Veering structures of the canonical decompositions of hyperbolic fibered two-bridge link complements

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ABSTRACT

In his previous work, the author proved that the canonical decompositions of hyperbolic fibered two-bridge link complements are layered. This implies that they admit taut structures. In this paper, we completely determine, for each hyperbolic fibered two-bridge link, whether the canonical decomposition of its complement is veering with respect to the taut structure.

Keywords: Two-bridge link; layered triangulation; hyperbolic 3-manifold; veering triangulation; canonical decomposition.

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1. Introduction

In [1], Agol has shown that every punctured surface bundle over S^1 with pseudo-Anosov monodromy, such that every complementary region of the stable lamination contains a puncture, admits a unique "veering" and layered ideal triangulation (see [1, Proposition 4.2]). He posed the following question: Are the veering ideal triangulations geometric, i.e. realized in the complete hyperbolic structure with all tetrahedra positively oriented? In [10] and [5], it is shown that veering triangulations admit strict angle structures, which is a necessary condition for an ideal triangulation to be geometric. However, Hodgson, Issa and Segerman [9] found non-geometric veering ideal triangulations through computer experiments.

In this paper, we consider the following natural problem: Which canonical decompositions (in the sense of Epstein-Penner [4] and Weeks [15]) are veering? For example, it is well-known that the canonical decompositions of once-punctured torus bundles over S^1 are veering and layered (cf. [7] and [2]). In this paper, we focus on the canonical decompositions of hyperbolic fibered two-bridge link complements. In the author's previous work [13], we have shown that these canonical decompositions are layered by using A'Campo's idea. The main purpose of this paper is

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to prove the following theorem, which completely determine, for each hyperbolic fibered two-bridge link, whether the canonical decomposition of its complement is veering.

Theorem 1.1. *The canonical decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0 < |r| < 1/2)$ is veering if and only if the slope r has the continued fraction expansion $\pm[2, 2, \ldots, 2]$.*

For the precise meaning of this theorem, see Theorem 5.7. To prove this theorem, we give a visual method for checking if a triangulation which satisfies A'Campo's criterion (Theorem 2.4) is veering in terms of the induced cusp triangulation (Proposition 2.9). We note that Hodgson, Issa and Segerman [9] have given an effective computer-based method for detecting if a given triangulation is veering (cf. [9, Appendix B]).

Both this work and the author's previous work [13] are part of the author's project which aims at extending the result of Dicks and Sakuma [3], on the relation between the Cannon-Thurston fractal tessellations and the canonical decompositions of punctured torus bundles, to hyperbolic fibered two-bridge links. In [8], Guéritaud has established a beautiful relation between Agol's veering triangulations of hyperbolic punctured surface bundles and the associated Cannon-Thurston maps. As a consequence of Guéritaud's result and Theorem 1.1, it follows that the Cannon-Thurston fractal tessellation and the canonical decomposition of the complement of the two-bridge link $K(r)$ with $r = \pm[2, 2, \ldots, 2]$ are intimately related.

This paper is organized as follows. In Section 2, we recall the notion of veering and layered triangulations, and give a method for checking if a triangulation which satisfies A'Campo's criterion is veering (Proposition 2.9). In Section 3, we review some properties of two-bridge links. In Section 4, we describe the canonical decompositions and their dual cell complexes of hyperbolic two-bridge link complements. In Section 5, we recall the result of [13] which describes the layered structures of the canonical decompositions of hyperbolic fibered two-bridge link complements. Finally, in Sections 6 and 7, we prove Theorem 1.1.

2. Veering structures of layered triangulations and A'Campo's criterion for detecting fiberedness

Let F be an oriented punctured surface, $h : F \to F$ an orientation-preserving homeomorphism, and $M := F \times \mathbb{R}/(x, t) \sim (h(x), t + 1)$ the F-bundle over S^1 with monodromy h. A *Whitehead move* of an ideal triangulation T of F is an operation which obtains a new ideal triangulation T' of F as follows. Take an ideal edge e which is shared by a pair of two distinct ideal triangles. The union of the ideal triangles forms a quadrilateral and the ideal edge e is one of the diagonals of the quadrilateral. We remove the ideal edge e and replace it with the other diagonal. Then we have a new ideal triangulation T' of F. Note that the Whitehead move

$T \to T'$ is realized by attaching an ideal tetrahedron along the pair of ideal triangles of T sharing e.

If we have a sequence of Whitehead moves $T = T_0 \to T_1 \to \cdots \to T_n = h(T)$, then, by gluing T_0 and T_n together by h, we obtain an ideal triangulation of M consisting of the ideal tetrahedra corresponding to the Whitehead moves. This triangulation is called a *layered triangulation* with respect to the fiber structure. The layered triangulation is said to *have no backtracking* if $T_{i+2} \neq T_i$ for any $0 \leq i \leq n-2$ and $T_{n-1} \neq h(T_1)$. We mainly consider layered triangulations which have no backtracking.

Now we recall Agol's result on the construction of a unique veering ideal triangulation of M when h is pseudo-Anosov.

Definition 2.1 ([8, Section 1.2 Definition])**.** Let $\mathcal{D}$ be an ideal triangulation of (the interior of) a compact oriented 3-manifold with non-empty toral boundary which contains just n ideal tetrahedra. For each ideal tetrahedron, the number of the dihedral angles of the tetrahedron is equal to 6. Then $\mathcal{D}$ has just $6n$ dihedral angles. A map $\varphi : \{6n \text{ dihedral angles}\} \to \{0, \pi\}$ is called a *taut structure* on $\mathcal{D}$ if the following hold:

(1) Each ideal tetrahedron of $\mathcal{D}$ has one pair of opposite angles mapped to π and all other angles mapped to 0 by φ.
(2) Each degree-k ideal edge of $\mathcal{D}$ is adjacent to precisely two angles mapped to π and $(k-2)$ angles mapped to 0 by φ.

We call an ideal triangulation with a taut structure a *taut triangulation*. The value of φ at a dihedral angle of an ideal tetrahedron is also called a "dihedral angle".

A layered triangulation $\mathcal{D}$ admits a natural taut structure as follows. Since the triangulation $\mathcal{D}$ is layered, each ideal tetrahedron has a pair of opposite ideal edges corresponding to a Whitehead move. We define a map φ so that the pair of dihedral angles at the pair of ideal edges are mapped to π and the other angles are mapped to 0. Then the map φ is a taut structure on the layered triangulation $\mathcal{D}$.

Definition 2.2 (cf. [8, Section 1.2 Definition])**.** (1) A taut structure on an ideal triangulation $\mathcal{D}$ of a compact oriented 3-manifold with non-empty toral boundary is said to be *veering* if there exists an assignment of two colors, red and blue, to all ideal edges of $\mathcal{D}$ so that every ideal tetrahedron can be sent by an orientation-preserving homeomorphism to the tetrahedron in Fig. 1(a). In this case, we also say that the ideal triangulation $\mathcal{D}$ (with the taut structure) is *veering*, and we also call the assignment of two colors the *veering structure* of $\mathcal{D}$.

(2) An ideal triangulation $\mathcal{D}$ with a layered structure is said to be *veering* if the taut structure of $\mathcal{D}$ induced by the layered structure is veering.

See [1, Definition 4.1] and [10, Proposition 1.4], for the meaning of the terminology "veering". For an ideal triangulation $\mathcal{D}$ of M with a taut structure, let $\mathcal{T}$
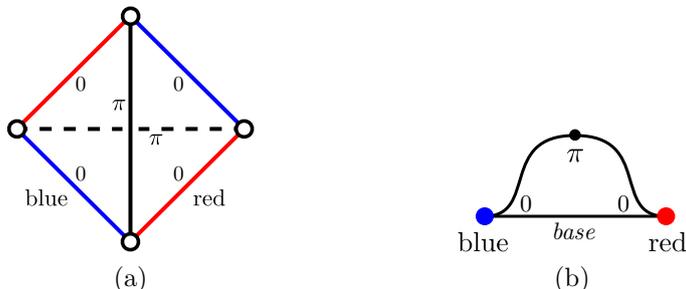
Fig. 1.   (a) An ideal tetrahedron with two colors. The diagonal edges (with angle π) might be any color. (b) A triangle in $\mathcal{T}$ with two colors induced by a veering ideal triangulation. Each triangle in $\mathcal{T}$ has a unique edge of the triangle such that its endpoints have angle 0. We call the edge the *base* of the triangle. Angles 0 and π are indicated by a graphical, train-track-like convention.

be the "angled triangulation" of ∂M induced by $\mathcal{D}$. (Namely, each corner of each triangle has either an angle 0 or π such that, for each vertex of $\mathcal{T}$, the sum of all the angles at the vertex is equal to 2π.) Here we identify ∂M with the link of the ideal vertices, and we assume that ∂M is oriented so that its normal vector points outward. An assignment of two colors to the ideal edges of $\mathcal{D}$ satisfies the condition in Definition 2.2 if and only if it satisfies the following condition:

- *Each (angled) triangle in $\mathcal{T}$ can be sent by an orientation-preserving homeomorphism to the (angled) triangle in Fig. 1(b). Here we assume that each vertex of $\mathcal{T}$ inherits the color of the edge of $\mathcal{D}$ containing it.*

In [1], Agol has shown that every punctured surface bundle over S^1 with pseudo-Anosov monodromy, such that every complementary region of the stable lamination contains a puncture, admits a unique veering layered (topological) ideal triangulation (see [1, Proposition 4.2]).

We now recall A'Campo's criterion for detecting fiberedness (see [13, Theorem 2.2]). Let $\mathcal{D}$ be an ideal triangulation of a compact oriented 3-manifold M with non-empty toral boundary, and let $\mathcal{F} = \mathcal{D}^*$ be the 2-dimensional cell complex *dual* to $\mathcal{D}$. Namely,

(1) Each edge (1-cell) of $\mathcal{F}$ is dual to an ideal triangle δ of $\mathcal{D}$. Namely, the edge joins the pair of (possibly identical) vertices (0-cell) of $\mathcal{F}$ dual to the pair of ideal tetrahedra of $\mathcal{D}$ sharing δ.
(2) Each face (2-cell) f of $\mathcal{F}$ is dual to an ideal edge $e = f^*$ of $\mathcal{D}$. Let δ_1, δ_2, …, δ_n be the ideal triangles of $\mathcal{D}$ sharing e which are arranged around e in this cyclic order. Then the boundary of the face f consists of the edges (1-cells) δ_1^*, δ_2^*, …, δ_n^* of $\mathcal{F}$ dual to δ_1, δ_2, …, δ_n of $\mathcal{D}$.

For an ideal tetrahedron t of $\mathcal{D}$ and an element x of $\mathcal{F}$ such that $x \cap t \neq \emptyset$, we call the closure of a component of $\mathrm{Int}(x) \cap \mathrm{Int}(t)$ a *germ* of the element x in t. It should be noted that each edge of $\mathcal{F}$ is shared by the germs of three faces of $\mathcal{F}$ and

that each vertex of $\mathcal{F}$ is shared by the germs of four edges and six faces of $\mathcal{F}$.

Let $C_*(\mathcal{F}, R) = \{C_i(\mathcal{F}, R)\}_{i \in \mathbb{Z}}$ and $C^*(\mathcal{F}, R) = \{C^i(\mathcal{F}, R)\}_{i \in \mathbb{Z}}$ be the chain complex and the cochain complex of $\mathcal{F}$, respectively, with coefficients in a commutative ring R. If R is the real number field $\mathbb{R}$, we drop the symbol R.

Definition 2.3. (1) A 1-cochain $\omega \in C^1(\mathcal{F})$ is said to be *balanced at a vertex* v of $\mathcal{F}$ if the values of ω at precisely two oriented edges with initial point v are positive and the values of ω at the other oriented edges with initial point v are negative.

(2) A 1-cochain $\omega \in C^1(\mathcal{F})$ is said to be *balanced* if ω is balanced at every vertex of $\mathcal{F}$.

Theorem 2.4 (A'Campo's criterion for detecting fiberedness). *Let $p : M \to S^1$ such that $p^*(1) \in H^1(\mathcal{F}; \mathbb{Z})$ is primitive, where 1 is the generator of $H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. Suppose that there exists a balanced 1-cocycle ω representing $p^*(1)$. Then the following hold:*

(1) There exists a fibration $p_\omega : M \to S^1$ which is homotopic to p.
(2) The ideal triangulation $\mathcal{D}$ of M is layered with respect to the fibration p_ω.

For the proof see [13]. We note that A'Campo's criterion does not necessarily guarantee that $\mathcal{D}$ has no backtracking. A'Campo has also introduced a notion of harmonic 1-cocycle and proved that each cohomology class of $H^1(\mathcal{F}; \mathbb{Z})$ is represented by a unique harmonic 1-cocycle. We call the harmonic 1-cocycle representing $p^*(1)$ the *canonical* 1-cocycle of $\mathcal{F}$. (If M is the exterior $E(K)$ of a hyperbolic link K, and if $\mathcal{D}$ is the canonical decomposition of (the interior of) $E(K)$ in the sense of Epstein-Penner [4] and Weeks [15], then we call the canonical 1-cocycle of $\mathcal{F}$ the canonical 1-cocycle of K.) In [13], we have shown that the canonical 1-cocycles of hyperbolic fibered two-bridge links are balanced, and hence the canonical decompositions of their complements are layered.

Let $\mathcal{D}$ be a layered triangulation of M satisfying A'Campo's criterion. We give a condition for $\mathcal{D}$ (with the natural taut structure) satisfying A'Campo's criterion to be veering. Let ω be the balanced 1-cocycle as in Theorem 2.4, $\mathcal{F}$ the 2-dimensional cell complex dual to $\mathcal{D}$, $\mathcal{T}$ the angled triangulation of ∂M induced by the layered ideal triangulation $\mathcal{D}$, and $\mathcal{T}^*$ the 2-dimensional cell decomposition of ∂M dual to $\mathcal{T}$.

Observation 2.5. There is a cellular map $\gamma : \mathcal{T}^* \to \mathcal{F}$ which satisfies the following conditions:

(1) $\gamma : |\mathcal{T}^*| \to |\mathcal{F}|$ extends to a deformation retraction $|\mathcal{D}| \to |\mathcal{F}|$.
(2) γ maps an i-cell of $\mathcal{T}^*$ to an i-cell of $\mathcal{F}$, and so it induces a map $\gamma^{(i)} : (\mathcal{T}^*)^{(i)} \to \mathcal{F}^{(i)}$ for $i = 0, 1, 2$. Moreover $\gamma^{(2)}$ is 2-1, $\gamma^{(1)}$ is 3-1, and $\gamma^{(0)}$ is 4-1. We say that a face $\widetilde{f}$ (resp. an edge $\widetilde{e}$, a vertex $\widetilde{v}$) of $\mathcal{T}^*$ *corresponds to* a face f (resp. an edge e, a vertex v) of $\mathcal{F}$ if $\gamma^{(2)}(\widetilde{f}) = f$ (resp. $\gamma^{(1)}(\widetilde{e}) = e$, $\gamma^{(0)}(\widetilde{v}) = v$).

Convention 2.6. We orient each edge e of $\mathcal{F}$ so that $\omega(e) > 0$, and orient each edge of $\mathcal{T}^*$ by that of the corresponding edge of $\mathcal{F}$.

For a face f in $\mathcal{F}$ and a vertex v of the boundary ∂f of f, we say that v is *maximal* (resp. *minimal*) in f with respect to ω, if the two edges in ∂f having v as an endpoint are oriented so that v is the terminal (resp. initial) point. Since ω is balanced, for each face f of $\mathcal{F}$, there is a unique vertex which is maximal (resp. minimal) in f with respect to ω (cf. [13, Proof of Theorem 2.2]). Consider a face $\widetilde{f}$ of $\mathcal{T}^*$ corresponding to a face f of $\mathcal{F}$. Then there is a unique vertex $\widetilde{v}_M$ (resp. $\widetilde{v}_m$) of $\mathcal{T}^*$ contained in $\partial \widetilde{f}$ such that $\widetilde{v}_M$ (resp. $\widetilde{v}_m$) corresponds to the maximal (resp. minimal) vertex in f. We also say that $\widetilde{v}_M$ (resp. $\widetilde{v}_m$) is *maximal* (resp. *minimal*) in the face $\widetilde{f}$ of $\mathcal{T}^*$ (with respect to ω). Let $\partial_R(\widetilde{f})$ (resp. $\partial_L(\widetilde{f})$) be the edge path in $\partial \widetilde{f}$ from $\widetilde{v}_m$ to $\widetilde{v}_M$, such that $\partial_R(\widetilde{f})$ (resp. $\partial_L(\widetilde{f})$) is coherent (resp. incoherent) with the orientation of $\partial \widetilde{f}$. (Recall the convention for the orientation of ∂M declared in the paragraph after Definition 2.2.)

