Stable extendibility of the tangent bundles over lens spaces

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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ABSTRACT. The purpose of this paper is to study the stable extendibility of the tangent bundle \( \tau_n(p) \) of the \((2n+1)\)-dimensional standard lens space \( L^n(p) \) for odd prime \( p \). We investigate the value of integer \( m \) for which \( \tau_n(p) \) is stably extendible to \( L^m(p) \) but not stably extendible to \( L^{m+1}(p) \), and in particular we completely determine \( m \) for \( p = 5 \) or 7. A stable splitting of \( \tau_n(p) \) and the stable extendibility of a Whitney sum of \( \tau_n(p) \) are also discussed.

1. Introduction

Let \( F \) be the real number field \( R \), the complex number field \( C \) or the quaternion number field \( H \). For a subspace \( A \) of a space \( X \), a \( t \)-dimensional \( F \)-vector bundle \( \zeta \) over \( A \) is said to be extendible to \( X \), if there is a \( t \)-dimensional \( F \)-vector bundle over \( X \) whose restriction to \( A \) is equivalent to \( \zeta \), that is, if \( \zeta \) is equivalent to the induced bundle \( i^*\eta \) of a \( t \)-dimensional \( F \)-vector bundle \( \eta \) over \( X \) under the inclusion map \( i: A \to X \). Instead, if \( i^*\eta \) is stably equivalent to \( \zeta \), namely \( i^*\eta + m \) is equivalent to \( \zeta + m \) for a trivial \( F \)-vector bundle \( m \) of dimension \( m \geq 0 \), \( \zeta \) is called stably extendible to \( X \) (cf. [10], p. 20 and [4], p. 273).

Let \( L^n(p) = S^{2n+1}/\mathbb{Z}_p \) denote the \((2n+1)\)-dimensional standard lens space mod \( p \). For an \( R \)-vector bundle \( \zeta \) over \( L^n(p) \), we define an integer \( s(\zeta) \) by

\[
s(\zeta) = \max\{m \mid m \geq n \text{ and } \zeta \text{ is stably extendible to } L^m(p) \},
\]

where \( s(\zeta) = \infty \) if \( \zeta \) is stably extendible to \( L^m(p) \) for every \( m \geq n \).

Let \( \tau_n(p) = \tau(L^n(p)) \) be the tangent bundle of \( L^n(p) \). Then, concerning \( s(\tau_n(p)) \), the following theorems have been obtained.

THEOREM ([7], Theorem 5.3). Let \( p \) be an integer \( > 1 \). Then, \( s(\tau_n(p)) = \infty \) if \( n = 0, 1 \) or 3.
THEOREM ([8], Theorem 4.3). Let $p$ be an odd prime. Then, $s(\tau_n(p)) < 2n + 2$, if $n \geq 2p - 2$.

THEOREM ([6], Theorem 1). $s(\tau_n(3)) = \infty$ if and only if $0 \leq n \leq 3$.

The purpose of this paper is to develop these results on the stable extendibility of the tangent bundle $\tau_n(p)$. Our main results are stated as follows.

**Theorem 1.** Let $p$ be an odd prime. Then, $s(\tau_n(p)) = 2n + 1$ if $n \geq 2p - 2$.

**Theorem 2.** (1) If $0 \leq n \leq 5$, then $s(\tau_n(5)) = \infty$.
(2) If $n \geq 6$, then $s(\tau_n(5)) = 2n + 1$.

**Theorem 3.** (1) If $0 \leq n \leq 7$, then $s(\tau_n(7)) = \infty$.
(2) If $n \geq 8$, then $s(\tau_n(7)) = 2n + 1$.

These theorems give support to our following conjecture.

**Conjecture.** For an odd prime $p$,

$s(\tau_n(p)) = \infty$ for $0 \leq n \leq p$, and $s(\tau_n(p)) = 2n + 1$ for $n \geq p + 1$.

We organize the paper as follows. In §2, we state some known results necessary to establish our results. In §3 we prove Theorem 1. In §4, we study $\tau_n(5)$ and $\tau_n(7)$ and prove Theorems 2 and 3. In §5, as a consequence of the preceding results, we give Theorem 4 concerning Schwarzenberger's property. In §6, we study the extendibility of the $m$-times Whitney sum $m\tau_n(p)$ of $\tau_n(p)$ for $m \geq 1$, and show in Proposition 6.1 the inequality

$s(m\tau_n(p)) \geq m(2n + 1)$ or $s(m\tau_n(p)) \geq m(2n + 1) - 1$

if $m$ is an odd or even integer respectively. Then, in Theorem 5 we give some condition for

$s(m\tau_n(p)) = m(2n + 1)$ or $m(2n + 1) - 1 \leq s(m\tau_n(p)) \leq m(2n + 1) + 1$

to hold according as $m$ is odd or even.

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2. Preliminary

For an odd prime $p$, the structures of the reduced $K$-ring $\tilde{K}(L^n(p))$ and the reduced $KO$-ring $\tilde{KO}(L^n(p))$ are determined by Kambe [5].

