Nondominated Coteries on Graphs

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Abstract—Let C and D be two distinct coteries under the vertex set V of a graph G = (V, E) that models a distributed system. Coterie C is said to G-dominate D (with respect to G) if the following condition holds: For any connected subgraph H of G that contains a quorum in D (as a subset of its vertex set), there exists a connected subgraph H' of H that contains a quorum in C. A coterie C on a graph G is said to be G-nondominated (G-ND) (with respect to G) if no coterie D ≠ C on G G-dominates C.

Intuitively, a G-ND coterie consists of irreducible quorums.

This paper characterizes G-ND coteries in graph theoretical terms, and presents a procedure for deciding whether or not a given coterie C is G-ND with respect to a given graph G, based on this characterization. We then improve the time complexity of the decision procedure, provided that the given coterie C is nondominated in the sense of Garcia-Molina and Barbara. Finally, we characterize the class of graphs G on which the majority coterie is G-ND.

Index Terms—Availability, coteries on graphs, distributed mutual exclusion problem, G-nondominatedness, majority consensus.

1 INTRODUCTION

The (distributed) mutual exclusion problem is widely recognized as a fundamental problem in distributed computing. Let us model a distributed system as an undirected graph; the vertices represent processes and the edges represent bidirectional communication links each connecting a pair of processes. In 1985, Garcia-Molina and Barbara [1] introduced the concept of coteries, and showed its usefulness for solving the mutual exclusion problem. A coterie is a set of mutually incomparable nonempty sets (called quorums) of vertices (i.e., processes) such that any two quorums intersect each other.

A coterie is used to solve the mutual exclusion problem as follows: When entering the critical section, a vertex is asked to gain permission from every vertex in a quorum and holds it until it leaves the critical section. Because of the intersection property of quorums, at most one vertex can be in the critical section, provided that a vertex never gives its permission to two vertices at a time.

Suppose that the graph (i.e., distributed system) on which the mutual exclusion algorithm mentioned above is implemented is unreliable so that fail-stop failures may occur on vertices and/or edges (i.e., processes and/or communication links). Then a vertex can enter the critical section only if there is a “surviving” quorum in the sense that all vertices in the quorum are being operational and for any pair of vertices in the quorum there is a path consisting of operating vertices and edges. Given the probabilities that a vertex and a link, respectively, are operational, the probability that there is a surviving quorum is called the availability of the coterie. Although the problem of finding an optimal coterie with respect to the availability is difficult and computing the availability of a given coterie on a given graph is known to be #P-hard in general [2], some studies have been done to reveal properties of optimal coteries on simple classes of graphs such as complete graphs, rings, and trees [3], [4], [5], [6], [7], [8].

In particular, Ibaraki, Nagamochi, and Kameda introduced the concept of G-domination as a central concept to calculate the availability of coteries on rings and trees, and showed that if a coterie C G-dominates a coterie D, then the availability of C is not smaller than that of D in general; we can thus discard D from the candidate list for optimal coteries [7]. They also characterized G-nondominated coteries on rings and trees. However, a characterization of G-nondominated coteries on general graphs is still open. This paper characterizes G-nondominated coteries in graph theoretical terms.

We first present a necessary and sufficient condition for a coterie on a graph to be G-nondominated. In order to check the condition, however, we need to test all trees appearing in the graph as a subgraph. Next, we show that if a coterie is nondominated in the sense of Garcia-Molina and Barbara [1], we can complete the test just by checking only so-called cut-trees. Finally, we discuss the majority coterie on graphs. We characterize the class of graphs G on which the majority coterie is G-nondominated, and derive an easy sufficient condition on G for the majority coterie to be G-nondominated.

2 PRELIMINARIES

Let G = (V, E) be an undirected graph that models a distributed system; each vertex v ∈ V represents a process of the


2. We will formally define the concept of G-domination in Section 2.

3. The number of trees is O(2^m), where m is the number of edges of the graph. Hence, testing the G-nondominatedness of a coterie based on this condition requires exponential time.
distributed system, and each edge \((u, v) \in E\) represents the bidirectional communication link between \(u\) and \(v\). The following definitions are by Garcia-Molina and Barbara [1].