For each vertex t^* of $\mathcal{F}$, there is a unique face f_M (resp. f_m) such that t^* is the maximal (resp. minimal) vertex in f_M (resp. f_m) with respect to ω. We call the edge of $\mathcal{D}$ dual to f_M (resp. f_m) the *valley* (resp. *ridge*) in the ideal tetrahedron t dual to t^* (see Fig. 2). By the construction of layered structure in Theorem 2.4 (cf. [13, Section 2]), we see that the dihedral angles of t at the valley and the ridge are π, and the other dihedral angles of t are 0.
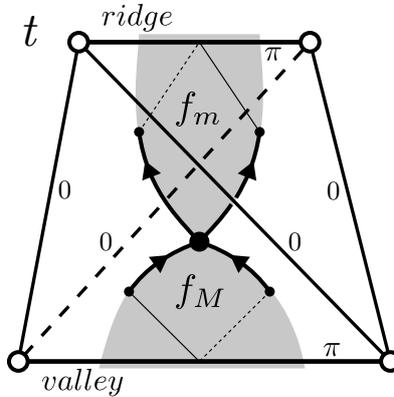
Fig. 2.   An ideal tetrahedron t of $\mathcal{D}$, and the four (germs of) edges of $\mathcal{F}$ incident on the vertex t^* of $\mathcal{F}$ dual to t.

Let Δ be a triangle of $\mathcal{T}$, Δ^* the vertex of $\mathcal{T}^*$ dual to Δ, and t the ideal tetrahedron of $\mathcal{D}$ containing Δ. Note that precisely three (germs of) edges of $\mathcal{T}^*$ contain the vertex Δ^*. We can observe that one of the following conditions holds:

(1) The triangle Δ intersects the ridge of t, and exactly one edge b of $\mathcal{T}^*$ points toward Δ^* (and the remaining two edges point away from Δ^*). The edge b is

dual to the base of Δ. Moreover, if $\mathcal{D}$ is veering, then the right (resp. left) vertex of the base of Δ with respect to the orientation of b is assigned red (resp. blue) color (see Fig. 3(b)).

(2) The triangle Δ intersects the valley of t, and exactly one edge b of $\mathcal{T}^*$ points away from Δ^* (and the remaining two edges point toward Δ^*). The edge b is dual to the base of Δ. Moreover, if $\mathcal{D}$ is veering, then the right (resp. left) vertex of the base of Δ with respect to the orientation of b is assigned blue (resp. red) color (see Fig. 3(c)).
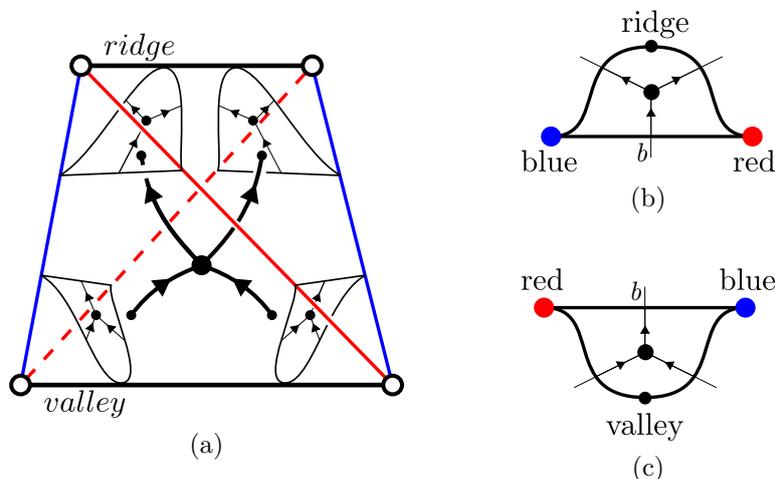
Fig. 3. An ideal tetrahedron of $\mathcal{D}$ and a part of the 1-skeleton of $\mathcal{F}$. Each of two triangles in $\mathcal{T}$ which intersects the ridge (resp. valley) of the ideal tetrahedron can be sent by an orientation-preserving homeomorphism to the angled triangle in (b) (resp. (c)).

Definition 2.7. Let $\widetilde{f}$ be a face of $\mathcal{T}^*$, and let ℓ be either $\partial_R(\widetilde{f})$ or $\partial_L(\widetilde{f})$.

(1) Let v be an interior vertex of ℓ, namely a vertex of ℓ different from the maximal and minimal vertices of $\widetilde{f}$. Then ℓ is said to be *attractive* (resp. *repulsive*) *at* v if the edge of $\mathcal{T}^*$, which is different from two successive (germs of) edges in ℓ passing through v, is oriented toward (resp. away from) v.

(2) The path ℓ is said to be *attractive* (resp. *repulsive*) if ℓ is attractive (resp. repulsive) at every interior vertex of ℓ

(3) The face $\widetilde{f}$ of $\mathcal{T}^*$ is called a *right-to-left face* (*RL-face* or *RL* in brief) if $\partial_R(\widetilde{f})$ is attractive and $\partial_L(\widetilde{f})$ is repulsive (see Fig. 4(a)). The face $\widetilde{f}$ of $\mathcal{T}^*$ is called a *left-to-right face* (*LR-face* or *LR* in brief) if $\partial_R(\widetilde{f})$ is repulsive and $\partial_L(\widetilde{f})$ is attractive (see Fig. 4(b)).

(4) The vertex of $\mathcal{T}$ dual to a face $\widetilde{f}$ of $\mathcal{T}^*$ is *RL* (resp. *LR*) if $\widetilde{f}$ is RL (resp. LR).

Remark 2.8. (1) Note that the above concept is preserved by the operation of replacing the balanced 1-cocycle ω with the balanced 1-cocycle $-\omega$.

(2) If $\mathcal{D}$ has no backtracking, then the following holds. Each face $\widetilde{f}$ of $\mathcal{T}^*$ has at least 3 edges, and so either $\partial_R(\widetilde{f})$ or $\partial_L(\widetilde{f})$ has an interior vertex. Hence $\widetilde{f}$ cannot be both RL and LR.

(a) RL-face   (b) LR-face
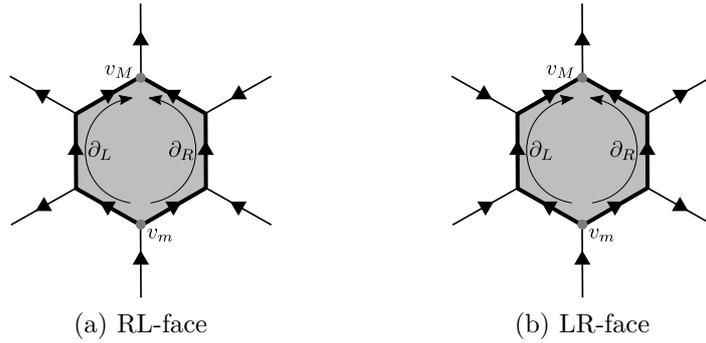
Fig. 4.   The grayed regions denote faces in $\mathcal{T}^*$, and the symbols v_M and v_m denote the maximal and minimal vertices of each face respectively. By Proposition 2.9, $\mathcal{D}$ is veering if and only if each face of $\mathcal{T}^*$ is either RL or LR. In this case, for each RL-face (resp. LR-face) of $\mathcal{T}^*$, an ideal edge which contains the vertex of $\mathcal{T}$ dual to the face is assigned red color (resp. blue color).

In the remainder of this section, we give conditions for a layered triangulation to be veering. The following proposition gives a method for checking whether a layered triangulation which satisfies A'Campo's criterion admits a veering structure.

Proposition 2.9. *Let $\mathcal{D}$ be a layered triangulation of a punctured surface bundle M over S^1 which has no backtracking. Assume that the layered structure comes from a balanced 1-cocycle ω satisfying A'Campo's criterion. Then $\mathcal{D}$ (with the taut structure induced by the layered structure) is veering if and only if each face of $\mathcal{T}^*$ is either RL or LR.*

To prove this proposition, we first show the following lemma.

Lemma 2.10. *Under the assumption of Proposition 2.9, the following holds. Let e be an ideal edge of $\mathcal{D}$, let f be the face of $\mathcal{F}$ dual to e, and let $\widetilde{f}_0$ and $\widetilde{f}_1$ be the faces of $\mathcal{T}^*$ corresponding to f. Then $\widetilde{f}_0$ is RL (resp. LR) if and only if $\widetilde{f}_1$ is RL (resp. LR).*

Proof. Let $\mathcal{D}$, e, $\widetilde{f}_0$ and $\widetilde{f}_1$ be as in the condition in the lemma. Since $\mathcal{D}$ has no backtracking, there is an ideal tetrahedron t of $\mathcal{D}$ containing e such that the dihedral angle of t at the ideal edge e is 0. Let Δ_0 and Δ_1 be the triangles of $\mathcal{T}$ contained in t intersecting the ideal edge e. We may assume that Δ_0 contains a part of the face

$\widetilde{f}_0$ of $\mathcal{T}^*$, i.e. Δ_0 contains a connected component of $\text{Int}(\widetilde{f}_0) \cap t$ whose boundary intersects the ideal edge e. Then Δ_1 contains a part of the face $\widetilde{f}_1$ of $\mathcal{T}^*$. Since the dihedral angle of t at e is 0, the ideal edge e is neither the valley nor the ridge of t. Hence we may assume, without loss of generality, that Δ_0 and Δ_1 intersect the valley and the ridge of t, respectively. Thus, by the observation about Fig. 3, the following hold:

(1) Among the three edges of $\mathcal{T}^*$ incident on the vertex Δ_0^* of $\mathcal{T}^*$, one edge points away from the vertex Δ_0^*, and two edges point toward the vertex Δ_0^*.
(2) Among the three edges of $\mathcal{T}^*$ incident on the vertex Δ_1^* of $\mathcal{T}^*$, one edge points toward the vertex Δ_1^*, and two edges point away from the vertex Δ_1^*.

Now assume that $\widetilde{f}_0$ is RL, i.e. $\partial_R(\widetilde{f}_0)$ is attractive and $\partial_L(\widetilde{f}_0)$ is repulsive. Then $\partial_R(\widetilde{f}_0)$ is attractive at the vertex Δ_0^* of $\mathcal{T}^*$. Hence $\partial_L(\widetilde{f}_1)$ is repulsive at the vertex Δ_1^* of $\mathcal{T}^*$. Since the same result holds for each ideal tetrahedron containing the ideal edge e such that the dihedral angle of t at the ideal edge e is 0, the face $\widetilde{f}_1$ is RL. By parallel argument, we can see that if $\widetilde{f}_0$ is LR, then $\widetilde{f}_1$ is also LR. □

Proof of Proposition 2.9. Suppose that $\mathcal{D}$ is veering, namely there exists an assignment of two colors to all ideal edges of $\mathcal{D}$ which satisfies the condition of Definition 2.2. This two-coloring induces a two-coloring of the vertices of $\mathcal{T}$, and hence it induces a two-coloring of the faces of $\mathcal{T}^*$. Let $\widetilde{f}$ be a face of $\mathcal{T}^*$. Assume that $\widetilde{f}$ is assigned red color (resp. blue color) by the two-coloring of the faces. Then we can see by the observation given by Fig. 3 that $\widetilde{f}$ is RL (resp. LR).

We will prove the converse. Suppose each face of $\mathcal{T}^*$ is either RL or LR. Pick an ideal edge e of $\mathcal{D}$. Let f be the face of $\mathcal{F}$ dual to e, and let $\widetilde{f}_0$ and $\widetilde{f}_1$ be the faces of $\mathcal{T}^*$ corresponding to f. Then, by Remark 2.8(2) and Lemma 2.10, exactly one of the following holds:

(1) Both $\widetilde{f}_0$ and $\widetilde{f}_1$ are RL,
(2) Both $\widetilde{f}_0$ and $\widetilde{f}_1$ are LR.

We assign two colors, red or blue, to all ideal edges e according to whether (1) or (2) holds. We show that the assignment of two colors satisfies the condition presented immediately after Definition 2.2, which guarantees that $\mathcal{D}$ is veering. To this end, pick an arbitrary triangle Δ of $\mathcal{T}$. Let t be the ideal tetrahedron of $\mathcal{D}$ containing Δ. Then Δ intersects either the ridge of t or the valley of t. Then, by the assignment of two colors to the edges of $\mathcal{D}$, we see that Δ can be sent by an orientation-preserving homeomorphism to the (angled) triangle in Fig. 3(b) or Fig. 3(c) according to whether Δ intersects the ridge of t or the valley of t, and so Δ can be also sent by an orientation-preserving homeomorphism to the (angled) triangle in Fig. 1(b). Hence the desired condition is satisfied, and hence $\mathcal{D}$ is veering. □

Remark 2.11. The assertion of Proposition 2.9 actually holds for any ideal triangulation $\mathcal{D}$ which admits a "taut" structure in the sense of [10, Definition 1.2], if each edge of $\mathcal{T}^*$ is oriented by using the "taut" structure in this sense.

In addition to Proposition 2.9, we also use the following lemma proved by [10, Lemma 2.3, Corollary 2.4].

Lemma 2.12. *Let $\mathcal{D}$ be as in Proposition 2.9, and suppose that $\mathcal{D}$ is veering. Then, for each face f of $\mathcal{F}$, no edge of f joins the maximal vertex and the minimal vertex of f with respect to ω. In particular, f has at least 4 edges.*

3. Hyperbolic fibered two-bridge links

In this section, we describe the hyperbolic fibered two-bridge links, following [13, Section 3]. Let $K = K(q/p)$ be a two-bridge link of slope $r = q/p$. We may assume that precisely one of p and q is odd, and $0 < |q| < p$. Moreover, by taking the mirror image and forgetting the orientation if necessary, we may assume $0 < q < p/2$. Thus $r = q/p$ has the following continued fraction expansion:

$$r = [2b_1, 2b_2, \ldots, 2b_m] = \cfrac{1}{2b_1 + \cfrac{1}{2b_2 + \cdots + \cfrac{1}{2b_m}}}, \tag{3.1}$$

where b_i is a non-zero integer. We denote the oriented link illustrated in Fig. 5 by $K[2b_1, 2b_2, \ldots, 2b_m]$. A two-bridge link $K[2b_1, 2b_2, \ldots, 2b_m]$ is the boundary of the surface obtained by successively plumbing the unknotted b_i-full twisted bands $(1 \leq i \leq m)$. Then $K(q/p) \cong K[2b_1, 2b_2, \ldots, 2b_m]$ is fibered if and only if b_i is equal to ± 1 for each $1 \leq i \leq m$. Moreover, $K(q/p)$ is hyperbolic if and only if $(b_1, b_2, b_3 \ldots, b_m) \neq \pm(1, -1, 1, \ldots, (-1)^{m-1})$ (cf. [12, Corollary 2]).
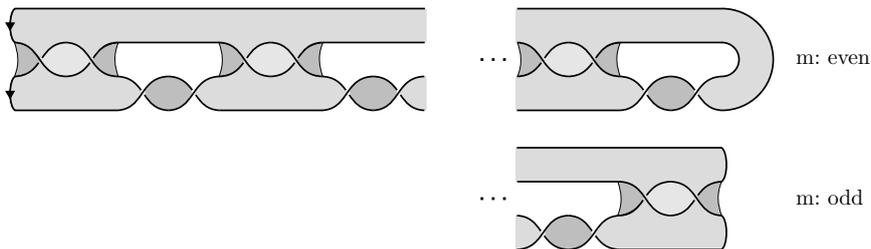
Fig. 5.   The hyperbolic two-bridge link $K[2b_1, 2b_2, \ldots, 2b_m]$ with $b_i = +1$ for each $1 \leq i \leq m$.

