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Varieties with non-linear Gauss fibers

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Abstract  For any given projective variety $Y$, we construct a projective variety $X \subset \mathbb{P}^N$ whose general fiber of the Gauss map with reduced scheme structure is isomorphic to $Y$ when the characteristic $> 0$.

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1 Introduction

Let $X \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field $K$. The Gauss map $\gamma$ from $X$ to the Grassmannian $G(\dim X, N)$ is the rational map, defined by $\gamma(p) = T_pX$ for a smooth point $p \in X$, and $T_pX \in G(\dim X, N)$ is the projective tangent space.

If the characteristic of $K$ is 0, then it is classically known that a general fiber of the Gauss map is a linear subspace of dimension $\dim X - \dim \gamma(X)$ (see, for example, [9]). If the characteristic of $K$ is positive, it is no longer true. Wallace ([8, Section 7]) pointed out that there exists a plane curve which has infinitely many multiple tangents, or equivalently, whose Gauss map has separable degree $> 1$ onto its image (see also [5, I-3]). Kaji ([3, Example 4.1],[4]) and Rathmann ([7, Example 2.13]) gave smooth curves with infinitely many multiple tangents. Noma constructed smooth or normal projective varieties whose Gauss maps have separable degree $> 1$ onto its image ([6]). In these cases, Gauss fibers are finite number of points. In [1], an example of a surface whose Gauss fibers are smooth conics is found. In the author’s best knowledge, this is the first example that Gauss fibers are not finite unions of linear subspaces. We are naturally led to the following question:

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Question 1 What kind of variety appears in the general fiber of the Gauss map?

Our answer in this paper is: “Any projective varieties can appear”. To be more precise, we will prove the following:

Theorem 1 Let charK > 0. For any positive integers k ≥ 2, r ≤ k and N ≥ k+r and any projective variety Y ⊂ P^k of codimension r, there exists a closed variety X ⊂ P^N of dimension k, not contained in any hyperplane, such that for a general point p ∈ X, Y is the fiber of the Gauss map γ^−1(γ(p)) ⊂ T_p X ∼= P^k set-theoretically (up to choices of coordinates).

Notation Unless otherwise stated, the base field K is an algebraically closed field of characteristic p > 0. G(k, N) is the Grassmannian of k-dimensional linear subspaces of P^N. Varieties are integral algebraic schemes. Points mean closed points. [v] ∈ P^N denotes the point of P^N corresponding to the equivalence class of v ∈ A^{N+1} \ {0}. Given a linear subspace V ⊂ A^{N+1}, P(V) ⊂ P^N means the linear subspace of P^N corresponding to V.

2 Construction

Let k ≥ 2, r ≤ k, N > k be positive integers and let Y ⊂ P^k be a closed variety of codimension r. Let B ⊂ G(k−r, N) be a closed variety of dimension r, I_B = {(x, E) ∈ P^N × B | x ∈ E} and let f : I_B → P^N, g : I_B → B be the natural projections. Let u_1, ..., u_r be a local parameter system of B, and let ρ_0, ..., ρ_{k−r} : U → A^{N+1} be a system of morphisms on some open set U ⊂ B such that P⟨(ρ_0(s), ..., ρ_{k−r}(s))⟩ is equal to the (k−r)-dimensional linear subspace given by s for all points s ∈ U. We assume that

1. f is generically étale onto its image,
2. dim \{ \frac{∂ω}{∂u_i} | i, j \} = r and τ_1, ..., τ_r form its base, and
3. \frac{∂}{∂u_i} (\frac{∂ω}{∂u_j}) = 0 for any i, j, l.

The conditions (1) and (2) imply that dim ⟨ρ_0, ..., ρ_{k−r}, τ_1, ..., τ_r⟩ = k + 1.

Let η : U × P^k → P^N,

(s) × (Y_0 : ... : Y_k) ↦

[Y_0 ρ_0(s) + ... + Y_{k−r} ρ_{k−r}(s) + Y_k τ_1(s) + ... + Y_k τ_r(s)],

and let X be the closure of η(U × Y). Then, X is the closed subvariety in P^N of dimension ≤ k. Let τ := η|_{(U × Y)} : U × Y → X.

Let Ỹ ⊂ A^{k+1} be the affine cone of Y ⊂ P^k. By changing the coordinate system if necessary, we may assume that Y_0 − y_0, ..., Y_{k−r} − y_{k−r} are a local parameter system of Ỹ at a smooth point (y_0, ..., y_k) ∈ Ỹ.