Let $\eta$ be the canonical $C$-line bundle over $L^n(p)$, the induced bundle from the canonical $C$-line bundle over the complex projective space $CP^n$ under
the projection $\pi : L^n(p) \to CP^n$, and $\sigma = \eta - 1$ its stable class in $\tilde{K}(L^n(p))$. Sometimes, we denote $\eta$ (resp. $\sigma$) by $\eta_n$ (resp. $\sigma_n$) to make it clear that $\eta$ (resp. $\sigma$) is over $L^n(p)$.

Let $r : \tilde{K}(X) \to \tilde{KO}(X)$ and $c : \tilde{KO}(X) \to \tilde{K}(X)$ be the homomorphisms induced by the real restriction and the complexification of the vector bundles, respectively. We set $\tilde{\sigma} = r(\sigma)$ in $\tilde{KO}(L^n(p))$. Also, let $L^n_0(p)$ denote the $2n$-skeleton of $L^n(p)$ as in [5].

Then, we shall use the following result, where $[x]$ denotes the largest integer $m$ with $m \leq x$ for a real number $x$.

**THEOREM 2.1** ([5], Theorem 2, Lemma 3.4).

1. We have the following isomorphism of abelian groups:
   
   $$
   \tilde{KO}(L^n(p)) \cong \begin{cases} 
   \tilde{KO}(L^n_0(p)) & \text{if } n \not\equiv 0 \mod 4, \\
   Z_2 + \tilde{KO}(L^n_0(p)) & \text{if } n \equiv 0 \mod 4.
   \end{cases}
   $$

2. Let $q = (p - 1)/2$ and $n = s(p - 1) + r$ ($0 \leq r < p - 1$). Then,
   
   $$
   \tilde{KO}(L^n_0(p)) = (Z_{p+1})^{[r/2]} + (Z_{p+1})^{q-[r/2]},
   $$
   
   and the direct summand $(Z_{p+1})^{[r/2]}$ and $(Z_{p+1})^{q-[r/2]}$ are generated additively by $\tilde{\sigma}^1, \ldots, \tilde{\sigma}^{[r/2]}$ and $\tilde{\sigma}^{[r/2]+1}, \ldots, \tilde{\sigma}^q$ respectively. Moreover, the ring structure is given by
   
   $$
   \tilde{\sigma}^{n+1} = \sum_{i=1}^{q} \binom{-2q + 1}{2i - 1} \binom{q + i - 1}{2i - 2} \tilde{\sigma}^i, \quad \tilde{\sigma}^{[n/2]+1} = 0,
   $$
   
   where $\binom{a}{b}$ denotes a binomial coefficient.

We also apply the following property.

**LEMMA 2.2** ([5], Lemma 3.5(2)). The homomorphism $c : \tilde{KO}(L^n_0(p)) \to \tilde{K}(L^n_0(p))$ is a monomorphism.

The following theorem due to Sjerve [11] is crucial in our proof, where $\pi_m : S^{2m+1} \to L^m(p)$ is the natural projection.

**THEOREM 2.3** ([11], Theorem A). If $\zeta \in \tilde{KO}(L^m(p)) \cap \ker \pi_m^*$, then the geometrical dimension of $\zeta$ satisfies $\text{g.dim} \zeta \leq 2\left[\frac{m}{2}\right] + 1$.

### 3. Proof of Theorem 1

By Theorem 2.3, we have the following.

**PROPOSITION 3.1.** For any $n \geq 1$, $s(\tau_n(p)) \geq 2n + 1$. 


PROOF. Let $m \geq n$ be an integer. Since $r(\eta_m) = r(\pi_m) - 2 \in \ker \pi_m \subseteq \tilde{KO}(L^n(p))$ for the projection $\pi_m : S^{2m+1} \to L^n(p)$, where $r : \tilde{K}(L^n(p)) \to \tilde{KO}(L^n(p))$ is the homomorphism mentioned in §2, we have
\[ g \dim ((n+1)(r(\eta_m) - 2)) \leq 2 \left\lfloor \frac{m}{2} \right\rfloor + 1 \]
by Theorem 2.3. Thus, there is a $(2\left\lfloor \frac{m}{2} \right\rfloor + 1)$-dimensional vector bundle $\beta$ over $L^n(p)$ satisfying that $(n+1)r(\eta_m)$ is stably equivalent to $\beta$. When $m = 2n + 1$, we have $2\left\lfloor \frac{m}{2} \right\rfloor + 1 = 2n + 1$ and thus $\beta$ is of dimension $2n + 1$. Hence, $(n+1)r(\eta_{2n+1})$ is stably equivalent to $\beta + 1$, and $\tau_n(p)$ is stably equivalent to $i^*\beta$ since $\tau_n(p) + 1 = (n+1)r(\eta_n)$ is stably equivalent to $i^*\beta + 1$. Therefore, $\tau_n(p)$ is stably extendible to $L^{2n+1}(p)$, and we have the required inequality $s(\tau_n(p)) \geq 2n + 1$.

PROOF OF THEOREM 1. By Theorem 4.3 in [8], we have $s(\tau_n(p)) < 2n + 2$ as we described in §1. Thus, by Proposition 3.1, we obtain the required result. \[ \square \]

REMARK 3.2. Proposition 3.1 is a special case of Theorem 4.2 in [7]. Therefore, Theorem 1 is originally due to Kobayashi-Maki-Yoshida ([7], [8]).