We also need to consider the continued fraction expansion

$$r = [a_1, a_2, \ldots, a_n], \tag{3.2}$$

where, for each $1 \leq i \leq n$, a_i is a positive integer and $a_n \geq 2$. In [13, Section 3], we gave a recipe transforming a continued fraction expansion (3.1) for fibered links into a continued fraction expansion (3.2).

Lemma 3.1 ([13, Lemma 3.1])**.** *Given a continued fraction $[2b_1, 2b_2, \ldots, 2b_m]$ ($b_1 = +1$ and $b_i = \pm 1$ for each $2 \leq i \leq m$), decompose the sequence $(2b_1, 2b_2, \ldots, 2b_m)$ into subsequences $(S_1, S_2, \ldots, S_n)$ so that*

(1) the entries of S_i have alternate signs, and
(2) the last entry of S_i and the first entry of S_{i+1} have the same sign.

For each S_i, we denote by $\ell(S_i)$ the length of S_i. Set

$$S_i' = \begin{cases} (2) & \text{if } \ell(S_i) = 1, \\ (1, \ell(S_i) - 1, 1) & \text{if } \ell(S_i) > 1. \end{cases}$$

Then we have the following identity of rational numbers

$$[2b_1, 2b_2, \ldots, 2b_m] = [S_1', S_2', \ldots, S_n'].$$

4. Description of the canonical decompositions of hyperbolic two-bridge link complements and the 2-cell complexes dual to the decompositions

In this section, we will describe combinatorial structures of the canonical decompositions of hyperbolic two-bridge link complements and the 2-cell complexes dual to the decompositions, following [7, 11, 14]. Hence we obtain a description of the cusp triangulations induced by $\mathcal{D}$ and its dual decompositions.

First of all, we consider the Farey tessellation of the hyperbolic plane $\mathbb{H}^2$. The vertex set of the Farey tessellation is equal to $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \subset \widehat{\mathbb{R}} \cong \partial\mathbb{H}^2$. Let $\sigma = (s_1, s_2, s_3)$ be a Farey triangle such that, for each $1 \leq i \leq 3$, s_i is a vertex of σ, and let H be the group of transformations on $\mathbb{R}^2$ generated by the π-rotations about the points in $\mathbb{Z}^2$. We obtain an H-invariant triangulation, $\widetilde{\mathrm{trg}(\sigma)}$, of $\mathbb{R}^2$ which is determined by the union of the lines of slopes $\{s_1, s_2, s_3\}$ in $\mathbb{R}^2$ passing through the integer lattice $\mathbb{Z}^2$. Then the Farey triangle σ determines a triangulation, $\mathrm{trg}(\sigma)$, of the sphere with four marked points $(S^2, P) = (\mathbb{R}^2, \mathbb{Z}^2)/H$. The subset $\{(0,0), (1,0), (0,1), (1,1)\}$ of $\mathbb{Z}^2$ is a system of representatives of $P = \mathbb{Z}^2/H$. Thus we identify P with this set. Note that each triangulation $\mathrm{trg}(\sigma)$ of (S^2, P) determines an ideal triangulation of the 4-times punctured sphere $S = S^2 \setminus P$. By abuse of notation, we also denote the ideal triangulation by $\mathrm{trg}(\sigma)$. The ideal triangulation $\mathrm{trg}(\sigma)$ consists of precisely six ideal edges and four ideal triangles. In particular, for each vertex s_i of the Farey triangle σ, there are exactly two ideal edges of slope s_i in $\mathrm{trg}(\sigma)$. We denote by $e_{s_i}^{(0)}$ the ideal edge of slope s_i which has the vertex $(0,0)$ of P as endpoint, and denote by $e_{s_i}^{(1)}$ the other ideal edge. Let σ' be a Farey triangle such that σ and σ' are adjacent, i.e. σ and σ' share precisely

two vertices, and let s (resp. s') be the vertex of σ (resp. σ') which is not contained in σ' (resp. σ). Then $\mathrm{trg}(\sigma')$ is obtained from $\mathrm{trg}(\sigma)$ by a pair of Whitehead moves, i.e. by replacing the edges $e_s^{(0)}$ and $e_s^{(1)}$ with the edges $e_{s'}^{(1)}$ and $e_{s'}^{(0)}$, respectively.

Let $K = K(q/p)$ be a hyperbolic two-bridge link. Then we may assume that p and q are relatively prime integers such that $1 < q < p/2$, and so q/p has the continued fraction expansion $[a_1, a_2, \ldots, a_n]$, where $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}_+)^n$, $a_1, a_n \geq 2$ and $n \geq 2$. Set $r = q/p$ and $c = \sum_{i=1}^{n} a_i$. Let $\Sigma(r) = (\sigma_1, \sigma_2, \ldots, \sigma_c)$ be the sequence of Farey triangles which intersect the hyperbolic geodesic joining ∞ and r in this order. For each $1 \leq l \leq c - 1$, the triangulation $\mathrm{trg}(\sigma_{l+1})$ is obtained from the triangulation $\mathrm{trg}(\sigma_l)$ by a pair of Whitehead moves. We can immerse a pair of ideal tetrahedra in $S \times \mathbb{R}$: the boundary of the immersed tetrahedra is made up of two *pleated surfaces* (top and bottom) homotopic to S and triangulated according to $\mathrm{trg}(\sigma_{l+1})$ and $\mathrm{trg}(\sigma_l)$, respectively. We denote the image of the immersion by $\mathrm{trg}(\sigma_l, \sigma_{l+1})$. The immersion is an embedding on the interior of the tetrahedra. Let s (resp. s') be the vertex of σ_l (resp. σ_{l+1}) which is not contained in σ_{l+1} (resp. σ_l). Then the edge $e_s^{(1)}$ and $e_{s'}^{(0)}$ are contained in the same tetrahedron, $t_{l-1}^{(0)}$, of $\mathrm{trg}(\sigma_l, \sigma_{l+1})$, and the edges $e_s^{(0)}$ and $e_{s'}^{(1)}$ are contained in the other tetrahedron, $t_{l-1}^{(1)}$, of $\mathrm{trg}(\sigma_l, \sigma_{l+1})$.

For each $1 \leq l \leq c - 2$, we can glue the top of the immersed pair $\mathrm{trg}(\sigma_l, \sigma_{l+1})$ onto the bottom of $\mathrm{trg}(\sigma_{l+1}, \sigma_{l+2})$ along $\mathrm{trg}(\sigma_{l+1})$ in $S \times \mathbb{R}$. Then the immersed pairs $\{\mathrm{trg}(\sigma_l, \sigma_{l+1})\}_{1 \leq l \leq c-1}$ can be stacked up to form a topological ideal triangulation, $\widehat{\mathcal{D}}(r)$, of $S \times [-1, 1]$. The restriction of $\widehat{\mathcal{D}}(r)$ to $S \times \{-1\}$ (resp. $S \times \{+1\}$) is $\mathrm{trg}(\sigma_1)$ (resp. $\mathrm{trg}(\sigma_c)$), and each $\mathrm{trg}(\sigma_l)$ can be regarded as (an ideal triangulation of) a pleated surface in $S \times [-1, 1]$. For each $0 \leq l \leq c - 3$ and $k, k' \in \{0, 1\}$, there exists a unique ideal triangle in $\mathrm{trg}(\sigma_{l+2})$ shared by the ideal tetrahedra $t_l^{(k)}$ and $t_{l+1}^{(k')}$. We denote the ideal triangle by $|t_l^{(k)}, t_{l+1}^{(k')}|$. $\mathcal{D}(r)$ denotes the topological ideal simplicial complex obtained from $\widehat{\mathcal{D}}(r)$ by collapsing each ideal edge of slopes ∞ and r into an ideal vertex. To be precise, $\mathcal{D}(r)$ is constructed as follows. Since each of the ideal edges $e_\infty^{(0)}$ and $e_\infty^{(1)}$ is collapsed into an ideal vertex, the subcomplex $\mathrm{trg}(\sigma_1)$ of $\widehat{\mathcal{D}}(r)$ is collapsed into a single edge, e_-, and $\mathrm{trg}(\sigma_2)$ is folded along the pair of ideal edges $e_{1/2}^{(0)}$ and $e_{1/2}^{(1)}$ to a pair of ideal triangles, $|t_1^{(0)}, t_1^{(1)}|_{(0)}$ and $|t_1^{(0)}, t_1^{(1)}|_{(1)}$, respectively. (Note that the slope $1/2$ is the vertex of σ_2 which is not contained in σ_1.) Put $r^* = [a_1, a_2, \ldots, a_n - 2]$. (Note that the slope r^* is the vertex of σ_{c-1} which is not contained in σ_c.) Similarly, since each of ideal edges $e_r^{(0)}$ and $e_r^{(1)}$ is collapsed into an ideal vertex, the subcomplex $\mathrm{trg}(\sigma_c)$ of $\widehat{\mathcal{D}}(r)$ is collapsed into a single edge, e_+, and $\mathrm{trg}(\sigma_{c-1})$ is folded along the pair of ideal edges $e_{r^*}^{(0)}$ and $e_{r^*}^{(1)}$ to a pair of ideal triangles, $|t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(0)}$ and $|t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(1)}$, respectively. In the special case, when $r = 2/5$ (i.e. $c = 4$), we denote the ideal triangle " $|t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(j)}$ " by $|t_{c-3}^{(0)}, t_{c-3}^{(1)}|'_{(j)}$ for each $j \in \{0, 1\}$, in order to distinguish it from $|t_1^{(0)}, t_1^{(1)}|_{(j)}$.

It is proved by Guéritaud [6] that $\mathcal{D}(r)$ is isotopic to the canonical decomposition of the complement of $K(r)$ (see also [7] and [2]). We see by the construction of $\mathcal{D}(r)$

that the 1-skeleton of the 2-dimensional cell complex, $\mathcal{F}(r)$, dual to $\mathcal{D}(r)$ is as illustrated in Fig. 6.
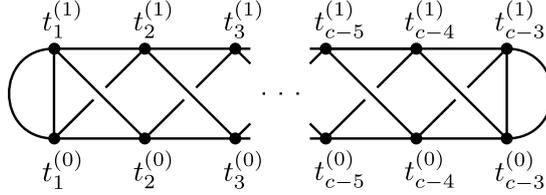
Fig. 6. The 1-skeleton of the 2-dimensional cell complex $\mathcal{F}(r)$ dual to $\mathcal{D}(r)$.

Notation 4.1. Let $\mathcal{S} = \mathcal{S}(r)$ be the set consisting of all vertices of Farey triangles in $\Sigma(r)$, and put $\mathcal{S}' = \mathcal{S}'(r) := \mathcal{S}(r) \setminus (\sigma_1^{(0)} \cup \sigma_c^{(0)})$. For each $0 \le l \le c - 2$, we denote by s_l the vertex of σ_{l+2} which is not contained in σ_{l+1}. We denote by $T(s)$ the subsequence of $\Sigma(r)$ consisting of the Farey triangles which contain the Farey vertex s.

Remark 4.2. The ideal simplices of the ideal triangulation $\mathcal{D} = \mathcal{D}(r)$ is described as follows. If $r \ne 2/5$, i.e. $K(r)$ is not the figure-eight knot, then

$$\mathcal{D}^{(1)} = \{e_s^{(k)} \mid s \in \mathcal{S}', \ k \in \{0,1\}\} \cup \{e_-, e_+\},$$

$$\mathcal{D}^{(2)} = \{|t_1^{(0)}, t_1^{(1)}|_{(j)}, \ |t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(j)} \mid j \in \{0,1\}\}$$
$$\cup \{|t_l^{(k)}, t_{l+1}^{(k')}| \mid 1 \le l \le c - 4, \ k, k' \in \{0,1\}\},$$

$$\mathcal{D}^{(3)} = \{t_l^{(k)} \mid 1 \le l \le c - 3, \ k \in \{0,1\}\},$$

where $\mathcal{D}^{(i)}$ is the set of all i-simplices in $\mathcal{D}$. If $r = 2/5 = [2,2]$, then the description of $\mathcal{D}$ needs to be replaced with the following:

$$\mathcal{D}^{(2)} = \{|t_1^{(0)}, t_1^{(1)}|_{(0)}, |t_1^{(0)}, t_1^{(1)}|_{(1)}, |t_1^{(0)}, t_1^{(1)}|'_{(0)}, |t_1^{(0)}, t_1^{(1)}|'_{(1)}\}.$$

Now we describe the induced cusp triangulation, i.e. the triangulation of the peripheral torus of $S^3 \backslash K(r)$ induced by $\mathcal{D}(r)$. To this end, we identify the underlying space of the subcomplex $\widehat{\mathcal{D}}_0(r) := \{\mathrm{trg}(\sigma_l, \sigma_{l+1})\}_{2 \le l \le c-2}$ of $\widehat{\mathcal{D}}(r)$ with $S \times [-1, 1]$. We first describe the triangulation of the peripheral annuli of $S \times [-1, 1]$ induced by $\widehat{\mathcal{D}}_0(r)$. For each ideal vertex $(i, j) \in P = \{(0,0), (1,0), (0,1), (1,1)\}$ of S, let $\widehat{\mathcal{A}}(i, j)$ be the triangulation of the peripheral annulus of $S \times [-1, 1]$ corresponding to (i, j). Since the combinatorics of the four peripheral annuli $\widehat{\mathcal{A}}(0,0)$, $\widehat{\mathcal{A}}(1,0)$, $\widehat{\mathcal{A}}(0,1)$ and $\widehat{\mathcal{A}}(1,1)$ are identical, let us focus on the single peripheral annulus $\widehat{\mathcal{A}} := \widehat{\mathcal{A}}(0,0)$. Since the ideal triangulation $\mathrm{trg}(\sigma_l)$ is an ideal triangulation of a level 4-punctured sphere, it induces a triangulation, $C(\sigma_l)$, of a core circle in $\widehat{\mathcal{A}}$. The triangulation $C(\sigma_l)$ consists of precisely three vertices and three edges. The region in $\widehat{\mathcal{A}}$ bounded

by $C(\sigma_l)$ and $C(\sigma_{l+1})$ consists of exactly two triangles. The family $\{C(\sigma_l)\}_{2 \le l \le c-1}$ forms the 1-skeleton of the triangulation of $\widehat{\mathcal{A}}$. Each d-simplex of $\widehat{\mathcal{A}}$ with $0 \le d \le 2$ is contained in a unique ideal $(d+1)$-simplex of $\mathcal{D}$.

Recall that, by folding of $\mathrm{trg}(\sigma_2)$ along the ideal edges of slope $1/2$, the ideal edges $e_{0/1}^{(0)}$, $e_{0/1}^{(1)}$, $e_{1/1}^{(0)}$, and $e_{1/1}^{(1)}$ in $\widehat{\mathcal{D}}$ are identified into a single ideal edge e_- in $\mathcal{D}$. Then the boundary line $C(\sigma_2)$ of a peripheral annulus is deformed into a zigzag circle which has a "hairpin curve" at the vertices contained in the ideal edges of slope $1/2$, and the vertices contained in the ideal edges of slope $0/1$ and $1/1$ are identified into a single vertex contained in e_-. Furthermore, since the folding identifies a puncture of $S \times [-1, 1]$ with another puncture, the resulting triangulation of the peripheral annulus is joined to the corresponding triangulation of another peripheral annulus. In particular, since the ideal edge $e_\infty^{(0)}$ (resp. $e_\infty^{(1)}$) of $\widehat{\mathcal{D}}(r)$ joins the punctures $(0, 0)$ and $(0, 1)$ (resp. $(1, 0)$ and $(1, 1)$) of S, the bottom circle of the underlying space $|\widehat{\mathcal{A}}(0,0)|$ (resp. $|\widehat{\mathcal{A}}(1,0)|$) and that of $|\widehat{\mathcal{A}}(0,1)|$ (resp. $|\widehat{\mathcal{A}}(1,1)|$) are identified. Similarly, the folding of the pleated surface $\mathrm{trg}(\sigma_{c-1})$ causes a similar effect on the other side of $\widehat{\mathcal{A}}(i, j)$. For each $(i, j) \in P$, we denote by $\mathcal{A}(i, j)$ the triangulated sub-annulus of the cusp triangulation, obtained in this way from the peripheral annulus $\widehat{\mathcal{A}}(i, j)$ of $S \times [-1, 1]$ corresponding to the puncture (i, j) of S. Then the cusp triangulation $\mathcal{T}$ is obtained from the union of $\mathcal{A}(0, 0)$, $\mathcal{A}(0, 1)$, $\mathcal{A}(1, 0)$ and $\mathcal{A}(1, 1)$. Each d-simplex of $\mathcal{A}(i, j)$ with $0 \le d \le 2$ is contained in a unique ideal $(d+1)$-simplex of $\mathcal{D}$. Thus we have a map λ_d

$$\{d\text{-simplex of } \mathcal{A}(i, j)\} \to \{\text{ideal } (d+1)\text{-simplex of } \mathcal{D}\}.$$

Definition 4.3. By the *label* of a d-simplex of $\mathcal{A}(i, j)$, we mean its image by λ_d.

