I=2 pion scattering length with the Wilson fermion

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The calculation of the I=2 pion scattering length in quenched lattice QCD is reexamined. The calculation is carried out with the Wilson fermion action employing Lüscher’s finite size scaling method at β=5.9, 6.1, and 6.3 corresponding to the range of lattice spacing a=0.12–0.07 fm. We obtain in the continuum limit a0/m=−2.09(35) 1/GeV2, which is consistent with the prediction of chiral perturbation theory a0/m=−2.265(51) 1/GeV2.

The energy eigenvalue of a two-pion system in a finite periodic box L3 is shifted by the finite size effect. Lüscher presented a relation between the energy shift ΔE and the S-wave scattering length a0, given by [8]

\[-ΔE \cdot \frac{m^2 L^2}{4 \pi^2} = T + C_1 \cdot T^2 + C_2 \cdot T^3 + O(T^4),\]

where T=a0/πL. The constants are C1=−8.9136 and C2=62.9205 computed from the geometry of the lattice. Since T has a small value, typically ~−10−2 in our simulations, we can safely neglect the higher order terms O(T4).

The energy shift ΔE can be obtained from the ratio R(t)=G(t)/D(t), where

\[G(t)=\langle \pi^+(t) \pi^-(t) W^-(t_1) W^-(t_2) \rangle,\]
\[D(t)=\langle \pi^+(t_1) W^-(t_1) \rangle \langle \pi^+(t_2) W^-(t_2) \rangle.\]

In order to enhance the signals against the noise we use wall sources for π−, which are denoted by W− in Eq. (2), by fixing the gauge configurations to the Coulomb gauge. The two wall sources are placed at different time slices t1 and t2 to avoid contaminations from Fierz-rearranged terms in the

![FIG. 1. The ratio R(t)=G(t)/D(t) at β=6.3 and κ=0.1513 corresponding to mπ=433(4) MeV. The wall sources are located at t=13 and 14.](image-url)
TABLE I. The results at $\beta = 5.9$. The four lines for each $m_\pi$ are results with the fitting functions Old, Exp, Lin, and Sqr, which are defined in Eqs. (4)–(6).