4. Stable extendibility of $\tau_n(5)$ and $\tau_n(7)$

Let $p$ be an odd prime. Hereafter, we use the same notation $\alpha$ to denote the stable class of $\alpha$ in $KO(L^n(p))$ (resp. $\tilde{K}(L^n(p))$) for a real (resp. complex) vector bundle $\alpha$ over $L^n(p)$. Also, we simply denote by $\alpha = \beta$ that vector bundle $\alpha$ and $\beta$ are stably equivalent.

Using ring structures of $KO(L^n(p))$ and $K(L^n(p))$ for an odd prime $p$, we have the following lemma, where and hereafter we denote $r(\eta)$ (resp. $c(r(\eta))$) simply by $r\eta$ (resp. $c(r\eta)$) for the homomorphisms $r : K(L^n(p)) \to KO(L^n(p))$ and $c : KO(L^n(p)) \to K(L^n(p))$.

**Lemma 4.1.** In $KO(L^n(p))$,
\[ (r\eta)^2 = r(\eta^2) + 2, \quad (r\eta)^3 = r(\eta^3) + 3r\eta. \]

**Proof.** Recall that $c(r\eta) = \eta + \eta^{-1}$ (cf. [3], Proposition 11.3, p. 191). Since $c : KO(L^n(p)) \to K(L^n(p))$ is a ring homomorphism, we have $c(r(\eta^2)) = \eta^2 + \eta^{-2}$ and $c((r\eta)^2) = (r\eta)^2 + (\eta + \eta^{-1})^2 = \eta^2 + \eta^{-2} + 2$. Then, by Lemma 2.2, $(r\eta)^2 = r(\eta^2) + 2$ in $KO(L^n(p))$. In the same way, $c(r(\eta^3)) = \eta^3 + \eta^{-3}$ and $c((r\eta)^3) = (r\eta)^3 + (\eta + \eta^{-1})^3 = \eta^3 + \eta^{-3} + 3(\eta + \eta^{-1})$. Thus, we have $(r\eta)^3 = r(\eta^3) + 3r\eta$, and complete the proofs. \[ \square \]

Since $\tau_n(p)$ is stably trivial for $n = 0$ or $1$ (cf. [7]), we have
LEMMA 4.2.

\[ s(\tau_n(p)) = \infty \quad \text{for } n = 0 \text{ or } 1. \]

Concerning \( \tau_n(5) \) for \( 2 \leq n \leq 5 \), we have the following.

PROPOSITION 4.3. The following stable equivalences hold:

\[ \tau_2(5) = 2r(\eta^2) + 1, \quad \tau_3(5) = r(\eta^2) + 5, \quad \tau_4(5) = 9 \quad \text{and} \quad \tau_5(5) = r\eta + 9. \]

Hence, \( s(\tau_n(5)) = \infty \) for \( 2 \leq n \leq 5 \).

PROOF. Let \( n = 2 \) or \( 3 \). Then, by Theorem 2.1, \( \tilde{KO}(L^n(5)) = Z_5\{\tilde{e}\} \) and \( \tilde{e}^2 = 0 \). Thus, we have \( 5r\eta - 10 = 0 \) and \( (r\eta)^2 - 4r\eta + 4 = 0 \). Then, using Lemma 4.1, we obtain \( r(\eta^2) + r\eta - 4 = 0 \). Since \( \tau_n(5) = (n + 1)r\eta - 1 \), we have

\[ \tau_2(5) = 3r\eta - 1 = -2r\eta + 9 = 2r(\eta^2) + 1; \]
\[ \tau_3(5) = 4r\eta - 1 = -r\eta + 9 = r(\eta^2) + 5. \]

Similarly, for \( n = 4 \) or \( 5 \), \( \tilde{KO}(L^n(5)) = Z_5\{\tilde{e}, \tilde{e}^2\} \) and thus \( 5r\eta - 10 = 0 \). Then, we have \( \tau_4(5) = 5r\eta - 1 = 9, \quad \tau_5(5) = 6r\eta - 1 = r\eta + 9 \). Thus, we have \( s(\tau_n(5)) = \infty \) for \( n = 2, 3, 4 \) or \( 5 \) as is required, since \( r(\eta^2) \) and \( r\eta \) over \( L^n(5) \) are extendible to \( L^m(5) \) for every \( m \geq n \).

REMARK 4.4. According to Yoshida [12], \( L^3(p) \) has a tangent 5-field. Hence, \( \tau_3(p) = \beta \oplus 5 \) for a 2-plane bundle \( \beta \) in general.

