proof. (1) is Lemma 2.10 in [2]. (2) is Lemma 1.4 and Corollary 1.8 in [25]. (3) is Lemma 1.10 in [25]. □

Recall that the edges of G_S are divided into at most three families of mutually parallel edges by [12, Lemma 5.1]. Let A, B and C be such three families, and let $|A|$ denote the number of edges in A, etc.

LEMMA 2.4. *If r is even, then G_S cannot contain three mutually parallel edges.*

Proof. Suppose that a family A, say, contains more than two edges. Since $|A|+|B|+|C|=rt/2$ is a multiple of t, the first and the last edge of A have a common label (at different sides of A). If $|A|$ is odd, then the middle edge of A has the same label on both endpoints, contradicting the parity rule. Thus $|A|\geq 4$, and hence A contains an extended S-cycle, which is impossible by Lemma 2.3(1). □

LEMMA 2.5. *r is odd.*

Proof. Assume not. By Lemma 2.4, $rt/2=|A|+|B|+|C|\leq 6$, and so $r\leq 3$. Thus $r=2$.

If G_S contains only one family of mutually parallel edges, then it contains an extended S-cycle, a contradiction. If G_S consists of two families A and B, then $t=4$ and $|A|=|B|=2$ by Lemma 2.4, since $|A|+|B|=t$. Then we may assume that G_S contains a $(1,2)$ S-cycle σ_1 and a $(3,4)$ S-cycle σ_2. Since the edges of an S-cycle does not lie in a disk on $\widehat{T}$ [14, Lemma 3.1] (recall that $K(r)$ is irreducible), the edges of σ_i form two disjoint essential loops on $\widehat{T}$ after shrinking the fat vertices into points. Let f_i be the face of σ_i for $i=1,2$. Shrinking $V_{1,2}$ to its core in $V_{1,2}\cup f_1$ gives a Möbius band B_1 such that ∂B_1 is the loop on $\widehat{T}$ formed by the edges of σ_1. Similarly we have another Möbius band B_2 from $V_{3,4}\cup f_2$. Since ∂B_1 and ∂B_2 bound an annulus R on $\widehat{T}$, the union of B_1, B_2 and R gives a Klein bottle in $K(r)$ meeting V in two meridian disks. (See the proof of Lemma 3.10 in [14].) This implies that $E(K)$ contains a twice-punctured Klein bottle. But it yields a closed non-orientable surface in $E(K)$ by attaching an annulus in $\partial E(K)$ to it. This is absurd.

Hence G_S consists of three families of mutually parallel edges. Since $t=|A|+|B|+|C|\leq 6$, we have $t=4$ or 6 by Lemma 2.4.

If $t=6$, then $|A|=|B|=|C|=2$. Then G_S contains three S-cycles with mutually disjoint label pairs, which is impossible by Lemma 2.3(3). Hence $t=4$. We may assume that $|A|=|B|=1$ and $|C|=2$. Then all faces of G_S are disks, and hence

$S \cap T$ contains no loop component. But this configuration contradicts the black-white coloring of the faces. Thus the case $t = 4$ is also impossible. □

LEMMA 2.6. *G_S cannot contain more than $t/2$ mutually parallel edges.*

Proof. Assume $|A| > t/2$. By Lemma 2.3(2), $|A| = t/2 + 1$ or $t/2 + 2$. If $|A| = t/2 + 1$, then the last edge of A has the same label at both endpoints by the symmetry of labels. This is impossible by the parity rule. Hence $|A| = t/2 + 2$, and so $t \equiv 0 \pmod 4$ by Lemma 2.3(2). Then the first two edges and the last two edges of A form S-cycles with disjoint label pairs. The edges of the two S-cycles form disjoint essential loops on $\widehat{T}$ after shrinking fat vertices into points. The torus $\widehat{T}$ is divided into two annuli F_1, F_2 by the two loops. The other $t - 4$ fat vertices of G_T make $(t-4)/2$ pairs corresponding to the edges of A. Thus each annulus F_i contains an even number of fat vertices in its interior. Then we have a Klein bottle in $K(r)$ meeting V in an even number of meridian disks as in the proof of Lemma 2.5. This implies that $E(K)$ contains a Klein bottle punctured an even number of times. It yields a closed non-orientable surface in $E(K)$ by attaching suitable annuli on $\partial E(K)$, a contradiction. □

PROPOSITION 2.7. *If $t \geq 4$, then $r = 1$.*

Proof. By Lemma 2.6, $rt/2 = |A| + |B| + |C| \leq 3t/2$, and hence $r \leq 3$. Thus $r = 1$ or 3.