Remark 4.4. The label of a vertex of $\mathcal{A}(0, 0)$ is equal to e_+, e_- or $e_{s_l}^{(0)}$ $(s_l \in \mathcal{S}')$. The labels of vertices and triangles of $\mathcal{A}(0, 0)$ are as depicted in Fig. 7.

The following lemma follows from the descriptions of $\mathcal{D}$ and $\mathcal{T}$, and Remark 4.4.

Lemma 4.5. *The labels of the edge set of the triangle $\Delta \in \mathcal{A}(0, 0) \subset \mathcal{T}$ with label $t_l^{(k)}$ is given as follows (see Fig. 7):*

$$\begin{cases} \{|t_{l-1}^{(0)}, t_l^{(0)}|, |t_l^{(0)}, t_{l+1}^{(0)}|, |t_l^{(0)}, t_{l+1}^{(1)}|\} & \text{if } 1 < l < c-3,\ k = 0, \\ \{|t_{l-1}^{(0)}, t_l^{(1)}|, |t_{l-1}^{(1)}, t_l^{(1)}|, |t_l^{(1)}, t_{l+1}^{(1)}|\} & \text{if } 1 < l < c-3,\ k = 1, \\ \{|t_1^{(0)}, t_1^{(1)}|_{(0)}, |t_1^{(0)}, t_2^{(0)}|, |t_1^{(0)}, t_2^{(1)}|\} & \text{if } l = 1,\ k = 0, \\ \{|t_1^{(0)}, t_1^{(1)}|_{(0)}, |t_1^{(0)}, t_1^{(1)}|_{(1)}, |t_1^{(1)}, t_2^{(1)}|\} & \text{if } l = 1,\ k = 1, \\ \{|t_{c-4}^{(0)}, t_{c-3}^{(0)}|, |t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(0)}, |t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(1)}\} & \text{if } l = c-3,\ k = 0, \\ \{|t_{c-4}^{(0)}, t_{c-3}^{(1)}|, |t_{c-4}^{(1)}, t_{c-3}^{(1)}|, |t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(0)}\} & \text{if } l = c-3,\ k = 1. \end{cases}$$
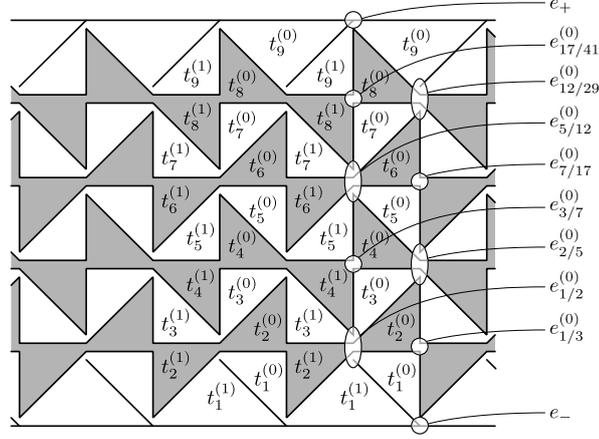
Fig. 7. A part of the infinite cyclic cover of $\mathcal{A}(0,0)$ with $r = [2,2,2,2,2,2]$.

Let $\mathrm{Aut}(\mathcal{D})$ be the group of the combinatorial automorphisms of the ideal simplicial complex $\mathcal{D}$, and let $\mathrm{Aut}^+(\mathcal{D})$ be the subgroup of $\mathrm{Aut}(\mathcal{D})$ consisting of the orientation-preserving automorphisms. Let $\widetilde{X}$ and $\widetilde{Y}$ be the homeomorphisms of $(\mathbb{R}^2, \mathbb{Z}^2) \times [-1,1]$ defined by, for each $(x,y) \in \mathbb{R}^2$,

$$\widetilde{X}((x,y),t) = ((x+1,y),t),$$
$$\widetilde{Y}((x,y),t) = ((x,y+1),t).$$

Note that $\widetilde{X}$ and $\widetilde{Y}$ induce the combinatorial isomorphisms between peripheral annuli. Both $\widetilde{X}$ and $\widetilde{Y}$ are H-equivariant, and they are compatible with the foldings of $\mathrm{trg}(\sigma_2)$ and $\mathrm{trg}(\sigma_{c-1})$. Hence $\widetilde{X}$ and $\widetilde{Y}$ induce orientation-preserving automorphisms of $\mathcal{D}$, which we denote by X and Y respectively. The subgroup $\langle X, Y \rangle$ of $\mathrm{Aut}^+(\mathcal{D})$ generated by X and Y is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (cf. [14, p.415]).

Notation 4.6. By Remark 4.4 and Lemma 4.5, we have the description of the triangulation $\mathcal{A}(0,0)$. The transformation $X^i Y^j \in \langle X, Y \rangle$ induces the combinatorial isomorphism from $\mathcal{A}(0,0)$ to $\mathcal{A}(i,j)$. The symbol $e_s^{(0)}(i,j)$ (resp. $e_\pm(i,j)$) denotes the vertex of $\mathcal{A}(i,j)$ obtained as the image, by $X^i Y^j$, of the vertex of $\mathcal{A}(0,0)$ with label $e_s^{(0)}$ (resp. $e_\pm$). Similarly, the symbol $t_l^{(k)}(i,j)$ denotes the triangle of $\mathcal{A}(i,j)$ obtained as the image, by $X^i Y^j$, of the triangle of $\mathcal{A}(0,0)$ with label $t_l^{(k)}$.

Observation 4.7. (1) The label of the vertex $e_s^{(0)}(i,j)$ of $\mathcal{A}(i,j)$ is as follows. Let p_s and q_s be relatively prime integers such that $s = q_s/p_s$, and let (p_s', q_s') be the mod 2 reduction of (p_s, q_s), i.e. $(p_s', q_s') \in P$. Then the label of $e_s^{(0)}(i,j)$ is equal to $e_s^{(0)}$ or $e_s^{(1)}$ according to whether $(i,j) \in \{(0,0), (p_s', q_s')\}$ or not.

(2) The label of the triangle $t_l^{(k)}(i,j)$ of $\mathcal{A}(i,j)$ is as follows. Let p_l and q_l be relatively prime integers such that $s_l = q_l/p_l$, and let (p_l', q_l') be the mod 2 reduction of (p_l, q_l). Recall that an ideal tetrahedron in $\mathrm{trg}(\sigma_{l+1}, \sigma_{l+2})$ is equal to $t_l^{(0)}$ or $t_l^{(1)}$

according to whether the tetrahedron contains the edge $e_{s_l}^{(0)}$ or not. Then the label of $t_l^{(k)}(i,j)$ is equal to $t_l^{(k)}$ or $t_l^{(k+1)}$ according to whether $(i,j) \in \{(0,0), (p'_l, q'_l)\}$ or not.

(3) The description of the vertices and the triangles in $\mathcal{A}(i,j)$ automatically determines the labels of the edges in $\mathcal{A}(i,j)$.

By abuse of notation, $e_s^{(k)}$ denotes the face of $\mathcal{F}$ dual to the ideal edge $e_s^{(k)}$ of $\mathcal{D}$. Similarly, $t_l^{(k)}$ and $|t_l^{(k)}, t_{l+1}^{(k')}|$ denote the vertex and the edge of $\mathcal{F}$ dual to the ideal tetrahedron $t_l^{(k)}$ and the ideal triangle $|t_l^{(k)}, t_{l+1}^{(k')}|$ respectively. It should be noted that the edge $|t_l^{(k)}, t_{l+1}^{(k')}|$ of $\mathcal{F}$ connects the vertex $t_l^{(k)}$ to the vertex $t_{l+1}^{(k')}$ in $\mathcal{F}$.

Now we describe the boundaries of the faces in $\mathcal{F}$. We denote by ∂_2 the boundary homomorphism $C_2(\mathcal{F}; \mathbb{Z}) \to C_1(\mathcal{F}; \mathbb{Z})$. Let f be a face of $\mathcal{F}$, and let f^* be one of the two vertices of the cusp torus $\mathcal{T}$ contained in the ideal edge of $\mathcal{D}$ dual to f. Consider a small oriented circle c in the cusp torus around the vertex f^*. Let $t_0, t_1, \ldots, t_{k-1}$ be the triangles in $\mathcal{T}$ which intersects c in this cyclic order. Then we have

$$\partial_2(f) = \sum_{i=0}^{k-1} \langle t_i, t_{i+1} \rangle.$$

Here, we assume that f is oriented so that it is coherent with c, and the symbol $\langle t_i, t_{i+1} \rangle$ denotes the oriented edge in $\mathcal{F}$ from the vertex t_i to the vertex t_{i+1}, where the indices are considered modulo k. By using this fact, we obtain the following description of ∂_2.

Lemma 4.8 ([13, Lemma 4.3])**.** *Suppose that $s \in \mathcal{S}$ and that $T(s) = (\sigma_d, \ldots, \sigma_{d+(k-1)})$. Then we have*

$$\partial_2(e_s^{(0)}) = B_{\mathrm{up}}(s) + B_{\mathrm{down}}(s) + B_{\mathrm{cap}^+}(s) + B_{\mathrm{cap}^-}(s),$$

where $B_{\mathrm{up}}(s)$, $B_{\mathrm{down}}(s)$, $B_{\mathrm{cap}^+}(s)$ and $B_{\mathrm{cap}^-}(s)$ are given as follows (see Fig. 8):

$$B_{\mathrm{up}}(s) = \begin{cases} \sum_{i=1}^{k-2} \langle t_{d+i-2}^{(0)}, t_{d+i-1}^{(0)} \rangle & \text{if } k > 2, \\ 0 & \text{if } k = 2, \end{cases}$$

$$B_{\mathrm{down}}(s) = \begin{cases} \sum_{i=1}^{k-2} \langle t_{d+i-1}^{(1)}, t_{d+i-2}^{(1)} \rangle & \text{if } k > 2, \\ 0 & \text{if } k = 2, \end{cases}$$

$$B_{\mathrm{cap}^+}(s) = \begin{cases} \langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(0)} & \text{if } s = r^*, \\ \langle t_{d+k-3}^{(0)}, t_{d+k-2}^{(1)} \rangle + \langle t_{d+k-2}^{(1)}, t_{d+k-3}^{(1)} \rangle & \text{if } s \neq r^*, \end{cases}$$

$$B_{\mathrm{cap}^-}(s) = \begin{cases} \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)} & \text{if } s = 1/2, \\ \langle t_{d-1}^{(1)}, t_{d-2}^{(0)} \rangle + \langle t_{d-2}^{(0)}, t_{d-1}^{(0)} \rangle & \text{if } s \neq 1/2. \end{cases}$$

Moreover, we have

$$\partial_2(e_s^{(1)}) = B'_{\mathrm{up}}(s) + B'_{\mathrm{down}}(s) + B'_{\mathrm{cap}^+}(s) + B'_{\mathrm{cap}^-}(s),$$

where $B'_{\text{up}}(s)$, $B'_{\text{down}}(s)$, $B'_{\text{cap}+}(s)$ and $B'_{\text{cap}-}(s)$ are given as follows (see Fig. 9):

$$B'_{\text{up}}(s) = \begin{cases} \sum_{i=1}^{k-2} \langle t_{d+i-2}^{(i)}, t_{d+i-1}^{(i+1)} \rangle & \text{if } k > 2, \\ 0 & \text{if } k = 2, \end{cases}$$

$$B'_{\text{down}}(s) = \begin{cases} \sum_{i=1}^{k-2} \langle t_{d+i-1}^{(i)}, t_{d+i-2}^{(i+1)} \rangle & \text{if } k > 2, \\ 0 & \text{if } k = 2, \end{cases}$$

$$B'_{\text{cap}+}(s) = \begin{cases} \langle t_{c-3}^{(k-1)}, t_{c-3}^{(k-2)} \rangle_{(1)} & \text{if } s = r^*, \\ \langle t_{d+k-3}^{(k-1)}, t_{d+k-2}^{(0)} \rangle + \langle t_{d+k-2}^{(0)}, t_{d+k-3}^{(k-2)} \rangle & \text{if } s \neq r^*, \end{cases}$$

$$B'_{\text{cap}-}(s) = \begin{cases} \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)} & \text{if } s = 1/2, \\ \langle t_{d-1}^{(0)}, t_{d-2}^{(1)} \rangle + \langle t_{d-2}^{(1)}, t_{d-1}^{(1)} \rangle & \text{if } s \neq 1/2. \end{cases}$$

Here the upper suffix k at $e_i^{(k)}$ is considered modulo 2.

(a)

(b)
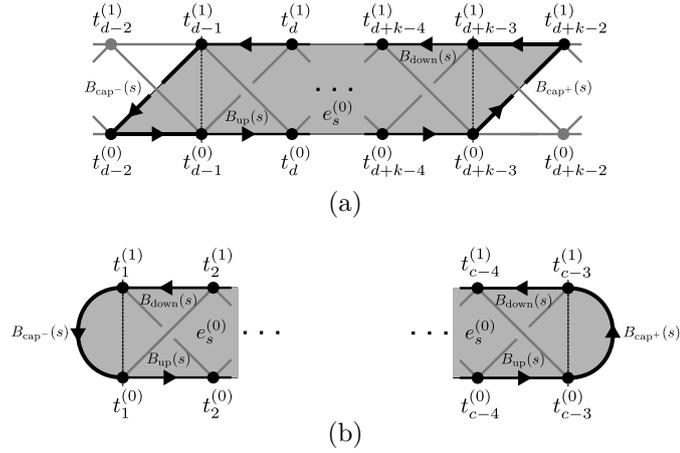
Fig. 8.   A face $e_s^{(0)}$ is shown as the grayed region in (a). (For $s = 1/2$, the term $B_{\text{cap}-}(s)$ is shown in the left in (b), and for $s = r^*$, the term $B_{\text{cap}+}(s)$ is shown in the right in (b).)
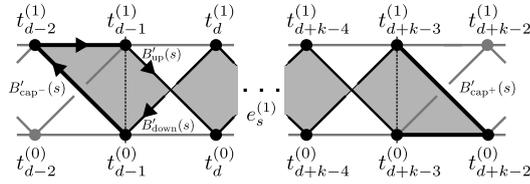
Fig. 9.   A face $e_s^{(1)}$ is shown as the grayed region. (The terms in $B'_{\text{up}}(s)$ come from the edges oriented from left to right, and the terms in $B'_{\text{down}}(s)$ come from the edges oriented from right to left.)

Remark 4.9. The formula for the cycle $\partial_2(e_s^{(k)})$ in the above lemma actually gives

the geometric boundary of $e_s^{(k)}$, namely the geometric boundary $\partial e_s^{(k)}$ of $e_s^{(k)}$ is the union of the underlying edges of the terms in the formula. By using Observation 4.7, we can obtain a formula describing the geometric boundaries of the 2-cells of $\mathcal{T}^*$.