PROPOSITION 4.5.

\[ s(\tau_n(5)) = 2n + 1 \quad \text{for } n = 6 \text{ or } 7. \]

PROOF. Let \( n = 6 \) or \( 7 \). By Proposition 3.1, we have \( s(\tau_n(5)) \geq 2n + 1 \). To establish the opposite inequality, we suppose that \( \tau_n(5) \) is stably extendible to \( L^{2n+2}(5) \), and derive a contradiction from the hypothesis. Thus, there is a \((2n + 1)\)-dimensional vector bundle \( \alpha \) over \( L^{2n+2}(5) \) satisfying that \( \tau_n(5) \) is stably equivalent to \( i^*\alpha \) for the inclusion map \( i : L^n(5) \to L^{2n+2}(5) \). By Theorem 2.1, \( \tilde{KO}(L^{2n+2}(5)) \) is generated additively by \( \tilde{e} \) and \( \tilde{e}^2 \) modulo a 2-torsion. Thus, we can put \( \alpha - (2n + 1) = a\tilde{e} + b\tilde{e}^2 + \delta \) in \( \tilde{KO}(L^{2n+2}(5)) \), where \( \delta \) is zero or a 2-torsion element. Then, since \( i^*\delta = 0 \) in \( \tilde{KO}(L^n(5)) = Z_5\{\tilde{e}\} + Z_5\{\tilde{e}^2\} \), we have \( i^*\alpha - (2n + 1) = a\tilde{e} + b\tilde{e}^2 \) in \( \tilde{KO}(L^n(5)) \).

Since \( i^*\alpha = \tau_n(5) \) and \( \tau_n(5) - (2n + 1) = (n + 1)\tilde{e} \), we have

\[
\begin{cases}
  a \equiv n + 1 \mod 5^2, \\
  b \equiv 0 \mod 5.
\end{cases}
\]
Hence, we can put
\[
\begin{cases}
  a = 5k + a_1 & \text{with } k \equiv 1 \mod 5, \\
  b = 5l
\end{cases}
\]
for some integers \(k\) and \(l\), where \(a_1 = 2\) and \(3\) when \(n = 6\) and \(7\) respectively.

Since \(K(L^{2n+2}(S))\) has no 2-torsion (cf. [5]), \(c\delta = 0\). Then, we have
\[
cx - (2n + 1) = acs + bcs^2 = a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4)
\]
\[
= (a - 4b)(\eta + \eta^{-1}) + b(\eta^2 + \eta^{-2}) - (2a - 6b).
\]

Let \(C_i(\gamma)\) denote the \(i\)-th Chern class of a complex vector bundle \(\gamma\), and \(C(\gamma) = 1 + C_1(\gamma) + \cdots\) the total Chern class. We denote \(C_i(\gamma)\) and \(C(\gamma)\) in the \(\mathbb{Z}_5\)-coefficient cohomology group by the same letters. Then, for \(x = C_1(\eta)\),
\[
\bigoplus_{i \geq 0} H^{2i}(L^{2n+2}(S); \mathbb{Z}_5) \cong \mathbb{Z}_5[x]/(x^{2n+1})
\]
as graded algebras (cf. [11]), and we have
\[
C(cx) = C(\eta + \eta^{-1})^{a-4b} C(\eta^2 + \eta^{-2})^b = (1 - x^2)^{a-4b} (1 - 4x^2)^b.
\]

Since \(a - 4b = 5(k - 4l) + a_1\) with \(k \equiv 1 \mod 5\) and \(b = 5l\), and since \(n = 6\) or \(7\),
\[
C(cx) = (1 - x^2)^{a_1}((1 - x^2)^5)^k ((1 - 4x^2)^5)^l
\]
\[
= (1 - x^2)^{a_1} (1 - x^{10})^{k-4l} (1 - 4^5 x^{10})^l
\]
\[
= (1 - x^2)^{a_1} (1 - (k - 4l)x^{10})(1 - 4^5 x^{10})
\]
\[
= (1 - x^2)^{a_1} (1 - kx^{10})
\]
\[
= (1 - x^2)^{a_1} (1 - x^{10})
\]
\[
= 1 - a_1 x^2 + \cdots + (-1)^{a_1+1} x^{10+2a_1}.
\]

Since \(10 + 2a_1 = 2n + 2\), we have \(C_{2n+2}(cx) \neq 0\) which contradicts that \(x\) is \((2n + 1)\)-dimensional. Thus, we have completed the proof.

**Proof of Theorem 2.** We obtain (1) by Lemma 4.2 and Proposition 4.3, and (2) by Theorem 1 and Proposition 4.5.

Next, we consider the proof of Theorem 3, but we can proceed similarly to Theorem 2.

**Proposition 4.6.** We have the following stable equivalences:
\[ \tau_2(7) = r(\eta^3) + r\eta + 1, \quad \tau_3(7) = r(\eta^3) + 2r\eta + 1, \]
\[ \tau_4(7) = 2r(\eta^3) + 2r(\eta^2) + 1, \quad \tau_5(7) = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1, \]
\[ \tau_6(7) = 13 \quad \text{and} \quad \tau_7(7) = r\eta + 13. \]

Hence, \( s(\tau_n(7)) = \infty \) for \( 2 \leq n \leq 7 \).