Proposition 1 τ is generically étale, and T_τ(s, y)X = η(s × P^k) for a general point s ∈ B and a general point y ∈ Y.
Remark 1

The conditions (1) and (2) force the ruling of $\mathbb{A}^{N+1}$ and $\mathbb{A}^1$.

Proof (Proof of Theorem 1)

We construct the tangent variety $Z = \mathbb{A}^N$ of dimension $k$, not contained in any hyperplane, such that for a general point $s \in Z$, $\gamma$ is the fiber of the Gauss map $\gamma^{-1}(\gamma(s)) \subset T_pZ \cong \mathbb{A}^k$ set-theoretically.

Proof

We construct $X \subset \mathbb{P}^N$ as above consideration. Let $\tau$ be as above. By Proposition 1, $\tau(s \times Y)$ is contained in $\gamma^{-1}(\gamma(\tau(s), y))$ for each general point $(s, y) \in B \times \bar{Y}$. If the tangential space $\mathbb{P}((p_0(s), \ldots, p_{k-r}(s), \tau_1(s), \ldots, \tau_r(s)))$ is uniquely determined from a general point of $B$ then $\tau$ is generically one-to-one, because a point contained in two distinct $Y$'s is a singular point of $X$. This implies that $\tau(s \times Y)$ coincides with the fiber $\gamma^{-1}(\gamma(\tau(s), y))$ for a general point $(s, y) \in B \times \bar{Y}$.

The nondegeneration of $X$ follows from the nondegeneration of the tangent variety $T_pZ$ of the ruled variety $Z = \bigcup_{E \subset B} E$, because $T_pZ$ is also the tangent variety of $X$.

Theorem 1 is given as the corollary of Theorem 2.

Proof (Proof of Theorem 1) If $N \geq k + r$ then we can take $B$ as the closure of the image of the morphism $\mathbb{A}^r \to \mathbb{G}(k-r, N); \ s \mapsto \mathbb{P}((v, p_1, \ldots, p_r))$, where $v$ is the morphism from $\mathbb{A}^r$ to $\mathbb{A}^{N+1}$ given by $v = (1, 0, \ldots, 0, u_1, \ldots, u_r, w_{p_1}, \ldots, w_{p_r}, w_{p_1}^{N-r-1})$ and $p_i \in \mathbb{A}^{N+1}$ is the point whose $l$-th coordinate is 0 for any $l \neq i$ and 1 for $l = i$. We have the result by Theorem 2.

Remark 1 The conditions (1) and (2) force the ruling of $B \subset \mathbb{G}(k-r, N)$ to be developable, i.e. tangent spaces of the ruled variety $\bigcup_{E \subset B} E \subset \mathbb{P}^N$ are constant on each linear subspace $E \subset B ([1])$. 

Proof Let $\hat{\tau} : U \times \hat{Y} \to \mathbb{A}^{N+1}$ be the affine lifting of $\tau$. By the assumption (3) and easy computation, we have

\[
\frac{\partial \hat{\tau}}{\partial \hat{u}_1} = Y_0 \frac{\partial u_1}{\partial \hat{u}_1} + \ldots + Y_{k-r} \frac{\partial u_{k-r}}{\partial \hat{u}_1} \\
\vdots \\
\frac{\partial \hat{\tau}}{\partial \hat{u}_r} = Y_0 \frac{\partial u_1}{\partial \hat{u}_1} + \ldots + Y_{k-r} \frac{\partial u_{k-r}}{\partial \hat{u}_1} \\
\frac{\partial \hat{\tau}}{\partial \hat{u}_{k-r}} = \rho_0 + \frac{\partial Y_1}{\partial Y_0} \tau_1 + \ldots + \frac{\partial Y_r}{\partial Y_0} \tau_r \\
\vdots
\]

By the assumptions (1) and (2), for a general point $(s, y) \in B \times \bar{Y}$, 

\[
\text{Im } d_{(s,y)} \hat{\tau} = (\tau_1(s), \ldots, \tau_r(s), \rho_0(s), \ldots, \rho_{k-r}(s)).
\]

This implies our assertion.
3 Examples

Example 1 Let char $K = 2$. We consider the hypersurface $X$ in $\mathbb{P}^3$ given by $F = X^6 + Y^6 + Z^6 + YZ^2W + Y^2Z^2W^2 + Y^4W^3$. For a general point $(x : y : z : w) \in X$, the tangent plane given by $wY + yW = 0$ because the Gauss map is given by $(\partial F/\partial X : \partial F/\partial Y : \partial F/\partial Z : \partial F/\partial W) = (0 : Z^4W + Y^2W^3 : 0 : YZ^4 + Y^3W^2) = (0 : W : 0 : Y)$. The intersection of $X$ and this plane is the plane curve $wY + yW = \sqrt{y}X^3 + \sqrt{y^3 + w^3}Y + \sqrt{y^2}Z^3 + \sqrt{y^2}YW^2 + \sqrt{y^2}Z^2 = 0$. This is the fiber of the Gauss map at $T_{(y : z : w)}X \subset \mathbb{P}^3$. This curve is smooth and of degree 3 (for a general point $(x : y : z : w) \in X$), hence this is an elliptic curve.