Set $r_{\mathbf{c}} = [a_1, \ldots, a_{n-1}]$. Note that the vertices in $\mathcal{T}$ corresponding to the ideal edges of slope $r_{\mathbf{c}}$ and $[a_1, \ldots, a_{n-1}, a_n - 1]$ are identified into the single vertex e_+. The meridian of $K(r)$ is described as follows.

Lemma 4.10 ([13, Lemma 4.4]). *Suppose that $T(0/1) = (\sigma_1, \ldots, \sigma_{k_-})$ and that $T(r_{\mathbf{c}}) = (\sigma_{c-(k_+-1)}, \ldots, \sigma_c)$. Then each of the following 1-cycles $\mu_{\pm}^{(0,0)}$ in $\mathcal{F}$ represents a meridian of the two-bridge link $K(r)$:*

$$\mu_{\pm}^{(0,0)} = M_{\text{up}}^{(0,0)}(\pm) + M_{\text{down}}^{(0,0)}(\pm) + M_{\text{cap}^+}^{(0,0)}(\pm) + M_{\text{cap}^-}^{(0,0)}(\pm),$$

where $M_{\text{up}}^{(0,0)}(\pm)$, $M_{\text{down}}^{(0,0)}(\pm)$, $M_{\text{cap}^+}^{(0,0)}(\pm)$ and $M_{\text{cap}^-}^{(0,0)}(\pm)$ are given as follows:

$$M_{\text{up}}^{(0,0)}(-) = \begin{cases} \sum_{i=1}^{k_- - 3} \langle t_i^{(1)}, t_{i+1}^{(1)} \rangle & \text{if } k_- > 3, \\ 0 & \text{if } k_- = 3, \end{cases}$$

$$M_{\text{down}}^{(0,0)}(-) = \begin{cases} \sum_{i=1}^{k_- - 3} \langle t_{i+1}^{(0)}, t_i^{(0)} \rangle & \text{if } k_- > 3, \\ 0 & \text{if } k_- = 3, \end{cases}$$

$$M_{\text{cap}^+}^{(0,0)}(-) = \begin{cases} \langle t_{k_- - 2}^{(1)}, t_{k_- - 1}^{(1)} \rangle + \langle t_{k_- - 1}^{(1)}, t_{k_- - 2}^{(0)} \rangle & \text{if } r^* \neq 0, \\ \langle t_{k_- - 2}^{(1)}, t_{k_- - 2}^{(0)} \rangle_{(0)} & \text{if } r^* = 0, \end{cases}$$

$$M_{\text{cap}^-}^{(0,0)}(-) = \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)},$$

$$M_{\text{up}}^{(0,0)}(+) = \begin{cases} \sum_{i=1}^{k_+ - 3} \langle t_{c-(i+3)}^{(1)}, t_{c-(i+2)}^{(1)} \rangle & \text{if } k_+ > 3, \\ 0 & \text{if } k_+ = 3, \end{cases}$$

$$M_{\text{down}}^{(0,0)}(+) = \begin{cases} \sum_{i=1}^{k_+ - 3} \langle t_{c-(i+2)}^{(0)}, t_{c-(i+3)}^{(0)} \rangle & \text{if } k_+ > 3, \\ 0 & \text{if } k_+ = 3, \end{cases}$$

$$M_{\text{cap}^+}^{(0,0)}(+) = \langle t_{c-3}^{(1)}, t_{c-3}^{(0)} \rangle_{(0)},$$

$$M_{\text{cap}^-}^{(0,0)}(+) = \begin{cases} \langle t_{c-k_+}^{(0)}, t_{c-(k_++1)}^{(0)} \rangle + \langle t_{c-(k_++1)}^{(0)}, t_{c-k_+}^{(1)} \rangle & \text{if } r_{\mathbf{c}} \neq 1/2, \\ \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)} & \text{if } r_{\mathbf{c}} = 1/2. \end{cases}$$

Furthermore, we have the following.

Lemma 4.11 ([13, Lemma 4.5]). *Suppose that $T(0/1) = (\sigma_1, \ldots, \sigma_{k_-})$ and that $T(r_{\mathbf{c}}) = (\sigma_{c-(k_+-1)}, \ldots, \sigma_c)$. Then each of the following 1-cycles $\mu_{\pm}^{(u,v)}$ in $\mathcal{F}$ represents a meridian of the two-bridge link $K(r)$:*

(1) For each $(u, v) \in \{(0,1), (1,0)\}$,

$$\mu_-^{(u,v)} = M_{\text{up}}^{(u,v)}(-) + M_{\text{down}}^{(u,v)}(-) + M_{\text{cap}^+}^{(u,v)}(-) + M_{\text{cap}^-}^{(u,v)}(-),$$

where $M_{\mathrm{up}}^{(u,v)}(-)$, $M_{\mathrm{down}}^{(u,v)}(-)$, $M_{\mathrm{cap}+}^{(u,v)}(-)$ and $M_{\mathrm{cap}-}^{(u,v)}(-)$ are given as follows. If $r^* \neq 0$, then

$$M_{\mathrm{up}}^{(u,v)}(-) = \begin{cases} \sum_{i=1}^{k_- - 3} \langle t_i^{(i+1)}, t_{i+1}^{(i)} \rangle & \text{if } k_- > 3 \text{ and } (u,v) = (0,1), \\ \sum_{i=1}^{k_- - 3} \langle t_i^{(0)}, t_{i+1}^{(0)} \rangle & \text{if } k_- > 3 \text{ and } (u,v) = (1,0), \\ 0 & \text{if } k_- = 3, \end{cases}$$

$$M_{\mathrm{down}}^{(u,v)}(-) = \begin{cases} \sum_{i=1}^{k_- - 3} \langle t_{i+1}^{(i+1)}, t_i^{(i)} \rangle & \text{if } k_- > 3 \text{ and } (u,v) = (0,1), \\ \sum_{i=1}^{k_- - 3} \langle t_{i+1}^{(1)}, t_i^{(1)} \rangle & \text{if } k_- > 3 \text{ and } (u,v) = (1,0), \\ 0 & \text{if } k_- = 3, \end{cases}$$

$$M_{\mathrm{cap}+}^{(u,v)}(-) = \begin{cases} \langle t_{k_- - 2}^{(k_- - 3)}, t_{k_- - 1}^{(0)} \rangle + \langle t_{k_- - 1}^{(0)}, t_{k_- - 2}^{(k_- - 2)} \rangle & \text{if } (u,v) = (0,1), \\ \langle t_{k_- - 2}^{(0)}, t_{k_- - 1}^{(1)} \rangle + \langle t_{k_- - 1}^{(1)}, t_{k_- - 2}^{(1)} \rangle & \text{if } (u,v) = (1,0), \end{cases}$$

$$M_{\mathrm{cap}-}^{(u,v)}(-) = \begin{cases} \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)} & \text{if } (u,v) = (0,1), \\ \langle t_1^{(1)}, t_1^{(0)} \rangle_{(1)} & \text{if } (u,v) = (1,0). \end{cases}$$

If $r^* = 0$, then we need to replace the formula for $M_{\mathrm{cap}+}^{(u,v)}(-)$ with the following:

$$M_{\mathrm{cap}+}^{(u,v)}(-) = \begin{cases} \langle t_{k_- - 2}^{(k_- - 3)}, t_{k_- - 2}^{(k_- - 2)} \rangle_{(1)} & \text{if } (u,v) = (0,1), \\ \langle t_{k_- - 2}^{(0)}, t_{k_- - 2}^{(1)} \rangle_{(0)} & \text{if } (u,v) = (1,0). \end{cases}$$

(2) Let (u,v) be the mod 2 reduction of (u^*, v^*), i.e. $(u,v) \in P$, where $r^* = v^*/u^*$. Then

$$\mu_+^{(u,v)} = M_{\mathrm{up}}^{(u,v)}(+) + M_{\mathrm{down}}^{(u,v)}(+) + M_{\mathrm{cap}+}^{(u,v)}(+) + M_{\mathrm{cap}-}^{(u,v)}(+),$$

where $M_{\mathrm{up}}^{(u,v)}(+)$, $M_{\mathrm{down}}^{(u,v)}(+)$, $M_{\mathrm{cap}+}^{(u,v)}(+)$ and $M_{\mathrm{cap}-}^{(u,v)}(+)$ are given as follows:

$$M_{\mathrm{up}}^{(u,v)}(+) = \begin{cases} \sum_{i=1}^{k_+ - 3} \langle t_{c-(i+3)}^{(i)}, t_{c-(i+2)}^{(i+1)} \rangle & \text{if } k_+ > 3, \\ 0 & \text{if } k_+ = 3, \end{cases}$$

$$M_{\mathrm{down}}^{(u,v)}(+) = \begin{cases} \sum_{i=1}^{k_+ - 3} \langle t_{c-(i+2)}^{(i)}, t_{c-(i+3)}^{(i+1)} \rangle & \text{if } k_+ > 3, \\ 0 & \text{if } k_+ = 3, \end{cases}$$

$$M_{\mathrm{cap}+}^{(u,v)}(+) = \langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(0)},$$

$$M_{\mathrm{cap}-}^{(u,v)}(+) = \begin{cases} \langle t_{c-k_+}^{(k_+ - 2)}, t_{c-(k_+ + 1)}^{(1)} \rangle + \langle t_{c-(k_+ + 1)}^{(1)}, t_{c-k_+}^{(k_+ - 3)} \rangle & \text{if } r_{\mathbf{c}} \neq 1/2, \\ \langle t_1^{(k_+ - 2)}, t_1^{(k_+ - 3)} \rangle_{(1)} & \text{if } r_{\mathbf{c}} = 1/2. \end{cases}$$

Furthermore, we obtain the following description of $\partial_2(e_\pm)$ in terms of the meridional 1-cycles.

Lemma 4.12 ([13, Lemma 4.6]). *Suppose that $(u,v) \in P$ such that $(u,v) \equiv (u^*, v^*)$*

mod 2, *where $u^*/v^* = r^*$. Then we have*

$$\partial_2(e_-) = \mu_-^{(0,0)} + \mu_-^{(0,1)},$$
$$\partial_2(e_+) = \mu_+^{(0,0)} + \mu_+^{(u,v)}.$$

Remark 4.13. The formulas in Lemma 4.12 do not give the geometric boundaries of e_- and e_+. In fact, the geometric boundary ∂e_- is given by the following formula which contains a cancelling pair:

$$\partial e_- = \mu_-^{(0,0)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(1)} + \mu_-^{(0,1)} + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)}.$$

Remark 4.14. For each hyperbolic two-bridge link $K(r)$ and for each slope s in $\mathcal{S}'(r)$, we see by Lemma 3.1 that the length of $T(s)$ is greater than or equal to 2. Hence, for each ideal edge e of $\mathcal{D}(r)$, the degree of e is at least three.

5. Balanced 1-cocycles of the hyperbolic fibered two-bridge link complements

In this section, we recall a description of the canonical 1-cocycle of a hyperbolic fibered two-bridge link $K(r)$, given by [13, Section 5].

In the remainder of this paper, we consider only hyperbolic fibered two-bridge link $K(r) = K[2b_1, 2b_2, \ldots, 2b_m]$. Recall that each b_i is either $+1$ or -1 and $(b_1, b_2, \ldots, b_m) \neq \pm(1, -1, \ldots, (-1)^{m-1})$ (cf. Section 3). The canonical 1-cocycle of $K[2b_1, 2b_2, \ldots, 2b_m]$ is denoted by $\omega[2b_1, 2b_2, \ldots, 2b_m]$.

Recall that the 1-skeleton of $\mathcal{F}$ is illustrated as in Fig. 6. By the *l-th block* of $\mathcal{F}$ (the *l*-th block, in brief), we mean the set $\{|t_1^{(0)}, t_1^{(1)}|_{(j)} \mid j = 0, 1\}$, $\{|t_l^{(k)}, t_{l+1}^{(k')}| \mid k, k' \in \{0, 1\}\}$ or $\{|t_{c-3}^{(0)}, t_{c-3}^{(1)}|_{(j)} \mid j = 0, 1\}$ according as $l = 0$, $1 \leq l \leq c - 4$ or $l = c - 3$, respectively. (Here the symbol c denotes the length of the sequence of Farey triangles $\Sigma(r)$.) For each $0 \leq l \leq c - 3$, the values of the canonical 1-cocycle ω at the oriented edges contained in the *l*-th block of $\mathcal{F}$ have the same absolute value (see [13, Section 5]). Thus ω has the following description:

(a) For $1 \leq l \leq c - 4$, there is a positive real number d_l and a quadruple $(\varepsilon_{00}, \varepsilon_{01}, \varepsilon_{10}, \varepsilon_{11}) \in \{+, -\}^4$ such that $\omega(\varepsilon_{kk'} \langle t_l^{(k)}, t_{l+1}^{(k')} \rangle) = d_l$.

(b) For $l = 0$, there is a positive real number d_0 and a couple $(\varepsilon_0, \varepsilon_1) \in \{+, -\}^2$ such that $\omega(\varepsilon_j \langle t_1^{(0)}, t_1^{(1)} \rangle_{(j)}) = d_0$. For $l = c - 3$, there is a positive real number d_{c-3} and a couple $(\varepsilon_0, \varepsilon_1) \in \{+, -\}^2$ such that $\omega(\varepsilon_j \langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(j)}) = d_{c-3}$.

In the remainder of this paper, we employ the following convention.

Convention 5.1. (1) Under the above assumption, if for each *l*-th block and for each $k, k', j \in \{0, 1\}$, $\varepsilon_{kk'} = +$ and $\varepsilon_j = +$, then ω is depicted as in Fig. 10, where each arrow in an edge represents the oriented edge $\langle t_1^{(0)}, t_1^{(1)} \rangle_{(j)}$, $\langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(j)}$ or $\langle t_l^{(k)}, t_{l+1}^{(k')} \rangle$ for each $1 \leq l \leq c - 4$. We say that the values of ω at the *l*-th block are

given by $(d_l; \varepsilon_{00}, \varepsilon_{01}, \varepsilon_{10}, \varepsilon_{11})$ or $(d_l; \varepsilon_0, \varepsilon_1)$. We also call d_l the *weight* of ω at the l-th block.

(2) In Figs. 11–15, 16(a), 18 and 19, each edge of $\mathcal{F}$ is oriented so that $\omega(e) > 0$ (cf. Convention 2.6). In Figs. 16(b), 20, 21 and 22, inherits the orientation of the corresponding edge in $\mathcal{F}$.
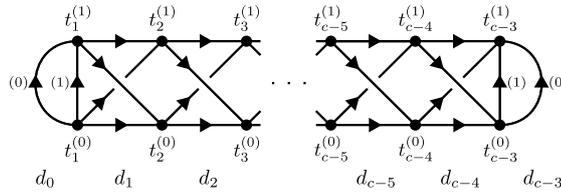
Fig. 10.   1-cochain ω, where $\varepsilon_{kk'} = +$ and $\varepsilon_j = +$ for every k, k' and j. The vertices $t_l^{(0)}$ are on the lower horizontal level, and the vertices $t_l^{(1)}$ are on the upper horizontal level.

In [13, Section 5], we proved that the canonical 1-cocycle of a hyperbolic fibered two-bridge link is obtained inductively from the following Propositions 5.2, 5.3 and 5.4.

Proposition 5.2 ([13, Proposition 5.6])**.** *The canonical 1-cocycle* $\omega[2b_1, 2b_1, 2b_3]$ *of* $K[2b_1, 2b_2, 2b_3]$ *is given by Fig. 11 under Convention 5.1.*

(a) $\omega[2, 2, 2]$

(b) $\omega[2, 2, -2]$ (c) $\omega[2, -2, -2]$
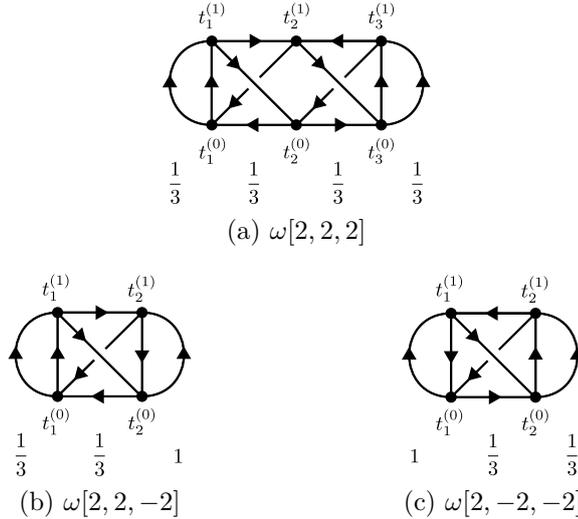
Fig. 11.   The canonical 1-cocycles of $K[2b_1, 2b_2, 2b_3]$.