Suppose $r = 3$. Then G_S contains three families A, B, C of mutually parallel edges, and $|A| = |B| = |C| = t/2$. But G_S contains an extended Scharlemann cycle of length 3, which is impossible by [14, Theorem 3.2]. (Note that [14, p.610] uses the fact $K(r)$ does not contain a Klein bottle. But the arguments there work for our situation, because the fat vertices of G_T make $t/2$ pairs as in the proof of Lemma 2.6.) □

2.2. Case $t = 2$. Let u_1 and u_2 be the vertices of G_T. By the parity rule, each edge of G_T connects different vertices u_1 and u_2. Then there are four *edge classes* in G_T, i.e., isotopy classes of edges of G_T in $\widehat{T}$ rel $u_1 \cup u_2$ (see [14, Figure 7.1]). We label an edge e of G_S by the class of the corresponding edge of G_T, and we call the label the *edge class label* of e.

LEMMA 2.8. *Two parallel edges in G_S have distinct edge class labels.*

Proof. If not, there are two edges which are parallel in both G_S and G_T. Then $E(K)$ would be cabled [12, Lemma 2.1]. □

PROPOSITION 2.9. *If $r = 4$, then K is a twist knot and it bounds a once-punctured Klein bottle whose boundary slope is r.*

Proof. Note that G_S has exactly four edges. By the possibility of coloring of faces, there are two possible configurations for G_S as shown in Figure 1 (after a homeomorphism of S). Here, the edges of the square are identified to form a torus in the usual way.

In the former case, the edges have mutually distinct edge class label by Lemma 2.8. Then the proof of Lemma 5.2 in [15] shows that $K(r)$ contains a Klein bottle. The conclusion follows from [22, Theorem 1.2].

In the latter case, we may assume that G_S contains two black bigons. Unless these two black bigons have the same pair of edge class labels, $K(r)$ contains a Klein

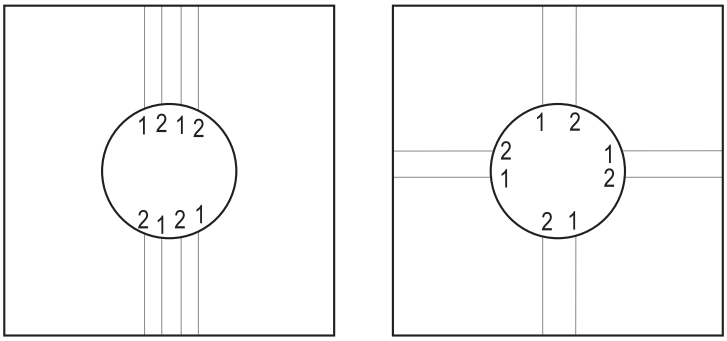

FIG. 1.

bottle again. Hence we suppose that the two black bigons have the same pair of edge class labels. Since all faces of G_S are disks, $S \cap T$ contains no loop component.

Now G_S contains a black S-cycle σ_1 and a white Scharlemann cycle σ_2 of length 4. Furthermore, all edges of these Scharlemann cycles lie in an essential annulus R on $\widehat{T}$. Let f_i be the face of σ_i for $i = 1, 2$, and let $M = N(R \cup V \cup f_1 \cup f_2)$. Then we can see that M is not a solid torus by calculating its first homology group. Also it is easy to show that R is essential in M, and both sides of R are irreducible. Hence M is irreducible, and so ∂M is incompressible in M. Since it is disjoint from V, it is compressible in $K(r)$ by the minimality of t. Thus ∂M is compressible in $E(K)$. It follows that $M' = \mathrm{cl}(K(r) - M)$ is a solid torus. Therefore $M' \cup N(f_2)$ is a handlebody of genus two, because we can regard $N(f_2)$ as a 1-handle attached to the solid torus M'.