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$m_\pi^2$ (GeV$^2$)</th>
<th>Fit</th>
<th>$\Delta E$ ($\times 10^{-3}$)</th>
<th>$E'$ ($\times 10^{-3}$)</th>
<th>$a_0/m_\pi$ (1/GeV$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1585</td>
<td>0.1560</td>
<td>Old 12.4(21)</td>
<td>$-0.84(12)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
</tr>
<tr>
<td>0.2529(56)</td>
<td>Exp 20.9(40)</td>
<td>$-1.29(20)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3468(49)</td>
<td>Lin 14.5(19)</td>
<td>$-0.96(11)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1580</td>
<td>0.2529</td>
<td>Sqr 23.1(74)</td>
<td>29(21)</td>
<td>1.40(35)</td>
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</tr>
<tr>
<td>0.3468(49)</td>
<td>Old 12.5(15)</td>
<td>$-0.822(84)$</td>
<td></td>
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<tr>
<td>0.1580</td>
<td>0.3468</td>
<td>Exp 19.9(27)</td>
<td>$-1.20(13)$</td>
<td>$-0.905(72)$</td>
<td></td>
</tr>
<tr>
<td>0.1580</td>
<td>0.3468</td>
<td>Lin 14.0(13)</td>
<td>$-0.905(72)$</td>
<td>$-1.62(17)$</td>
<td></td>
</tr>
<tr>
<td>0.1580</td>
<td>0.3468</td>
<td>Sqr 19.0(57)</td>
<td>14(15)</td>
<td>1.16(27)</td>
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<tr>
<td>0.1575</td>
<td>0.4369(48)</td>
<td>Old 12.1(12)</td>
<td>$-0.786(65)$</td>
<td>$-0.1565$</td>
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<tr>
<td>0.4369(48)</td>
<td>Exp 18.5(21)</td>
<td>$-1.108(98)$</td>
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<tr>
<td>0.4369(48)</td>
<td>Lin 13.3(11)</td>
<td>$-0.849(56)$</td>
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<tr>
<td>0.4369(48)</td>
<td>Sqr 16.3(50)</td>
<td>8(13)</td>
<td>$-1.00(24)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
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<td>0.1570</td>
<td>0.5337(49)</td>
<td>Old 11.5(10)</td>
<td>$-0.743(55)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
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<tr>
<td>0.5337(49)</td>
<td>Exp 17.0(17)</td>
<td>$-1.017(79)$</td>
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<tr>
<td>0.5337(49)</td>
<td>Lin 12.48(92)</td>
<td>$-0.794(47)$</td>
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<tr>
<td>0.5337(49)</td>
<td>Sqr 14.4(45)</td>
<td>5(12)</td>
<td>$-0.892(22)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
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<td>0.1565</td>
<td>0.6297(50)</td>
<td>Old 10.86(91)</td>
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<td>$-0.1565$</td>
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<td>0.6297(50)</td>
<td>Exp 15.6(15)</td>
<td>$-0.931(67)$</td>
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<tr>
<td>0.6297(50)</td>
<td>Lin 11.69(82)</td>
<td>$-0.741(42)$</td>
<td></td>
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<tr>
<td>0.6297(50)</td>
<td>Sqr 13.0(41)</td>
<td>3(10)</td>
<td>$-0.81(20)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
</tr>
<tr>
<td>0.1560</td>
<td>0.7279(51)</td>
<td>Old 10.19(82)</td>
<td>$-0.654(43)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
</tr>
<tr>
<td>0.7279(51)</td>
<td>Exp 14.2(13)</td>
<td>$-0.855(59)$</td>
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<tr>
<td>0.7279(51)</td>
<td>Lin 10.92(75)</td>
<td>$-0.692(38)$</td>
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<td></td>
</tr>
<tr>
<td>0.7279(51)</td>
<td>Sqr 11.9(37)</td>
<td>2.6(95)</td>
<td>$-0.74(19)$</td>
<td>$-0.1565$</td>
<td>$0.6297(50)$</td>
</tr>
</tbody>
</table>

The energy shift $\Delta E$ is obtained from the linear term in the expansion of $R(t)$:

$$R(t) = Z \cdot [1 - \Delta E \cdot \tau + O(\tau^2)],$$

where $\tau = t - t_2$. The quadratic and higher order terms have no simple relation to $\Delta E$ due to effects from intermediate off-shell two-pion states [2] and quenching effects [9]. We first attempt to fit the data with the form

$$(Sqr) \quad Z \cdot (1 - \Delta E \cdot \tau + E' \cdot \tau^2).$$

We find that this fit (Sqr) is quite ill determined, since the two terms correlate so strongly, resulting in unacceptably large errors in $\Delta E$ and $E'$. We then attempt to fit with

$$(Exp) \quad Z \cdot \exp(-\Delta E \cdot \tau),$$

$$(Lin) \quad Z \cdot (1 - \Delta E \cdot \tau).$$

These fitting forms give well-determined $\Delta E$, while it may be contaminated by contributions from the second order term. We also include a fit of the form

$$(Old) \quad Z \cdot \Delta E \cdot \tau$$

in an attempt for completeness, since this was used in our preliminary report [5]. Note, however, that this form is theoretically correct only when $Z$ is close to unity. The results for
TABLE III. The results at $\beta = 6.3$. The four lines for each $m_\pi$ are results with the fitting functions Old, Exp, Lin, and Sqr, which are defined in Eqs. (4)–(6).