**Proof.** First, let \( n = 2 \) or \( 3 \). Then, \( \widetilde{KO}(L^n(7)) = \mathbb{Z}_7\{\tilde{\sigma}\} \), \( \tilde{\sigma}^2 = 0 \) and \( \tilde{\sigma}^3 = 0 \) by Theorem 2.1. Thus, we have \( 7r\eta - 14 = 0 \), \( (r\eta)^2 - 4r\eta + 4 = 0 \) and \( (r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0 \). Then, using Lemma 4.1 and these three equations, we obtain \( r(\eta^3) + 5r\eta - 12 = 0 \). Since \( \tau_7(7) = (n + 1)r\eta - 1 \) in \( KO(L^n(7)) \), we have
\[ \tau_2(7) = 3r\eta - 1 = -4r\eta + 13 = r(\eta^3) + r\eta + 1; \]
\[ \tau_3(7) = 4r\eta - 1 = -3r\eta + 13 = r(\eta^3) + 2r\eta + 1. \]

Next, let \( n = 4 \) or \( 5 \). Then, \( \widetilde{KO}(L^n(7)) = \mathbb{Z}_7\{\tilde{\sigma}, \tilde{\sigma}^2\} \) and \( \tilde{\sigma}^3 = 0 \) by Theorem 2.1. Thus, we have \( 7r\eta - 14 = 0 \), \( 7(r\eta)^2 - 28r\eta + 28 = 0 \) and \( (r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0 \). Then, using Lemma 4.1 and these three equations, we obtain \( r(\eta^3) + r(\eta^2) + r\eta - 6 = 0 \). Since \( \tau_7(7) = (n + 1)r\eta - 1 \) in \( KO(L^n(7)) \), we have
\[ \tau_4(7) = 5r\eta - 1 = -2r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + 1; \]
\[ \tau_5(7) = 6r\eta - 1 = -r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1. \]

Similarly, for \( n = 6 \) or \( 7 \), we also have \( 7r\eta - 14 = 0 \) by Theorem 2.1. Thus, we have \( \tau_6(7) = 7r\eta - 1 = 13 \) and \( \tau_7(7) = 8r\eta - 1 = r\eta + 13 \). Hence, \( s(\tau_n(7)) = \infty \) for \( 2 \leq n \leq 7 \) as is required, since \( r(\eta^3) \), \( r(\eta^2) \) and \( r\eta \) over \( L^n(7) \) are extendible to \( L^m(7) \) for every \( m \geq n \).

**Proposition 4.7.**
\[ s(\tau_n(7)) = 2n + 1 \quad \text{for} \quad 8 \leq n \leq 11. \]

**Proof.** Let \( n = 8, 9, 10 \) or \( 11 \). By Proposition 3.1, we have \( s(\tau_n(7)) \geq 2n + 1 \). We suppose that \( \tau_n(7) \) is stably extendible to \( L^{2n+2}(7) \), and derive a contradiction from the hypothesis. Thus, there is a \( (2n+1) \)-dimensional vector bundle \( \alpha \) over \( L^{2n+2}(7) \) satisfying that \( \tau_n(7) \) is stably equivalent to \( i^*\alpha \). By Theorem 2.1, \( \widetilde{KO}(L^n(7)) \) and \( \widetilde{KO}(L^{2n+2}(7)) \) are both generated additively by \( \tilde{\sigma}, \tilde{\sigma}^2 \) and \( \tilde{\sigma}^3 \) modulo a 2-torsion. Thus, we can put \( \alpha - (2n + 1) = a\tilde{\sigma} + b\tilde{\sigma}^2 + d\tilde{\sigma}^3 + \delta \), where \( \delta \) is zero or a 2-torsion element. Then, since \( i^*\delta = 0 \) in \( \widetilde{KO}(L^n(7)) \), we have
\[ i^*\alpha - (2n + 1) = a\tilde{\sigma} + b\tilde{\sigma}^2 + d\tilde{\sigma}^3 \]
\[ \widetilde{KO}(L^n(7)) = \begin{cases} Z_2\{\tilde{\sigma}\} + Z_7\{\tilde{\sigma}^2, \tilde{\sigma}^3\} & n = 8 \text{ or } 9, \\
Z_2\{\tilde{\sigma}, \tilde{\sigma}^2\} + Z_7\{\tilde{\sigma}^3\} & n = 10 \text{ or } 11. \end{cases} \]
Since $i^*x = \tau_n(7)$ and $\tau_n(7) = (2n + 1) = (n + 1)\bar{s}$, we have
\[
\begin{align*}
    a &\equiv n + 1 \mod 7^2, \\
    b &\equiv 0 \mod 7 \quad (n = 8, 9), \mod 7^2 \quad (n = 10, 11), \\
    d &\equiv 0 \mod 7.
\end{align*}
\]
Hence, we can put
\[
\begin{align*}
    a &= 7k + a_1 \quad \text{with } k \equiv 1 \mod 7, \\
    b &= 7l, \\
    d &= 7h
\end{align*}
\]
for some integers $k$, $l$ and $h$, where $a_1 = 2, 3, 4$ or 5 according as $n = 8, 9, 10$ or 11. Consider the complexification of $x$. Then,
\[
cx - (2n + 1) = ac\bar{s} + bc\bar{s}^2 + dc\bar{s}^3
\]
\[
= a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4) \\
+ d((\eta + \eta^{-1})^3 - 6(\eta + \eta^{-1})^2 + 12(\eta + \eta^{-1}) - 8)
\]
\[
= (a - 4b + 15d)(\eta + \eta^{-1}) + (b - 6d)(\eta^2 + \eta^{-2}) + d(\eta^3 + \eta^{-3}) \\
- (2a - 6b + 20d).
\]
Recall that $\bigoplus_{i \geq 0} H^2(L^{2n+2}(7); Z_7) \simeq Z_7[x]/(x^{2n+3})$ as graded algebras, where $x = C_1(\eta)$. Then, we have
\[
C(cx) = C(\eta + \eta^{-1})^{a-4b+15d}C(\eta^2 + \eta^{-2})^{b-6d}C(\eta^3 + \eta^{-3})^d
\]
\[
= (1 - x^2)^{a-4b+15d}(1 - 4x^2)^{b-6d}(1 - 9x^2)^d.
\]
Since $a - 4b + 15d = 7(k - 4l + 15h) + a_1$ with $k \equiv 1 \mod 7$, $b - 6d = 7(l - 6h)$ and $d = 7h$, we have
\[
C(cx) = (1 - x^2)^{a_1}((1 - x^2)^7)^{k-4l+15h}((1 - 4x^2)^7)^{l-6h}((1 - 9x^2)^7)^h
\]
\[
= (1 - x^2)^{a_1} (1 - x^{14})^{k-4l+15h} (1 - 4x^{14})^{l-6h} (1 - 9x^{14})^h
\]
\[
= (1 - x^2)^{a_1} (1 - (k - 4l + 15h)x^{14}) (1 - 4(l - 6h)x^{14}) (1 - 2hx^{14})
\]
\[
= (1 - x^2)^{a_1} (1 - (k - 9h)x^{14}) (1 - 2hx^{14})
\]
\[
= (1 - x^2)^{a_1} (1 - (k - 7h)x^{14})
\]
\[
= (1 - x^2)^{a_1} (1 - x^{14})
\]
\[
= 1 - a_1x^2 + \cdots + (-1)^{a_1+1}x^{14+2a_1}.
\]
Since \( 14 + 2a_1 = 2n + 2 \), we have \( C_{2n+2}(cx) \neq 0 \), which contradicts that \( \alpha \) is \((2n + 1)\)-dimensional. Thus, we obtain the required result.