Proposition 5.3 ([13, Proposition 5.7]). *Suppose that $m \geq 4$, and that $K[2b_1, \ldots, 2b_m]$ is a hyperbolic fibered two-bridge link such that $K[2b_1, \ldots, 2b_{m-1}]$ is a torus link, namely $(b_1, \ldots, b_{m-1})$ is alternating. Then the canonical 1-cocycle $\omega[2b_1, \ldots, 2b_m]$ is given by Fig. 12 under Convention 5.1, where the number of blocks with weight $1/2$ is equal to $m - 3$.*

Proposition 5.4 ([13, Proposition 5.8]). *Suppose that $m \geq 4$, and that $K[2b_1, \ldots, 2b_m]$ is a hyperbolic fibered two-bridge link such that $K[2b_1, \ldots, 2b_{m-1}]$ is not a torus link. Let N be the number of blocks of $\mathcal{F}[2b_1, \ldots, 2b_{m-1}]$. Then the following hold:*

(1) If $b_{m-1} = b_m$, then the number of blocks of $\mathcal{F}[2b_1, \ldots, 2b_{m-1}, 2b_m]$ is equal to $N + 2$.

   (a) If $b_{m-2} = b_{m-1}$, then the values of $\omega_{m-1} = \omega[2b_1, \ldots, 2b_{m-1}]$ at the last two blocks are given by $(1/3; \alpha, -\alpha, \alpha, -\alpha)$ and $(1/3; \alpha, \alpha)$ for some $\alpha \in \{+, -\}$ (see Fig. 13(a)). The values of $\omega_m = \omega[2b_1, \ldots, 2b_{m-1}, 2b_m]$ at the l-th block are equal to those of ω_{m-1} at the l-th block for any $1 \leq l \leq N - 2$. Moreover, the values of ω_m at the last four blocks are given by $(1/4; \alpha, -\alpha, \alpha, -\alpha)$, $(1/4; \alpha, \alpha, -\alpha, -\alpha)$, $(1/3; -\alpha, \alpha, -\alpha, \alpha)$ and $(1/3; -\alpha, -\alpha)$ (see Fig. 14(a)).

   (b) If $b_{m-2} \neq b_{m-1}$, then the values of ω_{m-1} at the last block are given by $(1; \alpha, -\alpha)$ for some $\alpha \in \{+, -\}$ (see Fig. 13(b)). The values of ω_m at the l-th block are equal to those of ω_{m-1} at the l-th block for any $1 \leq l \leq N - 1$. Moreover, the values of ω_m at the last three blocks are given by $(1/2; -\alpha, \alpha, \alpha, -\alpha)$, $(1/3; \alpha, -\alpha, \alpha, -\alpha)$ and $(1/3; \alpha, \alpha)$ (see Fig. 14(b)).

(2) If $b_{m-1} \neq b_m$, then the number of blocks of $\mathcal{F}[2b_1, \ldots, 2b_{m-1}, 2b_m]$ is equal to $N + 1$.

   (a) If $b_{m-2} = b_{m-1}$, then the values of ω_{m-1} at the last two blocks are given by $(1/3; \alpha, -\alpha, \alpha, -\alpha)$ and $(1/3; \alpha, \alpha)$ for some $\alpha \in \{+, -\}$ (see Fig. 13(a)). The values of ω_m at the l-th block are equal to those of ω_{m-1} at the l-th block for any $1 \leq l \leq N - 2$. Moreover, the values of ω_m at the last three blocks are given by $(1/4; \alpha, -\alpha, \alpha, -\alpha)$, $(1/4; \alpha, \alpha, -\alpha, -\alpha)$ and $(1; -\alpha, \alpha)$ (see Fig. 14(c)).

   (b) If $b_{m-2} \neq b_{m-1}$, then the values of ω_{m-1} at the last block are given by $(1; \alpha, -\alpha)$ for some $\alpha \in \{+, -\}$ (see Fig. 13(b)). The values of ω_m at the l-th block are equal to those of ω_{m-1} at the l-th block for any $1 \leq l \leq N - 1$. Moreover, the values of ω_m at the last two blocks are given by $(1/2; -\alpha, \alpha, \alpha, -\alpha)$ and $(1; \alpha, -\alpha)$ (see Fig. 14(d)).

By using Propositions 5.2 and 5.4, we obtain the following corollary, which plays an important role in the proof of the main result.

Corollary 5.5. *The canonical 1-cocycle $\omega = \omega[2, \ldots, 2]$ has the following property. For each $1 \leq l \leq c - 6$ and $k, k' \in \{0, 1\}$, the signs of $\omega(\langle t_l^{(k)}, t_{l+1}^{(k')} \rangle)$ and*
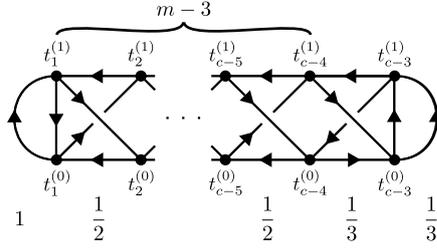
Fig. 12.   The canonical 1-cocycle of $K[2, -2, 2, -2, \ldots, \pm 2, \pm 2]$.

(a) $b_{m-2} = b_{m-1}$ (b) $b_{m-2} \neq b_{m-1}$

Fig. 13.   The values of ω_{m-1} at the last block(s) for $\alpha = +$.

(a) $b_{m-2} = b_{m-1} = b_m$ (b) $b_{m-2} \neq b_{m-1} = b_m$

(c) $b_{m-2} = b_{m-1} \neq b_m$ (d) $b_{m-2} \neq b_{m-1} \neq b_m$
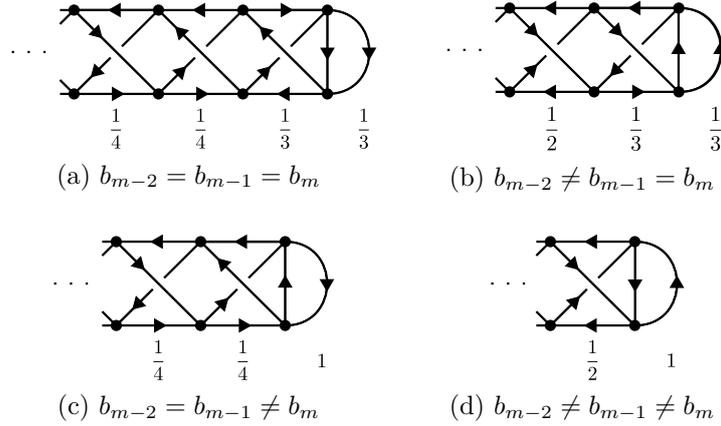
Fig. 14.   Replacing the values of ω_m at the last block(s) in ω_{m-1} with one of these values for $\alpha = +$.

$\omega(\langle t_{l+2}^{(k)}, t_{l+3}^{(k')} \rangle)$ *are different (see Fig. 15).*

Lemma 5.6. *For $K[2, \ldots, 2]$, the following holds. For $X, Y \in \langle X, Y \rangle < \mathrm{Aut}(\mathcal{D})$, we have $X^*(\omega) = Y^*(\omega) = -\omega$, where X^* and Y^* are the induced automorphisms of $\mathrm{H}^1(\mathcal{D})$.*

Proof. Note that if $q/p = [2, \ldots, 2]$, then $(p, q) \equiv (1, 0)$ or $(0, 1) \mod 2$. By using this fact, the automorphisms X^* and Y^* of $\mathrm{H}^1(\mathcal{D})$ have the following description:

(1) If p is odd, the $X^* = Y^* = -\mathrm{id}_{\mathrm{H}^1(\mathcal{D})}$.
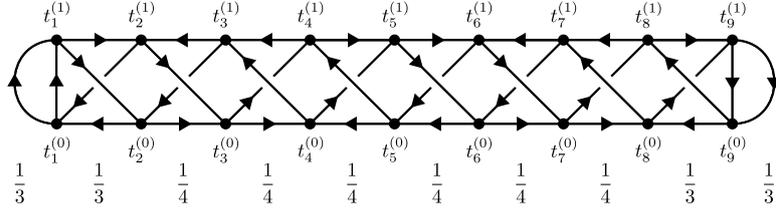
Fig. 15.   The canonical 1-cocycle $\omega[2, 2, 2, 2, 2, 2]$.

(2) If p is even, then

$$X^*(\mu_1^*, \mu_2^*) = (-\mu_2^*, -\mu_1^*),$$
$$Y^*(\mu_1^*, \mu_2^*) = (-\mu_1^*, -\mu_2^*),$$

where $\{\mu_1^*, \mu_2^*\}$ is the dual basis of the meridian pair $\{\mu_1, \mu_2\}$ of the 2-component oriented link $K(r)$.

Since ω is the unique harmonic representative of $p^*(1)$, we have $X^*(\omega) = Y^*(\omega) = -\omega$.   □

At the end of this section, we give the precise meaning of Theorem 1.1.

Theorem 5.7. *Let $K(r) = K[2b_1, \ldots, 2b_m]$ be a hyperbolic fibered two-bridge link, where $b_i \in \{\pm 1\}$ and $(b_1, b_2, \ldots, b_m) \neq \pm(1, -1, \ldots, (-1)^{m-1})$. Then the canonical decomposition of the complement of $K(r)$ is veering with respect to the layered structure given by the canonical 1-cocycle ω, if and only if $(b_1, b_2, \ldots, b_m) = \pm(1, 1, \ldots, 1)$.*

6. Proof of the "only if" part of Theorem 5.7

In this section, we prove the "only if" part of Theorem 5.7. Let $K(r) = K[2b_1, \ldots, 2b_m]$ and $\mathcal{D} = \mathcal{D}[2b_1, \ldots, 2b_m]$ be as in Theorem 5.7, and let $\omega = \omega[2b_1, \ldots, 2b_m]$ be the canonical 1-cocycle of $K[2b_1, \ldots, 2b_m]$.

The "only if" part of Theorem 5.7 follows from the following Lemmas 6.1 and 6.2.

Lemma 6.1. $\mathcal{D}[2, 2, -2]$ *and* $\mathcal{D}[2, -2, -2]$ *are not veering.*

Proof. Since $K[2, 2, -2]$ is equivalent to $K[2, -2, -2]$, we treat only $\mathcal{D}[2, 2, -2]$. Set $r = [2, 2, -2]$. Note that the sequence of Farey triangles $\Sigma(r) = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ by Lemma 3.1, where $\sigma_1 = (\infty, 1/1, 0/1)$, $\sigma_2 = (1/1, 1/2, 0/1)$, $\sigma_3 = (1/2, 1/3, 0/1)$, $\sigma_4 = (1/2, 2/5, 1/3)$, and $\sigma_5 = (2/5, 3/8, 1/3)$. Recall the symbols $\mathcal{S}(r)$ and $T(s)$ ($s \in \mathcal{S}(r)$) introduced in Notation 4.1. Then we have $T(1/2) = (\sigma_2, \sigma_3, \sigma_4)$, and therefore, by Lemma 4.8 and Remark 4.9, the geometric boundary of $e_{1/2}^{(0)}$ is given by the following formula (see Fig. 16):

$$\partial_2(\langle e_{1/2}^{(0)} \rangle) = \langle t_1^{(0)}, t_2^{(0)} \rangle + \langle t_2^{(1)}, t_1^{(1)} \rangle + \langle t_2^{(0)}, t_2^{(1)} \rangle_{(0)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)}.$$

Moreover, by Proposition 5.2, we have

$$\omega(\langle t_1^{(0)}, t_2^{(0)} \rangle) = -1/3 < 0,$$
$$\omega(\langle t_2^{(1)}, t_1^{(1)} \rangle) = -1/3 < 0,$$
$$\omega(\langle t_2^{(0)}, t_2^{(1)} \rangle_{(0)}) = +1 > 0 \text{ and}$$
$$\omega(\langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)}) = -1/3 < 0.$$

Hence, $t_2^{(1)}$ and $t_2^{(0)}$, respectively, are the maximal vertex and minimal vertex of the face $e_{1/2}^{(0)}$ of $\mathcal{F}$ with respect to ω (see Fig. 16). Thus the edge $|t_2^{(0)}, t_2^{(1)}|_{(0)}$ of $e_{1/2}^{(0)}$ in $\mathcal{F}$ connects the maximal vertex and the minimal vertex. Hence, $\mathcal{D}[2, 2, -2]$ is not veering by Lemma 2.12. □
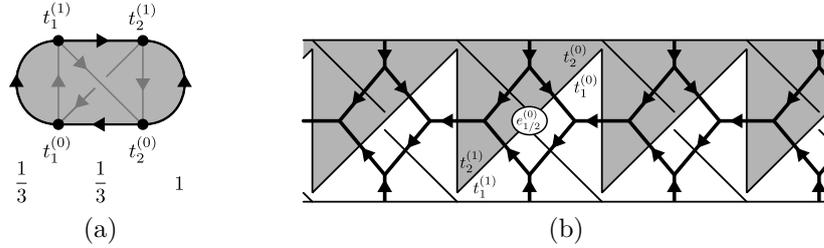
Fig. 16. (a) The face $e_{1/2}^{(0)}$ of $\mathcal{F}[2, 2, -2]$. (b) A part of the infinite cyclic cover of the annulus $\mathcal{A}(0, 0)$ for $\mathcal{D}[2, 2, -2]$.

Lemma 6.2. *If $m \geq 4$ and $b_i \neq b_j$ for some i, j, then $\mathcal{D}[2b_1, \ldots, 2b_m]$ is not veering.*

Proof. Let $[a_1, \ldots, a_n]$ be the continued fraction expansion of $r = [2b_1, \ldots, 2b_m]$ into positive integers with $a_n \geq 2$. Set $r^* = [a_1, \ldots, a_{n-1}, a_n - 2]$.

Case 1: $b_{m-1} \neq b_m$.

Subcase 1-a: $b_{m-2} \neq b_{m-1}$. By Lemma 3.1, we see $a_{n-1} = 1$ and $a_n \geq 3$ (see Fig. 17(a)). Thus $T(r^*) = (\sigma_{c-2}, \sigma_{c-1})$ and therefore we see by Lemma 4.8 that the following holds (see Fig. 18(a)):

$$\partial_2(\langle e_{r^*}^{(0)} \rangle) = \langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(0)} + \langle t_{c-3}^{(1)}, t_{c-4}^{(0)} \rangle + \langle t_{c-4}^{(0)}, t_{c-3}^{(0)} \rangle.$$

Hence $\mathcal{D}$ is not veering by Lemma 2.12 and Remark 4.9.

Subcase 1-b: $b_{m-2} = b_{m-1}$. By Lemma 3.1, we see $a_{n-1} = 1$ and $a_n = 2$ (see Fig. 17(b)). Thus $r^* = [a_1, \ldots, a_{n-2}]$ and $T(r^*) = (\sigma_{c-3}, \sigma_{c-2}, \sigma_{c-1})$. Hence, by Lemma 4.8 and Remark 4.9, the geometric boundary of $e_{r^*}^{(0)}$ is given by the following formula:

$$\partial_2(\langle e_{r^*}^{(0)} \rangle) = \langle t_{c-4}^{(0)}, t_{c-3}^{(0)} \rangle + \langle t_{c-3}^{(1)}, t_{c-4}^{(1)} \rangle + \langle t_{c-3}^{(0)}, t_{c-3}^{(1)} \rangle_{(0)} + \langle t_{c-4}^{(1)}, t_{c-5}^{(0)} \rangle + \langle t_{c-5}^{(0)}, t_{c-4}^{(0)} \rangle.$$

On the other hand, we can see by using Proposition 5.4 that $\{t^{(0)}_{c-3}, t^{(1)}_{c-3}\}$ forms the minimal and maximal vertices of $e^{(0)}_{r^*}$ (see Fig. 18(b)). Since the edge $|t^{(0)}_{c-3}, t^{(1)}_{c-3}|_{(0)}$ of $e^{(0)}_{r^*}$ in $\mathcal{F}$ connects these two vertices, $\mathcal{D}$ is not veering by Lemma 2.12.