\[
\begin{array}{cccc}
\kappa & \beta = 6.3 \\
& m_\pi^2 (\text{GeV}^2) & \Delta E & E'/a_0/m_\pi \\
\hline
& & (\times 10^{-3}) & (\times 10^{-5}) & (1/\text{GeV}^2) \\
0.15130 & Old & 5.97(60) & - & -1.21(11) \\
0.1876(36) & Exp & 8.19(89) & - & -1.58(14) \\
& Lin & 6.71(60) & - & -1.34(10) \\
& Sqr & 7.9(18) & 2.4(36) & -1.54(29) \\
0.15115 & Old & 5.79(48) & - & -1.160(83) \\
0.2399(36) & Exp & 7.78(71) & - & -1.48(11) \\
& Lin & 6.43(49) & - & -1.267(81) \\
& Sqr & 7.7(14) & 2.6(28) & -1.48(22) \\
0.15100 & Old & 5.63(42) & - & -1.115(70) \\
0.2924(36) & Exp & 7.42(60) & - & -1.400(93) \\
& Lin & 6.19(42) & - & -1.206(69) \\
& Sqr & 7.3(13) & 2.3(24) & -1.39(19) \\
0.15075 & Old & 5.33(36) & - & -1.042(59) \\
0.3815(38) & Exp & 6.87(51) & - & -1.282(76) \\
& Lin & 5.80(36) & - & -1.118(58) \\
& Sqr & 6.5(11) & 1.5(21) & -1.23(16) \\
0.15050 & Old & 5.01(33) & - & -0.973(54) \\
0.4728(40) & Exp & 6.34(45) & - & -1.177(67) \\
& Lin & 5.42(33) & - & -1.038(52) \\
& Sqr & 5.81(99) & 0.8(19) & -1.10(15) \\
0.15000 & Old & 4.36(30) & - & -0.842(48) \\
0.6634(45) & Exp & 5.37(39) & - & -0.996(58) \\
& Lin & 4.70(30) & - & -0.894(46) \\
& Sqr & 4.72(89) & 0.0(17) & -0.90(14) \\
\end{array}
\]

$\Delta E$ [and $E'$ in the case (Sqr)] are given in Table I for $\beta = 5.9$, Table II for $\beta = 6.1$, and Table III for $\beta = 6.3$. We take the same fitting range for the four fits, $t = 21–42$ for $\beta = 5.9$, $t = 25–50$ for $\beta = 6.1$, and $t = 27–62$ for $\beta = 6.3$. The value of $\chi^2$ for each fitting is always small, and does not discriminate among fits. We do not consider the case (Sqr) further because of very large errors, although the resulting central values for the energy shift are consistent with those from (Exp) and (Lin). The problem we must consider is whether we can remove contamination of the second order term for $\Delta E$ from (Exp) and (Lin).

Figure 2 shows $a_0/m_\pi$ as a function of the pion mass obtained at each $\beta$, with their numerical values tabulated in Tables I, II, and III. We observe a large difference between (Exp) and (Lin), indicating that contributions from the $O(\alpha)$ term are indeed non-negligible and greatly affect the determination of $\Delta E$. In all figures of $a_0/m_\pi$ versus $m_\pi$ the data show a behavior linear in $m_\pi^2$. We then fit

\[a_0/m_\pi = A + B \cdot m_\pi^2\]

(7)

to extract the value $A$ in the chiral limit. From the view point of CHPT we may in principle have a term $m_\pi^2 \log(m_\pi^2/\Lambda^2)$ added to Eq. (7). If we include this term with a free coefficient into the fit, however, the coefficients correlate so strongly that the fit is invalidated, producing a large error also for $A$. It is difficult to distinguish $m_\pi^2$ and $m_\pi^2 \log(m_\pi^2)$ within the range of $m_\pi^2$ that concerns us and the limited statistics. Since we do not see any significant curvature in the figure of $a_0/m_\pi$ versus $m_\pi$ we simply drop this logarithmic term which itself vanishes at the chiral limit. We also note that for the Wilson fermion action the term proportional to $1/m_\pi^2$ may also exist, arising from explicit breaking of chiral symmetry, and also from quenching effects [9]. We do not see a $1/m_\pi^2$ effect, as our simulation is perhaps well away from $m_\pi^2 = 0$ and such a term is already damped into noise for the range of our simulation. Hence we do not include this term into our fit. In order to detect these two additional terms a simulation is needed close to the chiral limit with much higher statistics.