**Definition 1.** Let \(V\) be a universal set of vertices. A set \(C\) of nonempty subsets of \(V\) is said to be a coterie under \(V\) if both of the following conditions hold:

1. (Intersection Property) \(\forall p, q \in C [p \cap q \neq \emptyset],\) and
2. (Minimality Property) \(\forall p, q \in C [p \not\subseteq q].\)

An element of a coterie is called a quorum.

**Definition 2.** Let \(C\) and \(D\) be two distinct coteries under \(V\). \(C\) is said to dominate \(D\) if for any quorum \(p \in D\) there exists a quorum \(q \in C\) such that \(q \subseteq p\). A coterie \(C\) is said to be nondominated (ND, for short) if no coterie dominates \(C\).

The reliability of a vertex (edge) is the probability that the vertex (edge) is operational. Then the availability of a coterie \(C\) on \(G\), denoted by \(A_C(G)\), is defined as the probability that there is a connected subgraph \(G' = (V', E')\) of \(G\) consisting only of operating vertices and edges such that \(q \subseteq V'\) for some \(q \in C\), given the reliabilities of a vertex and an edge. If a coterie \(C\) dominates a coterie \(D\), then \(A_C(G) \geq A_D(G)\) by definition. Thus we can essentially assume that optimal coteries (with respect to the availability) are ND. However, the nondominatedness is obviously not sufficient to pursue an optimal coterie on a graph, since the availability of the coterie depends heavily on the graph. The following concepts are introduced by Ibaraki, Nagamochi, and Kameda to analyze the availability of a coterie on a graph [7].

**Definition 3.** Let \(G = (V, E)\) and \(C\) be a graph and a coterie under \(V\), respectively. The set of all connected minimal subgraphs \(h = (V_h, E_h)\) of \(G\) such that \(q \subseteq V_h\) for some \(q \in C\) is denoted by \(H_C(G)\), where \(h\) is “minimal” in the sense that no proper subgraph of \(h\) satisfies the above condition any more. Hence, \(H_C(G)\) is a set of trees.

Let \(H_C^*(G)\) denote the subset of \(H_C(G)\) constructed from \(H_C(G)\) by repeatedly removing a tree whose proper subtree is in \(H_C(G)\). Then, for any two distinct trees, \(g, h \in H_C^*(G), g \not\subseteq h\). This is called the minimality property of \(H_C^*(G)\).

**Definition 4.** Let \(G = (V, E)\) be a graph, and let \(C\) and \(D\) be two coteries under \(V\). Coterie \(C\) is said to be G-dominant (with respect to \(G\)) if \(H_C^*(G) \neq H_D^*(G)\), and for any \(g \in H_D^*(G)\), there is an \(h \in H_C^*(G)\) such that \(h\) is a subtree of \(g\). A coterie is said to be G-nondominated (G-ND, for short) (with respect to \(G\)) if no coterie G-dominates \(C\) with respect to \(G\).

**Definition 5.** Let \(G = (V, E)\) and \(C\) be a graph and a coterie under \(V\), respectively. By \(C_G(C)\), we denote the set of all subsets \(q \subseteq V\) such that for some \(h = (V_h, E_h) \in H_C^*(G)\), \(q = V_h\) holds.

Let \(C^*_G(C)\) be the subset of \(C_G(C)\) constructed from \(C_G(C)\) by repeatedly removing an element whose proper subset is in \(C_G(C)\). Then, for any distinct elements \(p, q \in C_G^*(C), p \not\subseteq q\). This is called the minimality property of \(C_G^*(C)\).