The above surface $X$ is given by our method in Section 2: $B$ is the closure of the image of the morphism $\mathbb{A}^1 \to \mathbb{G}(1,3) \subset \mathbb{P}^4; u \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (1 : u : w^2 : 0 : 0 : 0)$ where $p_{ij}$ are Pl"{u}cker coordinates, and $Y \subset \mathbb{P}^2$ is the hypersurface given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$ where $Y_0, Y_1, Y_2$ are coordinates on $\mathbb{P}^2$.

Example 2 Let char $K = 3$. Let $B$ be the closure of the image of the morphism $\mathbb{A}^1 \to \mathbb{G}(1,3) \subset \mathbb{P}^4; u \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (1 : u : u^3 : 0 : 0 : 0)$, and $Y \subset \mathbb{P}^2$ be the hypersurface given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$. Then we have the surface $X$ given by $X^{12} + Y^{12} + Z^{12} + 2Y^2Z^2W + 2Y^6Z^3W^3 + Y^8W^4$. By easy computation, we can check that the general fibers of the Gauss map of this surface are smooth curves of genus 3.

Example 3 Now, we give an example of a 3-fold $X \subset \mathbb{P}^4$ whose general fibers of the Gauss map are twisted cubic curves.

Let char $K = 2$. Let $B \subset \mathbb{G}(1,4)$ be the closure of the image of the morphism $\mathbb{A}^2 \to \mathbb{G}(1,4), (u, v) \mapsto \mathbb{P}((\rho_0, \rho_1))$ where

$\rho_0 = (1 0 0 0 u^2)$
$\rho_1 = (0 1 0 v v^2),$

and let $Y \subset \mathbb{P}^3$ be a curve given by $Y_0Y_2 - Y_1^2, Y_1Y_3 - Y_2^2, Y_0Y_3 - Y_1Y_2$. Then our construction in Section 2 gives the hypersurface $X \subset \mathbb{P}^4$ whose defining polynomial is $X_0X_1 + X_0^3 + X_0X_1X_2 + X_2^3X_2 + X_0X_1X_2$ (by using Groebner Basis).

Example 4 We give an example of a hypersurface $X \subset \mathbb{P}^9$ whose general Gauss fibers are abelian surfaces.

Let char $K = 2$. Let $B \subset \mathbb{G}(1,4)$ be the closure of the image of the morphism $\mathbb{A}^3 \to \mathbb{G}(2,9), (u_1, u_2, u_3, u_4, u_5, u_6) \mapsto \mathbb{P}((\rho_0, \rho_1, \rho_2))$ where

$\rho_0 = (1 0 0 u_4 0 0 u_2 0 0 u_3^2 + u_5^2)$
$\rho_1 = (0 1 0 u_3 0 0 u_4 0 0 u_3^3 + u_4^3)$
$\rho_2 = (0 0 1 0 0 u_5 0 0 u_6 u_5^3 + u_6^3),$

and let $Y$ be the surface $E \times E \subset \mathbb{P}^2 \times \mathbb{P}^2$ where $E$ is the elliptic curve in $\mathbb{P}^2$ given by $x_0^2 + x_1^2 + x_2^2 = 0$. We embed $Y$ to $\mathbb{P}^8$ by Segre embedding. We have the hypersurface $X \subset \mathbb{P}^9$ as the closure of the image of the morphism $U \times Y \to \mathbb{P}^9$ (where $U$ is an open subset of $B$), $(s) \times (x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2 :)$
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\[ y_1 : y_2 \mapsto [x_0 y_0 \rho_0(s) + x_1 y_1 \rho_1(s) + x_2 y_2 \rho_2(s) + x_0 y_1 \frac{\partial \rho_0}{\partial u_1}(s) + x_1 y_0 \frac{\partial \rho_1}{\partial u_1}(s) + x_1 y_2 \frac{\partial \rho_1}{\partial u_2}(s) + x_2 y_0 \frac{\partial \rho_2}{\partial u_3}(s) + x_2 y_1 \frac{\partial \rho_2}{\partial u_4}(s)]. \]

Recently, varieties with non-constant Gauss fibers are found, which result will be published in [2].

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References