**Proof of Theorem 3.** We have (1) by Lemma 4.2 and Proposition 4.6, and (2) by Theorem 1 and Proposition 4.7.

5. **Application to stably splitting problem**

A splitting (resp. stably splitting) problem of vector bundles can be stated: When is a given \( k \)-plane bundle equivalent (resp. stably equivalent) to a sum of \( k \) line bundles? Concerning this, the following result is called Schwarzenberger's property.

**Theorem** ([1], [2], [9], [10]). Let \( F = \mathbb{C} \) or \( \mathbb{R} \). If a \( k \)-dimensional \( F \)-vector bundle \( \xi \) over \( FP^n \) is extendible to \( FP^m \) for every \( m > n \), then \( \xi \) is stably equivalent to the Whitney sum of \( k \) numbers of \( F \)-line bundles.

We remark that the theorem is also valid if the condition for extendibility is changed to that for stably extendibility (cf. [8], [4]). Then, some related results are shown as follows:

**Theorem** ([4], Theorem B). If a \( k \)-dimensional \( H \)-vector bundle \( \xi \) over \( HP^n \) is stably extendible to \( HP^m \) for every \( m > n \) and its top non-zero Pontrjagin class is not zero mod 2, then \( \xi \) is stably equivalent to the Whitney sum of \( k \) numbers of \( H \)-line bundles provided \( k \leq n \).

**Theorem** ([8], Theorem B). If a \( k \)-dimensional vector bundle \( \xi \) over \( L^n(3) \) is stably extendible to \( L^m(3) \) for every \( m > n \), then \( \xi \) is stably equivalent to the Whitney sum of \( \lfloor \frac{k}{2} \rfloor \) numbers of 2-plane bundles.

We have another answer from Lemma 5.2 in [7], Theorems 2 and 3 and Propositions 4.3 and 4.6.

**Theorem 4.** Let \( p = 5 \) or \( 7 \) and \( n \geq 1 \). Then, \( \tau_n(p) \) is stably equivalent to the Whitney sum of \( \lfloor \frac{2n+1}{2} \rfloor \) numbers of 2-plane bundles if and only if \( s(\tau_n(p)) = \infty \) holds.

6. **Study on \( m\tau_n(p) \)**

Let \( m\tau_n(p) \) be the \( m \)-times Whitney sum of the tangent bundle \( \tau_n(p) \). We have the following in the similar way to the proof of Proposition 3.1.

**Proposition 6.1.** Let \( m \geq 1 \). Then, for any \( n \geq 1 \), we have

\[
s(m\tau_n(p)) \geq m(2n + 1) \quad \text{or} \quad s(m\tau_n(p)) \geq m(2n + 1) - 1
\]

if \( m \) is an odd or even integer respectively.
PROOF. For any integer \( k \geq 1 \), we have

\[
\text{g.dim}(m(n+1)(\eta_k - 2)) \leq 2\left[\frac{k}{2}\right] + 1
\]

by Theorem 2.3. Thus, there is a \((2\left[\frac{k}{2}\right] + 1)\)-dimensional vector bundle \( \beta \) satisfying that \( m(n+1)\eta_k \) is stably equivalent to \( \beta \). Let \( m \) be an odd (resp. even) integer. When \( k = m(2n+1) \) (resp. \( k = m(2n+1) - 1 \)), we have \( 2\left[\frac{k}{2}\right] + 1 = m(2n+1) \) (resp. \( = m(2n+1) - 1 \)). Thus, \( m(n+1)\eta_m(2n+1) \) (resp. \( m(n+1)\eta_{m(2n+1)-1} \)) is stably equivalent to \( \gamma + m \) for the \( (2n+1) \)-dimensional vector bundle \( \gamma = \beta \) (resp. \( = \beta + 1 \)). Then, \( m\tau_n(p) \) is stably equivalent to \( i^*(\gamma) \) since \( m\tau_n(p) + m = m(n+1)\eta_n \), and thus we have the required inequality \( s(m\tau_n(p)) \geq m(2n+1) \) (resp. \( s(m\tau_n(p)) \geq m(2n+1) - 1 \)).