**Lemma 1.** \(C_G^*(C)\) is a coterie.

**Proof.** \(C_G^*(C)\) satisfies the minimality property by Definition 5. As for the intersection property, by Definition 5, \(C_G^*(C) \subseteq C_G^*(C)\) implies that for each \(q \in C_G^*(C)\), there exists an \(h \in H_C^*(G)\) such that \(q = V_h\). Hence, it is sufficient to show that \(V_h \cap V_f = \emptyset\) for any two trees \(h, f \in H_C^*(G)\). By Definition 3, there exist quorums \(p_h\) and \(p_f\) in \(C\) such that \(p_h \subseteq V_h\) and \(p_f \subseteq V_f\). Since \(p_h \cap p_f = \emptyset\), we have \(V_h \cap V_f = \emptyset\).

By the definition of \(H_C^*(C)\), we may rephrase the availability of a coterie \(C\) on \(G\) as follows: The availability is the probability that there is an \(h \in H_C^*(G)\) consisting only of operating vertices and edges. Hence, if \(C\) is G-dominated, then there is a G-ND coterie whose availability is not smaller than that of \(C\).

Coterie \(C\) is said to be closed under \(G\), if \(C = C_G^*(C)\). If \(C\) is not closed, by definition \(C_G^*(C)\) is dominated by \(C\). They have the same availability though. As you will see in the proof of Theorem 1, \(C_G^*(C)\) plays an important role in finding coteries that G-dominate \(C\).

**Example 1.** Consider a graph \(G = (V, E)\) in Fig. 1, and let \(C\) be a coterie under \(V\) defined as follows:

\[
C = \{[a, b], [a, d], [b, d]\}.
\]

Fig. 2 illustrates all the elements in \(H_C^*(G)\). Note that a subgraph \(\{(a, b, c), ([a, b], (b, c))\}\) of \(G\), for instance, contains quorum \(\{a, b\}\) but is not an element of \(H_C^*(G)\), since it is not minimal.

We construct from \(H_C^*(G)\),

\[
C_G(C) = \{[a, b], [a, b, c], [a, d, e], [b, c, d], [b, d, e]\}.
\]

\(C_G(C)\) is not a coterie, since it does not satisfy the minimality property; \(\{a, b, e\}\) is a superset of \(\{a, b\}\). We construct a coterie \(C_G^*(C)\) from \(C_G(C)\) by removing \(\{a, b, e\}\):

\[
C_G^*(C) = \{[a, b], [a, d, e], [b, c, d], [b, d, e]\}.
\]

Observe that \([b, d]\) in \(C\) is not an element of \(C_G^*(C)\), since the subgraph of \(G\) induced by \([b, d]\) is not connected. It is replaced by minimal supersets, \([b, c, d]\) and \([b, d, e]\), that leave the induced subgraph connected.

### 3 Characterizing G-Nondominated Coteries

In this section, we characterize G-ND coteries. In what follows, notation \(f \subseteq g \) \((f \not\subseteq g)\) denotes that \(f\) is a subgraph (proper subgraph) of \(g\). Also, by \(T(G)\), we denote the set of all connected acyclic (not necessarily spanning) subgraphs of a graph \(G\).
THEOREM 1. For any $\alpha \in \mathcal{H}_C^v(C)$ there exists an $\alpha \in \mathcal{H}_C^v(C)$ satisfying $q = V_\alpha$. Since $V_\alpha \not\subseteq V_f$ for any $\alpha \in \mathcal{H}_C^v(C)$, $q \not\subseteq V_f$ for any $\alpha \in \mathcal{C}_G^v(C)$. Now, we define a new coterie $D$ as the set constructed from $\mathcal{C}_G^v(C) \cup \{V_f\}$ by repeatedly removing a quorum that is a superset of $V_f$ so that the resulting set $D$ satisfies the minimality property of coterie. In the rest, we show that $D$ $G$-dominates $C$.