Case 2: $b_{m-1} = b_m$. Let i_0 be the maximum integer i such that $b_{i-1} \neq b_i$ and $b_i = b_{i+1}$. Then, by Lemma 3.1, there exists an integer n' with $1 < n' < n$ such that $[a_1, \ldots, a_{n'}] = [2b_1, \ldots, 2b_{i_0}]$. Set $s = [a_1, \ldots, a_{n'-1}]$. Note that the number of triangles in the sequence $\Sigma([2b_1, \ldots, 2b_{i_0}])$ is equal to $c' = \sum_{i=1}^{n'} a_i$, and that if $i_0 = m - 1$, then $s = r^*$ (see Fig. 17(c)). By using Lemma 3.1, we have $T(s) = (\sigma_{c'-1}, \sigma_{c'}, \sigma_{c'+1})$. Thus, by Lemma 4.8 and Remark 4.9, the geometric boundary of $e^{(0)}_s$ is given by the following formula:

$$\partial_2(\langle e^{(0)}_s \rangle) = \begin{cases} \langle t^{(0)}_{c'-2}, t^{(0)}_{c'-1} \rangle + \langle t^{(1)}_{c'-1}, t^{(1)}_{c'-2} \rangle \\ \quad + \langle t^{(0)}_{c'-1}, t^{(1)}_{c'-1} \rangle_{(0)} + \langle t^{(1)}_{c'-2}, t^{(0)}_{c'-3} \rangle + \langle t^{(0)}_{c'-3}, t^{(1)}_{c'-2} \rangle & \text{if } i_0 = m - 1, \\ \langle t^{(0)}_{c'-2}, t^{(0)}_{c'-1} \rangle + \langle t^{(1)}_{c'-1}, t^{(1)}_{c'-2} \rangle + \langle t^{(0)}_{c'-1}, t^{(1)}_{c'} \rangle \\ \quad + \langle t^{(1)}_{c'}, t^{(1)}_{c'-1} \rangle + \langle t^{(1)}_{c'-2}, t^{(0)}_{c'-3} \rangle + \langle t^{(0)}_{c'-3}, t^{(1)}_{c'-2} \rangle & \text{if } i_0 < m - 1. \end{cases}$$

On the other hand, since $b_{i_0-1} \neq b_{i_0}$ and $b_{i_0} = b_{i_0+1}$, the set $\{t^{(0)}_{c'-2}, t^{(1)}_{c'-2}\}$ forms the minimal and maximal vertices of $e^{(0)}_s$ by Proposition 5.4 (see Fig. 19). Let $\widetilde{e}^{(0)}_s$ be the face of $\mathcal{T}^*$ contained in $|\mathcal{A}(0,0)|$ corresponding to the face $e^{(0)}_s$ of $\mathcal{F}$ (in the sense of Observation 2.5). By abuse of notation, we denote the edges of $\mathcal{T}^*$ contained in $|\mathcal{A}(0,0)|$ corresponding to the edges $|t^{(0)}_{c'-2}, t^{(1)}_{c'-1}|$ and $|t^{(0)}_{c'-4}, t^{(0)}_{c'-3}|$ of $\mathcal{F}$ by the same symbols. By Lemma 4.5, the edges $|t^{(0)}_{c'-2}, t^{(1)}_{c'-1}|$ and $|t^{(0)}_{c'-4}, t^{(0)}_{c'-3}|$ of $\mathcal{T}^*$ are adjacent to the boundary $\partial \widetilde{e}^{(0)}_s$ of $\widetilde{e}^{(0)}_s$ different from the minimal and maximal vertices of $\widetilde{e}^{(0)}_s$. Let ℓ and ℓ', respectively, be the edge-paths in $\partial \widetilde{e}^{(0)}_s$ connecting the minimal and maximal vertices of $\widetilde{e}^{(0)}_s$. We may assume that $|t^{(0)}_{c'-2}, t^{(1)}_{c'-1}|$ is adjacent to ℓ and $|t^{(0)}_{c'-4}, t^{(0)}_{c'-3}|$ is adjacent to ℓ'. Note that we have $\{\ell, \ell'\} = \{\partial_L(\widetilde{e}^{(0)}_s), \partial_R(\widetilde{e}^{(0)}_s)\}$. On the other hand, we can see by using Proposition 5.4 that both $|t^{(0)}_{c'-2}, t^{(1)}_{c'-1}|$ and $|t^{(0)}_{c'-4}, t^{(0)}_{c'-3}|$ of $\mathcal{F}$ are oriented either toward or away from the boundary of $e^{(0)}_s$ (see Fig. 19), and so precisely one of the following holds:

(1) The edge-paths ℓ and ℓ' are attractive at the vertices $t^{(1)}_{c'-1}$ and $t^{(1)}_{c'-3}$ in $\mathcal{T}^*$, respectively.

(2) The edge-paths ℓ and ℓ' are repulsive at the vertices $t^{(1)}_{c'-1}$ and $t^{(1)}_{c'-3}$ in $\mathcal{T}^*$, respectively.

Hence $\mathcal{D}$ is not veering by Proposition 2.9. □

7. Proof of the "if" part of Theorem 5.7

Throughout this section, we consider the hyperbolic two-bridge links $K(r)$ with $r = [2, \ldots, 2]$. For each $0 \leq i, j \leq 1$, $\mathcal{T}^*(i, j)$ denotes the subcomplex of the 1-skeleton of $\mathcal{T}^*$ made up of the edges of $\mathcal{T}^*$ which intersect $|\mathcal{A}(i, j)|$. Put $\mathcal{S}'' =$

(a) $b_{m-2} \neq b_{m-1} \neq b_m$ (b) $b_{m-2} = b_{m-1} \neq b_m$

(c) $b_{m-1} = b_m$
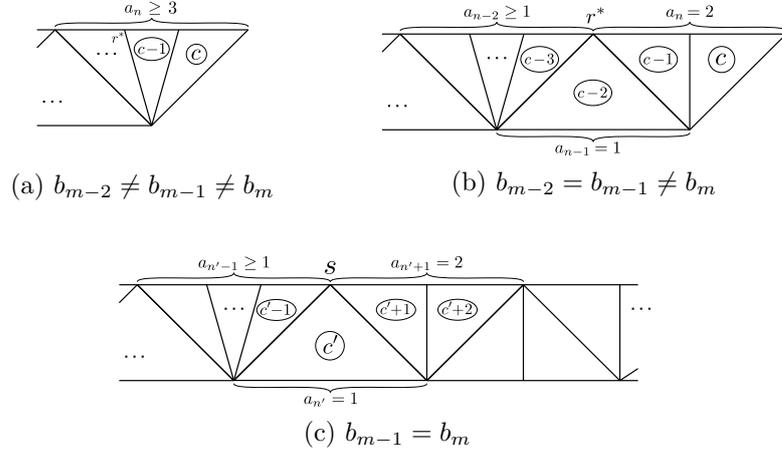
Fig. 17.   The sequence of Farey triangles $\Sigma([2b_1, \ldots, 2b_m])$. In (c), if $b_{m-2} \neq b_{m-1}$, then we have $n' = n - 1$, and so $s = r^*$.

(a) $b_{m-2} \neq b_{m-1} \neq b_m$ (b) $b_{m-2} = b_{m-1} \neq b_m$
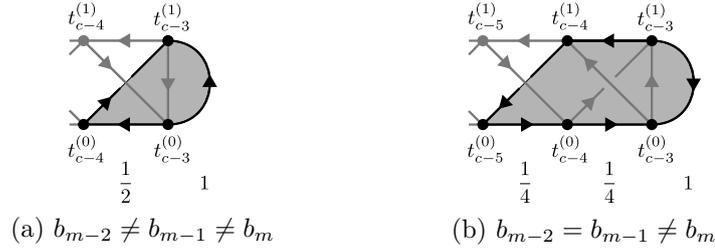
Fig. 18.   The face $e_{r^*}^{(0)}$ is shown as the grayed region.

$\mathcal{S}''(r) = \mathcal{S}'(r) \setminus (\sigma_2^{(0)} \cup \sigma_{c-1}^{(0)})$. Recall that the cusp cross section is oriented as stated in the paragraph after Definition 2.2.

By Proposition 2.9, the "if" part of the main theorem follows from the following Lemmas 7.1 and 7.2.

Lemma 7.1. *Suppose $m = 2$ and $b_1 = b_2 = +1$, i.e. $K(r)$ is the figure-eight knot. Then $\mathcal{D}$ is veering by the assignment of two colors to the ideal edges $\{e_-, e_+\}$ of $\mathcal{D}$ defined by*

$$\begin{cases} e_- : red, \\ e_+ : blue. \end{cases}$$

Though this lemma is well-known, we give a proof based on Proposition 2.9, so that the readers become familiar with Proposition 2.9.

Proof. Since there is an orientation-reversing automorphism of $\mathcal{D}[2, 2]$ which maps e_+ to e_- (cf. [14, Section II.3]), we may only prove that each of the vertices of $\mathcal{T}$

(a) $b_{i_0-2} = b_{i_0-1}$, $i_0 = m-1$

(c) $b_{i_0-2} = b_{i_0-1}$, $i_0 < m-1$

(b) $b_{i_0-2} \neq b_{i_0-1}$, $i_0 = m-1$

(d) $b_{i_0-2} \neq b_{i_0-1}$, $i_0 < m-1$
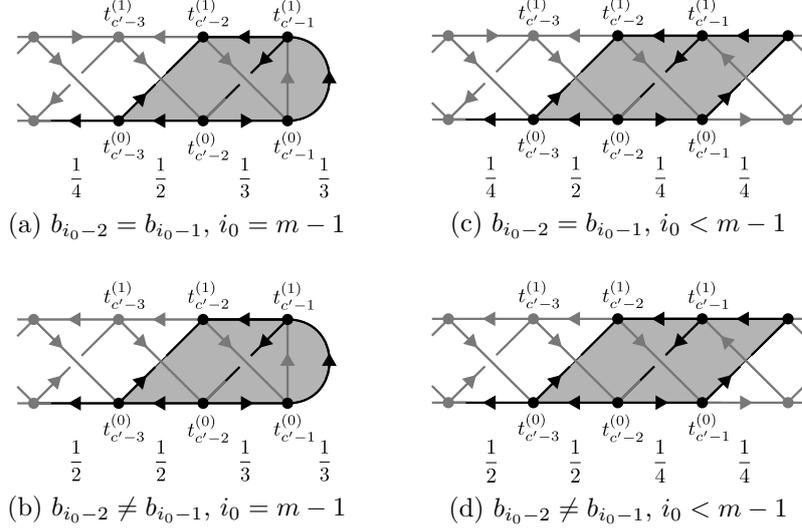
Fig. 19.   The face $e_s^{(0)}$ is shown as the grayed region.

with label e_- is RL in the sense of Definition 2.7(4).

The sequence of Farey triangles $\Sigma([2,2])$ is equal to $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_1 = (1/0, 1/1, 0/1)$, $\sigma_2 = (1/1, 1/2, 0/1)$, $\sigma_3 = (1/2, 1/3, 0/1)$ and $\sigma_4 = (1/2, 2/5, 1/3)$. Then we have $T(0/1) = (\sigma_1, \sigma_2, \sigma_3)$. Hence, by Lemmas 4.10 and 4.11, we have

$$\mu_-^{(0,0)} = \langle t_1^{(1)}, t_1^{(0)} \rangle'_{(0)} + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)},$$
$$\mu_-^{(0,1)} = \langle t_1^{(0)}, t_1^{(1)} \rangle'_{(1)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)}.$$

Now, let $v = e_-(0,0)$ be the vertex of $\mathcal{A}(0,0)$ with label e_-, and let v^* be the face of $\mathcal{T}^*$ dual to the vertex v. Then, by Lemma 4.12 and Remark 4.13, the geometric boundary ∂v^* is given by the following formula (see Fig. 20):

$$\partial_2(v^*) = \langle t_1^{(1)}, t_1^{(0)} \rangle'_{(0)} + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(1)}$$
$$+ \langle t_1^{(0)}, t_1^{(1)} \rangle'_{(1)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)} + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)}.$$

By [13, Example 5.5] and Fig. 20(b), the edge-paths $\partial_R(v^*)$ and $\partial_L(v*)$ are given by the following formulas:

$$\partial_R(v^*) = \left(\langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)}, \langle t_1^{(1)}, t_1^{(0)} \rangle'_{(0)}, \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)} \right),$$
$$\partial_L(v^*) = \left(\langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)}, \langle t_1^{(1)}, t_1^{(0)} \rangle'_{(1)}, \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)} \right).$$

By using Lemma 4.5 and Observation 4.7, the edges $|t_1^{(0)}, t_1^{(1)}|_{(0)}$ and $|t_1^{(0)}, t_1^{(1)}|'_{(1)}$ (contained in $\mathcal{T}^*(0,0)$) are adjacent to $\partial_R(v^*)$ at interior vertices of $\partial_R(v^*)$ (see Fig. 20(b)). Since the edges are oriented toward $\partial_R(v^*)$, the edge-path $\partial_R(v^*)$ is attractive. Similarly, the edges $|t_1^{(0)}, t_1^{(1)}|'_{(0)}$ and $|t_1^{(0)}, t_1^{(1)}|_{(0)}$ (contained in $\mathcal{T}^*(0,1)$)

are adjacent to $\partial_L(v^*)$ at interior vertices of $\partial_L(v^*)$. Since the edges are oriented away from $\partial_L(v^*)$, the edge-path $\partial_L(v^*)$ is repulsive. Hence the vertex v of $\mathcal{A}(0,0)$ is RL, and so the vertices of $\mathcal{T}$ with label e_- are RL by Lemma 2.10. $\qquad\square$
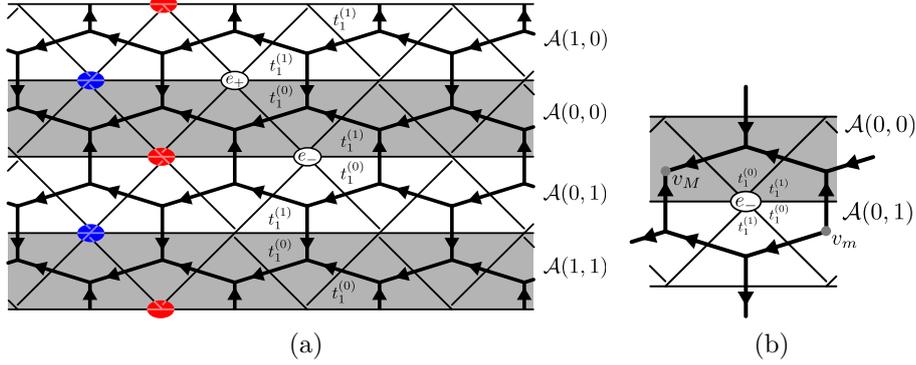
Fig. 20. (a) A part of the universal cover of $\mathcal{T}[2,2]$. (b) A neighborhood of the vertex of $\mathcal{A}(0,0)$ with label e_-. The symbols v_M and v_m denote the maximal and minimal vertices of the dual face e_-, respectively.

Lemma 7.2. *Suppose that $m \geq 3$ and $b_1 = \cdots = b_m = +1$. Then $\mathcal{D}$ is veering by the assignment of two colors to the ideal edges of $\mathcal{D}$ defined by*

$e_- : red,$

$$e_s^{(k)} : \begin{cases} red & \text{if } s \in \mathcal{S}'(r) \text{ is equal to } [2b_1, \ldots, 2b_{2m'}] \text{ or } [2b_1, \ldots, 2b_{2m'}, 1], \\ blue & \text{if } s \in \mathcal{S}'(r) \text{ is equal to } [2b_1, \ldots, 2b_{2m'-1}] \text{ or } [2b_1, \ldots, 2b_{2m'-1}, 1], \end{cases}$$

$$e_+ : \begin{cases} red & \text{if } m \text{ is odd, i.e. } K(r) \text{ is a two component link,} \\ blue & \text{if } m \text{ is even, i.e. } K(r) \text{ is a knot,} \end{cases}$$

for any $k \in \{0,1\}$.