Now, in order to consider the case when \( s(m\tau_n(p)) = m(2n+1) \) or \( s(m\tau_n(p)) \leq m(2n+1) + 1 \) holds in Proposition 6.1, we first define an integer \( \varepsilon_p(t, l) \).

DEFINITION. For a non-negative integer \( t \) and a positive integer \( l \), define an integer \( \varepsilon_p(t, l) \) as follows.

\[
\varepsilon_p(t, l) = \min \left\{ 2j \left| 2\left[\frac{t}{2}\right] + 1 < 2j \text{ and } \left(\left[\frac{t}{2}\right] + l\right) \neq 0 \mod p \right. \right\}.
\]

Then, we have \( t < \varepsilon_p(t, l) \leq 2\left[\frac{t}{2}\right] + 2l \) and \( \varepsilon_p(t, 1) = 2\left[\frac{t}{2}\right] + 2 \), and the following lemma.

LEMMA 6.2. Let \( p \) be an odd prime and \( \zeta \) a \( t \)-dimensional vector bundle over \( L^n(p) \). If there is a positive integer \( l \) with \( \varepsilon_p(t, l) \leq n \), then \( \zeta \) is not stably equivalent to \( (\left[\frac{t}{2}\right] + l)\eta \).

PROOF. We write simply \( \varepsilon(t, l) \) instead of \( \varepsilon_p(t, l) \). For the Pontrjagin class of \( (\left[\frac{t}{2}\right] + l)\eta \), we have

\[
P_{\varepsilon(t, l)/2}\left(\left(\left[\frac{t}{2}\right] + l\right)\eta\right) = \left(\frac{\left[\frac{t}{2}\right] + l}{\varepsilon(t, l)}\right) \in H^{2\varepsilon(t, l)}(L^n(p); Z),
\]

which is not zero by the definition of \( \varepsilon(t, l) \) and the assumption \( \varepsilon(t, l) \leq n \). However, since \( \zeta \) is of dimension \( t \) and \( \left[\frac{t}{2}\right] < \varepsilon(t, l) \), we have \( P_{\varepsilon(t, l)/2}(\zeta) = 0 \). Thus, \( \zeta \) is not stably equivalent to \( (\left[\frac{t}{2}\right] + l)\eta \), as is required.

The following is also obtained using the calculation in the proof of Theorem 1.1 in [7].

PROPOSITION 6.3. Let \( p \) be an odd prime, and \( \zeta \) a \( t \)-dimensional vector bundle over \( L^n(p) \). Assume that there is a positive integer \( l \) satisfying
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(1) \( \zeta \) is stably equivalent to \( \left( \left\lfloor \frac{t}{2} \right\rfloor + 1 \right) \eta \), and

(2) \( p^{n/(p-1)} \) > \( \left\lfloor \frac{t}{2} \right\rfloor + 1 \).

Then, \( s(\zeta) < \varepsilon_p(t, l) \).

PROOF. Here, we put \( h = \left\lfloor \frac{t}{2} \right\rfloor + 1 \), and write \( \varepsilon(t, l) \) instead of \( \varepsilon_p(t, l) \). Then, by Lemma 6.2, \( n < \varepsilon(t, l) \). Now, we suppose that \( \zeta \) is stably extendible to \( L^{\varepsilon(t, l)}(p) \), and derive a contradiction from the hypothesis. Thus, there exists a \( t \)-dimensional vector bundle \( \alpha \) over \( L^{\varepsilon(t, l)}(p) \) satisfying that \( i^*\alpha \) is stably equivalent to \( h \eta \).

Now, we apply the same methods used in the proof of Theorem 1.1 in [7]. The integers \( c_i \) used there are \( c_1 = h \) and \( c_i = 0 \) for \( 2 \leq i \leq p - 1 \) in our case. Then, the total Pontrjagin class of \( j^*\alpha \), where \( j \) is the inclusion map \( j: L^{\varepsilon(t, l)}(p) \rightarrow L^{\varepsilon(t, l)}(p) \), is given as

\[
P(j^*\alpha) = (1 + x^2)^h \text{ in } H^*(L^{\varepsilon(t, l)}(p); \mathbb{Z}).
\]