Let γ be an arc on f_1 connecting the two arcs $f_1 \cap V$. Since $N(\gamma) = N(f_1)$ and $N(R \cup V \cup f_1) \cong N(V \cup f_1)$, $\mathrm{cl}(E(K) - N(\gamma)) \cong \mathrm{cl}(K(r) - N(V \cup f_1)) \cong M' \cup N(f_2)$. Thus K has an unknotting tunnel γ which lies in S. Then K is 2-bridge [11, Lemma 6.1]. According to [3], genus one 2-bridge knots with toroidal slope 4 are twist knots, and 4 is the boundary slope of a once-punctured Klein bottle bounded by such a knot. This completes the proof. □

LEMMA 2.10. *Assume* $r \neq 4$. *Then* G_S *cannot have more than three mutually parallel edges.*

Proof. Suppose that there are four mutually parallel edges. Then there are two bigons with the same color among these edges. If these bigons have distinct pairs of edge class labels, then $K(r)$ contains a Klein bottle as before and $r = 4$ by [22]. Thus the two bigons have the same pair of edge class labels. This implies that there are two parallel edges with the same edge class label, which contradicts Lemma 2.8. □

Let F be a family of mutually parallel edges in G_S.

LEMMA 2.11. *Assume* r *is odd. If* F *is not empty, then* $|F| = 1$ *or* 3.

Proof. By Lemma 2.10, $|F| \leq 3$. If $|F| = 2$, then each edge of F has the same label at its endpoints by the symmetry of labels. This is impossible by the parity rule. □

LEMMA 2.12. $r = 3$ *is impossible.*

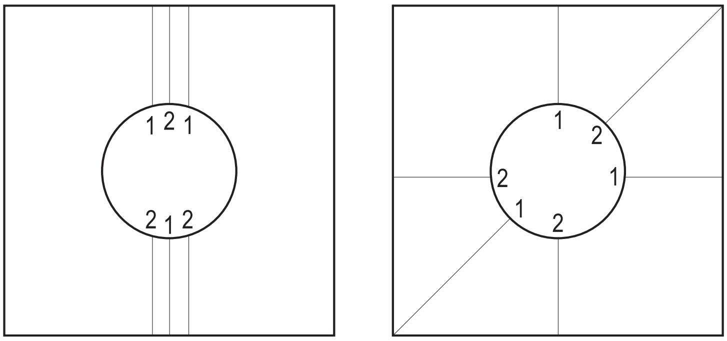

FIG. 2.

Proof. Note that G_S has just 3 edges. By Lemma 2.11 and the possibility of coloring of faces, there are only two possibilities for G_S as shown in Figure 2.

For the first case of Figure 2, $S \cap T$ contains loops, because of the coloring of the faces of G_S. Recall that such a loop does not bound a disk in either S or T. On the other hand, the edges have mutually distinct edge class labels by Lemma 2.8. Hence any loop of $S \cap T$ bounds a disk in T, a contradiction.

For the second case, G_S contains a black Scharlemann cycle τ_1 and a white Scharlemann cycle τ_2, both of length 3. The edges of τ_1 (and hence τ_2) lie in an essential annulus R on $\widehat{T}$ ([14, Lemma 3.7]). Let f_i be the face of τ_i, and let $M = N(R \cup V \cup f_1 \cup f_2)$. We may assume that R divides M into two pieces M_1 and M_2, where $f_i \subset M_i$. By [14, Lemma 3.7], M_i is a solid torus and the core of R runs three times along the core of M_i for $i = 1, 2$. Then M is a Seifert fibered manifold over the disk with two exceptional fibers of index 3. In particular, ∂M is incompressible in M. Then, as in the proof of Proposition 2.9, the complement $M' = \mathrm{cl}(K(r) - M)$ is a solid torus. Hence $K(r)$ is a Seifert fibered manifold over the 2-sphere with at most three exceptional fibers. (Recall that $K(r)$ is irreducible.) But this contradicts that $K(r)$ is toroidal. □