Proof. We begin by proving the following claims.

Claim 1. *Let s be an element of $\mathcal{S}'(r)$. Let f (resp. f') be the face of $\mathcal{T}^*$ dual to one of the two vertices of $\mathcal{T}$ with label $e_s^{(0)}$ (resp. $e_s^{(1)}$). Then f is RL or LR according to whether f' is RL or LR, and vice versa.*

Proof of Claim 1. We treat only the case when f is RL. Let (i,j) be a pair $(0,1)$ or $(1,0)$ such that $(i,j) \not\equiv (p_s, q_s) \mod 2$, where $s = q_s/p_s$. Then the ideal edge $e_s^{(1)}$ intersects $\mathcal{A}(i,j)$ by Observation 4.7. By Lemma 2.10, we may assume $f \subset |\mathcal{A}(0,0)|$ and $f' \subset |\mathcal{A}(i,j)|$. Then the orientation-preserving automorphism

$X^i Y^j$ of $\mathcal{D}$ maps f to f'. Note that $(X^i Y^j)^*(\omega) = -\omega$ by Lemma 5.6. Hence $X^i Y^j$ induces an orientation-preserving isomorphism form $\mathcal{A}(0,0)$ to $\mathcal{A}(i,j)$, which "reverse" the orientation of the edges of $\mathcal{T}^*$. To be precise, if e is an edge of $\mathcal{A}(0,0)$, then the orientation of its copy $X^i Y^j(e)$ in $\mathcal{A}(i,j)$ determined by ω is opposite to the orientation of e determined by ω. Hence, by Remark 2.8(1), the face f is RL if and only if f' is RL. $\qquad\square$

Claim 2. *Let (s_l, s_{l+2}) be a pair of elements of $\mathcal{S}''(r)$. Let f (resp. f') be the face of $\mathcal{T}^*$ dual to the vertex of $\mathcal{A}(0,0) \subset \mathcal{T}$ with label $e_{s_l}^{(0)}$ (resp. $e_{s_{l+2}}^{(0)}$). Then f is RL or LR according to whether f' is LR or RL, and vice versa.*

Proof of Claim 2. Since $b_1 = \cdots = b_m = +1$ and since (s_l, s_{l+2}) is a pair of elements of $\mathcal{S}''(r)$, we see from the construction of $\mathcal{D}$ and $\mathcal{T}$ that there is an orientation-reversing (combinatorial) isomorphism τ from the 2-cell $f \subset |\mathcal{A}(0,0)|$ to the 2-cell $f' \subset |\mathcal{A}(0,0)|$, which maps an edge $\langle t_l^{(k)}, t_{l+1}^{(k')} \rangle$ in the boundary of f to the edge $\langle t_{l+2}^{(k)}, t_{l+3}^{(k')} \rangle$ in the boundary of f'. Moreover, we see by using Corollary 5.5 that τ reverses the orientation of the edges (specified by ω) which has a vertex in the boundary of f. Hence f is RL if and only if f' is LR. $\qquad\square$

By [14, Section II.3], there is an involution ι of $\mathcal{D}[2b_1, \ldots, 2b_m]$ which interchanges e_+ with e_- and $e_{r*}^{(0)}$ with $e_{1/2}^{(0)}$. Moreover, ι is orientation-preserving or orientation-reversing according to whether m is odd or even. By this fact and Claims 1 and 2, it is enough to show that each of the vertices of $\mathcal{T}$ with label e_- or $e_{2/5}^{(0)}$ is RL, and each of the vertices of $\mathcal{T}$ with label $e_{1/2}^{(0)}$ or $e_{1/3}^{(0)}$ is LR. (When $m = 3$, $e_{2/5}^{(0)}$ should be ignored since such an ideal edge does not exist.)

We first prove that the vertices with label e_- are RL. Let $v = e_-(0,0)$ be the vertex of $\mathcal{A}(0,0)$ with label e_-, and let v^* be the face of $\mathcal{T}^*$ dual to the vertex v. Since $T(0/1) = (\sigma_1, \sigma_2, \sigma_3)$, we see by Lemma 4.12 and Remark 4.13 that the geometric boundary of v^* is given by the following formula:

$$\partial_2(v^*) = \langle t_1^{(1)}, t_2^{(1)} \rangle + \langle t_2^{(1)}, t_1^{(0)} \rangle + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)} + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(1)}$$
$$+ \langle t_1^{(0)}, t_2^{(0)} \rangle + \langle t_2^{(0)}, t_1^{(1)} \rangle + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)} + \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)}.$$

By Propositions 5.2 and 5.4 (see Fig. 22(a)), we have

$$\partial_R(v^*) = \left(\langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)}, \langle t_1^{(1)}, t_2^{(1)} \rangle, \langle t_2^{(1)}, t_1^{(0)} \rangle, \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)} \right),$$
$$\partial_L(v^*) = \left(\langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)}, \langle t_1^{(1)}, t_2^{(0)} \rangle, \langle t_2^{(0)}, t_1^{(0)} \rangle, \langle t_1^{(0)}, t_1^{(1)} \rangle_{(1)} \right).$$

Note that $\partial_R(v^*)$ and $\partial_L(v^*)$ are edge-paths of $\mathcal{T}^*(0,0)$ and $\mathcal{T}^*(0,1)$ respectively. We see by Propositions 5.2 and 5.4, Lemma 4.5 and Observation 4.7 that $\partial_R(v^*)$ is attractive and $\partial_L(v^*)$ is repulsive (see Fig. 22(a)). Hence v is RL, and so the vertices of $\mathcal{T}$ with label e_- are RL by Lemma 2.10.
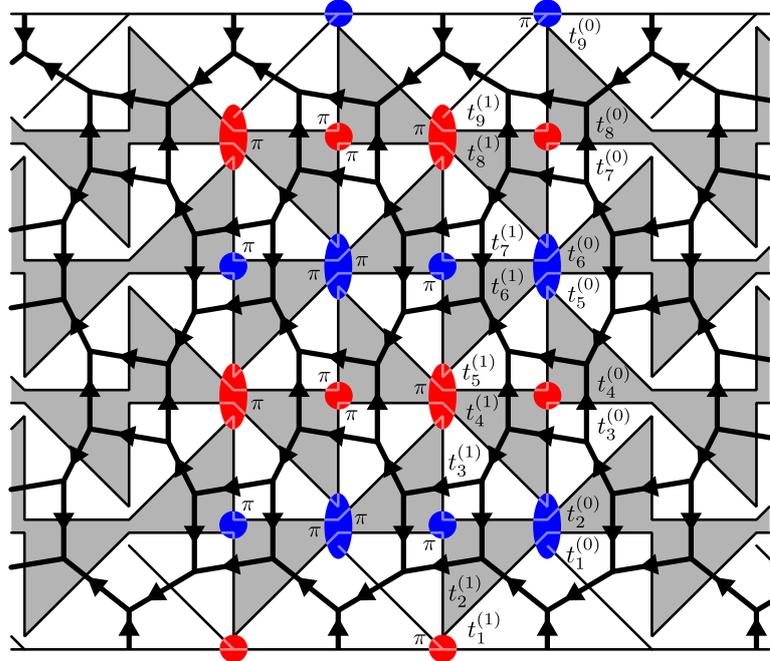
Fig. 21.   A part of the infinite cyclic cover of the annulus $\mathcal{A}(0,0)$ for $\mathcal{D}[2,2,2,2,2,2]$.

(a) e_-

(b) $e_{1/2}^{(0)}$

(c) $e_{1/3}^{(0)}$
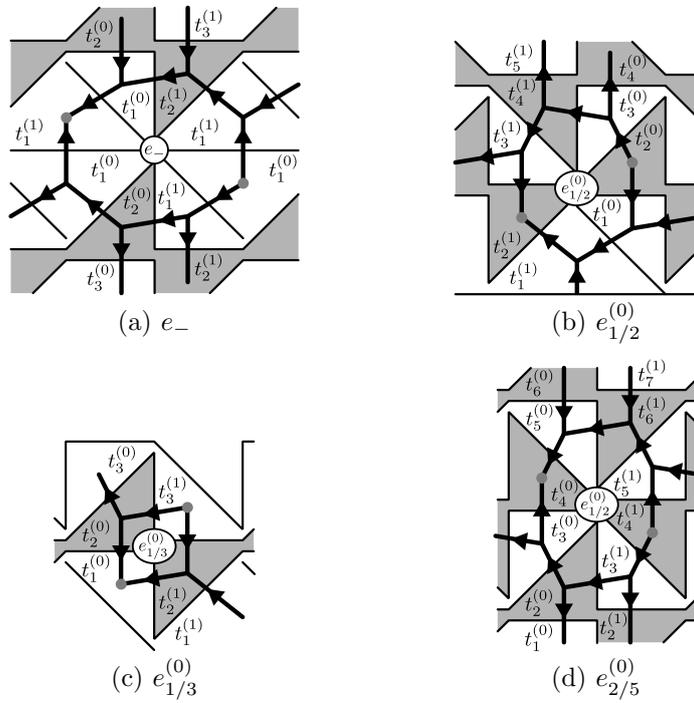
(d) $e_{2/5}^{(0)}$

Fig. 22.   Neighborhoods of dual faces e_-, $e_{1/2}^{(0)}$, $e_{1/3}^{(0)}$ and $e_{2/5}^{(0)}$.

Next, we shall prove that the vertices of $\mathcal{T}$ with label $e_{1/2}^{(0)}$ are LR. Let $v = e_{1/2}^{(0)}(0,0)$ be the vertex of $\mathcal{A}(0,0)$ with label $e_{1/2}^{(0)}$, and let v^* be the face of $\mathcal{T}^*$ dual to the vertex v. Since $T(1/2) = (\sigma_2, \sigma_3, \sigma_4, \sigma_5)$, we see by Lemma 4.8 and Remark 4.9 that the geometric boundary of v^* is given by the following formula when $m \geq 4$:

$$\partial_2(v^*) = \langle t_1^{(0)}, t_2^{(0)} \rangle + \langle t_2^{(0)}, t_3^{(0)} \rangle + \langle t_3^{(1)}, t_2^{(1)} \rangle + \langle t_2^{(1)}, t_1^{(1)} \rangle$$
$$+ \langle t_3^{(0)}, t_4^{(1)} \rangle + \langle t_4^{(1)}, t_3^{(1)} \rangle + \langle t_1^{(1)}, t_1^{(0)} \rangle_{(0)}.$$

Hence, by Propositions 5.2 and 5.4 (see Figs. 21 and 22(b)), we have the following when $m \geq 4$:

$$\partial_R(v^*) = \left(\langle t_2^{(0)}, t_3^{(0)} \rangle, \langle t_3^{(0)}, t_4^{(1)} \rangle, \langle t_4^{(1)}, t_3^{(1)} \rangle, \langle t_3^{(1)}, t_2^{(1)} \rangle \right),$$
$$\partial_L(v^*) = \left(\langle t_2^{(0)}, t_1^{(0)} \rangle, \langle t_1^{(0)}, t_1^{(1)} \rangle_{(0)}, \langle t_1^{(1)}, t_2^{(1)} \rangle \right).$$

Note that each edge in these edge-paths is an element of $\mathcal{T}^*(0,0)$. (When $m = 3$, the two edges $|t_3^{(0)}, t_4^{(1)}|$ and $|t_3^{(1)}, t_4^{(1)}|$ become identified.) We see by Propositions 5.2 and 5.4 and Lemma 4.5 that $\partial_R(v^*)$ is repulsive and $\partial_L(v^*)$ is attractive (see Figs. 21 and 22(b)). Hence the vertex v is LR, and so the vertices of $\mathcal{T}$ with label $e_{1/2}^{(0)}$ are LR by Lemma 2.10.

Next, we shall show that the vertices of $\mathcal{T}^*$ with label $e_{1/3}^{(0)}$ are LR. Let $v = e_{1/3}^{(0)}(0,0)$ be the vertex of $\mathcal{A}(0,0)$ with label $e_{1/3}^{(0)}$, and let v^* be the face of $\mathcal{T}^*$ dual to the vertex v. Since $T(1/3) = (\sigma_3, \sigma_4)$, we see by Lemma 4.8 and Remark 4.9 that the geometric boundary of v^* is given by the following formula:

$$\partial_2(v^*) = \langle t_2^{(0)}, t_3^{(1)} \rangle + \langle t_3^{(1)}, t_2^{(1)} \rangle + \langle t_2^{(1)}, t_1^{(0)} \rangle + \langle t_1^{(0)}, t_2^{(0)} \rangle.$$

Hence, by Propositions 5.2 and 5.4 (see Figs. 21 and 22(c)), we have

$$\partial_R(v^*) = \left(\langle t_2^{(0)}, t_1^{(0)} \rangle, \langle t_1^{(0)}, t_2^{(1)} \rangle \right),$$
$$\partial_L(v^*) = \left(\langle t_2^{(0)}, t_3^{(1)} \rangle, \langle t_3^{(1)}, t_2^{(1)} \rangle \right).$$

Note that each edge in these edge-paths is an element of $\mathcal{T}^*(0,0)$. We see by Propositions 5.2 and 5.4 and Lemma 4.5 that $\partial_R(v^*)$ is repulsive and $\partial_L(v^*)$ is attractive (see Figs. 21 and 22(c)). Hence the vertex v is LR, and so the vertices of $\mathcal{T}$ with label $e_{1/3}^{(0)}$ are LR by Lemma 2.10.

Finally, we shall show that the vertices of $\mathcal{T}$ with label $e_{2/5}^{(0)}$ are RL. Let $v = e_{2/5}^{(0)}(0,0)$ be the vertex of $\mathcal{A}(0,0)$ with label $e_{2/5}^{(0)}$, and let v^* be the face of $\mathcal{T}^*$ dual to the vertex v. Since $T(2/5) = (\sigma_4, \sigma_5, \sigma_6, \sigma_7)$, we see by Lemma 4.8 and Remark 4.9 that the geometric boundary of v^* is given by the following formula:

$$\partial_2(v^*) = \langle t_3^{(0)}, t_4^{(0)} \rangle + \langle t_4^{(0)}, t_5^{(0)} \rangle + \langle t_5^{(1)}, t_4^{(1)} \rangle + \langle t_4^{(1)}, t_3^{(1)} \rangle$$
$$+ \langle t_5^{(0)}, t_6^{(1)} \rangle + \langle t_6^{(1)}, t_5^{(1)} \rangle + \langle t_3^{(1)}, t_2^{(0)} \rangle + \langle t_2^{(0)}, t_3^{(0)} \rangle.$$

Hence, by Propositions 5.2 and 5.4 (see Figs. 21 and 22(d)), we have

$$\partial_R(v^*) = \left(\langle t_4^{(1)}, t_5^{(1)} \rangle, \langle t_5^{(1)}, t_6^{(1)} \rangle, \langle t_6^{(1)}, t_5^{(0)} \rangle, \langle t_5^{(0)}, t_4^{(0)} \rangle \right),$$

$$\partial_L(v^*) = \left(\langle t_4^{(1)}, t_3^{(1)} \rangle, \langle t_3^{(1)}, t_2^{(0)} \rangle, \langle t_2^{(0)}, t_3^{(0)} \rangle, \langle t_3^{(0)}, t_4^{(0)} \rangle \right).$$

Note that each edge in these edge-paths is an element of $\mathcal{T}^*(0, 0)$. We see by Propositions 5.2 and 5.4 and Lemma 4.5 that $\partial_R(v^*)$ is attractive and $\partial_L(v^*)$ is repulsive (see Figs. 21 and 22(d)). Hence the vertex v is RL, and so the vertices of $\mathcal{T}$ with label $e_{2/5}^{(0)}$ are RL by Lemma 2.10. □

Acknowledgements

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