![FIG. 2. The mass dependence of $a_0/m_\pi (1/\text{GeV}^2)$ at each lattice spacing.](image)

TABLE IV. The values of $a_0/m_\pi (1/\text{GeV}^2)$ in the chiral limit for each fitting function for $R(t)$ at each $\beta$ and those in the continuum limit obtained by linear extrapolation in the lattice spacing. The fitting functions of $R(t)$ are defined in Eqs. (4)–(6).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a(1/\text{GeV})$</th>
<th>Old</th>
<th>Exp</th>
<th>Lin</th>
<th>Sqr</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.9</td>
<td>0.493(7)</td>
<td>-0.96(10)</td>
<td>-1.51(16)</td>
<td>-1.093(90)</td>
<td>-1.58(36)</td>
</tr>
<tr>
<td>6.1</td>
<td>0.378(6)</td>
<td>-1.185(59)</td>
<td>-1.653(80)</td>
<td>-1.335(55)</td>
<td>-1.78(22)</td>
</tr>
<tr>
<td>6.3</td>
<td>0.302(5)</td>
<td>-1.335(76)</td>
<td>-1.745(99)</td>
<td>-1.466(74)</td>
<td>-1.77(21)</td>
</tr>
</tbody>
</table>

\[a \to 0 \quad -1.92(25) \quad -2.09(35) \quad -2.07(24) \quad -2.04(78)\]
correct to the first order in $t$

negligible

$\sim$

different from the other two in the continuum limit, indicating the

Old

hand, the extrapolation with $(\text{Old})$ gives a value somewhat

large on finite lattices, vanishes approaching the continuum limit. This shows that the second

$E$

order term $O(\tau^2)$ included in Eq. (3) becomes irrelevant as

$\Delta E \cdot \tau$ becomes sufficiently small; one may use any formula correct to the first order in $\tau$ to extract $\Delta E$. On the other hand, the extrapolation with $(\text{Old})$ gives a value somewhat different from the other two in the continuum limit, indicating that the departure of $Z$ from unity could be non-negligible (although at $1.2\sigma - 1.5\sigma$).

As our final value for the scattering length in the continuum limit at the physical pion mass we take the result from $(\text{Exp})$, which agrees with that from $(\text{Lin})$ but has a larger statistical error:

$$a_0/m_\pi = -2.09(35) \text{ 1/GeV}^2,$$

where a rather large error arises from the continuum extrapolation. This result is compared with the CHPT prediction

$$a_0/m_\pi = -2.265(51) \text{ 1/GeV}^2.$$ (9)

The scattering length we derived at the continuum limit agrees well with the prediction of CHPT. The difference seen in the fitting formulas $(\text{Old})$ and $(\text{Lin})$ accounts for the $1.5\sigma$ difference of the lattice result from the CHPT prediction mentioned in our preliminary report, which is based on the incorrect extrapolation formula $(\text{Old})$.

We remark that our results also agree with those of Liu et al. [6]:

$$a_0/m_\pi = -1.75(38) \text{ 1/GeV}^2 \text{ for scheme I},$$ (10)

$$a_0/m_\pi = -2.34(46) \text{ 1/GeV}^2 \text{ for scheme II},$$ (11)

where the two values (schemes I and II) refer to their two different treatments for the finite volume corrections.

In this Brief Report we have reported a calculation of the scattering length for the $I=2$ $S$-wave two-pion system. We have shown that the result in the continuum limit is virtually independent of the choice of fitting functions used to extract $\Delta E$ from the ratio $R(t)$, and that it is consistent with the prediction of CHPT within our 15% statistical error.