LEMMA 2.13. $r = 5$ *is impossible.*

Proof. By Lemmas 2.11 and the possibility of coloring of faces, there is only one possibility for G_S as shown in Figure 3. Since r is integral, the points of $\partial S \cap u_i$ appear successively on both ∂S and u_i for $i = 1, 2$. By using this observation, we can determine G_T as shown in Figure 3. Then G_S contains a black S-cycle σ and a white Scharlemann cycle τ of length 3, such that the edges of σ and τ are contained in the same essential annulus on $\widehat{T}$. Then the argument in the proof of Lemma 2.12 works. □

Proof. [Proof of Theorem 1.2] By Lemmas 2.1, 2.2, 2.12 and 2.13, and Proposition 2.7, we have $|r| = 0, 1, 2$ or 4. Also, if $|r| = 2$ or 4, then $t = 2$ by Proposition 2.7. Finally, if $|r| = 4$, then we have the desired conclusion by Proposition 2.9. □

3. Alternating case. In this section, we prove Theorem 1.3. Let K be an alternating hyperbolic knot, and let r be a toroidal slope.

Proof. [Proof of Theorem 1.3] By Lemmas 3.1 and 3.3 of [1], r is an integer divisible by 4, and K is either a 2-bridge knot or a pretzel knot of type $(1/q_1, 1/q_2, 1/q_3)$,

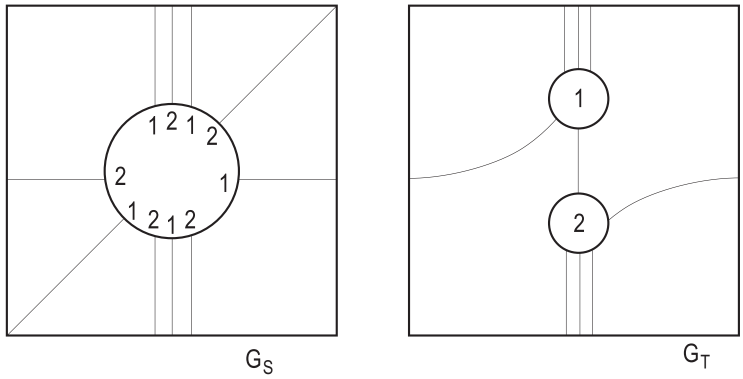

FIG. 3.

where all $q_i \geq 2$. (See also [19].) When K is 2-bridge, [3] shows that K is either genus one or of the form $[b_1, b_2]$ with $|b_1|, |b_2| > 2$. In either case, we see that $|r| \leq 4g(K)$, and that if $r = 4g(K)$ then K bounds a once-punctured Klein bottle.

Next assume that K is a pretzel knot of type $(1/q_1, 1/q_2, 1/q_3)$. If all q_i are odd, then K has genus one, and 0 is the only toroidal slope [1, Lemma 3.3]. If one of q_i, q_1 say, is even, then K bounds a once-punctured Klein bottle whose boundary slope is $2(q_2 + q_3)$, as a checkerboard surface in the standard diagram. Let $s = 2(q_2 + q_3)$. By [9, 10], K has genus $s/4$, that is, $s = 4g(K)$.

CLAIM 3.1. *$K(s)$ is toroidal.*

Proof. [Proof of Claim 3.1] Since $K(s)$ contains a Klein bottle, there are three possibilities for $K(s)$: toroidal, reducible, or a Seifert fibered manifold with finite fundamental group (more precisely, a prism manifold). First, $K(s)$ is irreducible, because K is strongly invertible ([7]). Also non-trivial surgery on an alternating knot yields a manifold with infinite fundamental group [5]. Therefore $K(s)$ must be toroidal. □

Since s is the only toroidal slope of K [1, Lemma 3.3], this completes the proof of Theorem 1.3. □

There are examples of alternating hyperbolic knots showing that the estimate of Theorem 1.3 is best possible for each genus. In case of genus one, twist knots give such examples. For $g \geq 2$, the 2-bridge knot $[b_1, 2g]$ ($b_1 \geq 3$, odd) has genus g and a toroidal slope $4g$ ([3]). Also it bounds a once-punctured Klein bottle whose boundary slope is $4g$.

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