Here, the following equality is used to calculate the above Pontrjagin class as in [7]:

\[
(1 + i^2) x^{2^i} p^{n/(p-1)} = 1 + i^{2p^{n/(p-1)}} x^{2^{p^{n/(p-1)}}} = 1 \text{ in } H^*(L^{\varepsilon(t, l)}(p); \mathbb{Z})
\]

for \( 1 \leq i \leq \frac{p-1}{2} \), and it holds because \( p^{n/(p-1)} > h \) from the assumption (2) and \( 2h \geq \varepsilon(t, l) \) as mentioned above. Then, from the total Pontrjagin class of \( j^*\alpha \) and by the definition of \( \varepsilon(t, l) \), we have

\[
P_{\varepsilon(t, l)/2}(j^*\alpha) = \left( \frac{h}{\varepsilon(t, l)/2} \right) x^{\varepsilon(t, l)} \neq 0 \text{ in } H^{2\varepsilon(t, l)}(L^{\varepsilon(t, l)}(p); \mathbb{Z}),
\]

which contradicts that \( j^*\alpha \) is of dimension \( t \) and \( t < \varepsilon(t, l) \). Thus, we have completed the proof. \( \square \)

Then, we have the following.

THEOREM 5. Let \( m \geq 1 \) and \( n \geq 1 \) be integers.

(1) If \( m \) is odd,

\[
p^{n/(p-1)} > m(n+1) \quad \text{and} \quad \left( \frac{m(n+1)}{m(n+1)-m-1} \right) \neq 0 \mod p,
\]

then \( s(m \tau_n(p)) = m(2n+1) \).

(2) If \( m \) is even,

\[
p^{n/(p-1)} > m(n+1) \quad \text{and} \quad \left( \frac{m(n+1)}{m(n+1)+m} \right) \neq 0 \mod p,
\]

then \( s(m \tau_n(p)) = m(2n+1) - 1, m(2n+1) \) or \( m(2n+1) + 1 \).
First, we assume that \( m \) is odd, and prove (1). By Proposition 6.1, we have \( s(m \tau_n(p)) \geq m(2n + 1) \). Thus, we assume further that 

\[
p^{\lceil n/(p-1) \rceil} > m(n + 1) \quad \text{and} \quad \left( \frac{m(n + 1)}{m(2n+1)+1} \right) = \left( \frac{m(n + 1)}{m(n + 1) - \frac{m}{2}} \right) \not\equiv 0 \mod p,
\]

and prove the inequality \( s(m \tau_n(p)) \leq m(2n + 1) \). Consider \( e_p(m(2n + 1), \frac{m}{2}) \). Since \( 2 \left\lceil \frac{m(2n+1)}{2} \right\rceil + 1 < m(2n + 1) + 1 \), and by the latter assumption above, we have \( e_p(m(2n + 1), \frac{m}{2}) \not\equiv m(2n + 1) + 1 \). Hence, by Proposition 6.3, we have \( s(m \tau_n(p)) < e_p(m(2n + 1), \frac{m}{2}) \leq m(2n + 1) + 1 \), and thus we have proved (1).

Next, we assume that \( m \) is even, and prove (2). By Proposition 6.1, we have \( s(m \tau_n(p)) \geq m(2n + 1) - 1 \). Thus, we further assume that 

\[
p^{\lceil n/(p-1) \rceil} > m(n + 1) \quad \text{and} \quad \left( \frac{m(n + 1)}{m(2n+1)+1} \right) = \left( \frac{m(n + 1)}{mn + 1 + \frac{m}{2}} \right) \not\equiv 0 \mod p,
\]

and prove \( s(m \tau_n(p)) \leq m(2n + 1) + 1 \). Then, since \( 2 \left\lceil \frac{m(2n+1)}{2} \right\rceil + 1 < m(2n + 1) + 2 \), and by the last assumption above, we have \( e_p(m(2n + 1), \frac{m}{2}) \not\equiv m(2n + 1) + 2 \). Hence, by Proposition 6.3, \( s(m \tau_n(p)) < e_p(m(2n + 1), \frac{m}{2}) \) \leq m(2n + 1) + 1, and thus we have proved (2) and completed the proof of Theorem 5.

We illustrate the results of Theorems 5 for \( p = 5 \) or \( 7 \) and for \( 2 \leq m \leq 5 \).

**Example.** Let \( n \geq 1 \), and \( p = 5 \) or \( 7 \).

1. If \( n \geq 2p - 2 \), then \( s(2 \tau_n(p)) = 4n + 1, 4n + 2 \) or \( 4n + 3 \).
2. Assume that \( n \geq 3p - 3 \) and \( n + 1 \not\equiv 0 \mod p \) for \( p = 5 \), \( n = 12, 14, 15 \) or \( n \geq 3p - 3 \) and \( n + 1 \not\equiv 0 \mod p \) for \( p = 7 \).

Then, \( s(3 \tau_n(p)) = 6n + 3 \).
3. Assume that \( n \geq 3p - 3 \) and \( n + 1 \not\equiv 0 \mod p \). Then, \( s(4 \tau_n(p)) = 8n + 3, 8n + 4 \) or \( 8n + 5 \).
4. Assume that \( n \geq 3p - 3 \). For \( p = 5 \), we have no information on \( s(5 \tau_n(5)) \) from Theorem 5. For \( p = 7 \), if \( \frac{1}{2} (5n + 4)(5n + 5) \not\equiv 0 \mod 7 \), then \( s(5 \tau_n(7)) = 10n + 5 \).

**References**

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