By construction, $D$ satisfies the minimality property. Since $V_h \cap V_f \neq \emptyset$ for any $\alpha \in \mathcal{H}_C^v(D)$, the intersection property also holds. Thus $D$ is certainly a coterie under $V$. That $D$ $G$-dominates $C$ follows from the fact that $D$ $G$-dominates $C^v(C)$.

**Only if part:** Let $D$ be any coterie under $V$ that $G$-dominates $C$. There are two cases to consider: the case $\mathcal{H}_C^v(C) \subset \mathcal{H}_C^v(D)$ and the case $\mathcal{H}_C^v(C) \supset \mathcal{H}_C^v(D)$.

1) Suppose $\mathcal{H}_C^v(C) \subset \mathcal{H}_C^v(D)$. Then there exists an $\alpha \in \mathcal{H}_C^v(D) - \mathcal{H}_C^v(C)$. Now, we show that (1) holds for this $\alpha$. Suppose otherwise that (1) does not hold for this $\alpha$, i.e., there is an $\alpha \in \mathcal{H}_C^v(C)$ such that either $\alpha \nsubseteq f$ or $V_\alpha \cap V_f = \emptyset$ holds. If $\alpha \nsubseteq f$, $\mathcal{H}_C^v(D)$ contains both $\alpha$ and $f$, which contradicts the minimality of $\mathcal{H}_C^v(D)$. If $V_\alpha \cap V_f = \emptyset$, $\alpha$ and $f$ do not intersect each other, which contradicts the intersection property of $\mathcal{H}_C^v(D)$.

2) Suppose $\mathcal{H}_C^v(C) \supset \mathcal{H}_C^v(D)$. Then there exists a $g \in \mathcal{H}_C^v(C) - \mathcal{H}_C^v(D)$. Since $D$ $G$-dominates $C$, there exists an $\alpha \in \mathcal{H}_C^v(D)$ satisfying $f \subseteq g$. $f$ must be in $\mathcal{H}_C^v(D) - \mathcal{H}_C^v(C)$, since otherwise $C$ contains both $f$ and $g$, which contradicts the minimality of $\mathcal{H}_C^v(C)$. Now we show that (1) holds for this $\alpha$. Suppose otherwise that the formula does not hold for this $\alpha$, i.e., there is an $\alpha \in \mathcal{H}_C^v(C)$ such that either $\alpha \nsubseteq f$ or $V_\alpha \cap V_f = \emptyset$ holds. If $\alpha \nsubseteq f$, $\mathcal{H}_C^v(C)$ contains both $\alpha$ and $g$, which contradicts the minimality of $\mathcal{H}_C^v(C)$. Finally, if $V_\alpha \cap V_f = \emptyset$, then there is an $f' \in \mathcal{H}_C^v(D)$ such that $f' \subseteq h$ (because $D$ $G$-dominates $C$), a contradiction since $f'$ and $f$ do not intersect each other. (It contradicts the intersection property of $\mathcal{H}_C^v(D)$.)

**Example 2.** Consider graph $G$ in Fig. 1 and coterie $C$ given in Example 1. A subtree $(\{b, c\}, \{\{b, c\}\})$ of $C$ satisfies (1). As in the proof of Theorem 1, we can construct a new coterie $D$ from $\mathcal{C}_G^v(C) \cup \{\{b, c\}\}$ by removing $\{b, d, e\}$ that is a superset of $\{b, c\}$:

$$D = \{(a, b), \{b, e\}, \{a, d, e\}, \{b, d, e\}\}.$$ 

Fig. 3 illustrates all the elements in $\mathcal{H}_C^v(D)$. Comparing Fig. 2 with Fig. 3, coterie $D$ $G$-dominates coterie $C$. 

**Lemma 2.** Let $G=(V, E)$ and $C$ be a graph and a coterie under $V$, respectively. Let $f = (V_f, E_f)$ be any tree in $\mathcal{T}(G)$. Then $f \subseteq \alpha$ for some $\alpha \in \mathcal{H}_C^v(C)$, if $q \subseteq V_f$ for some $q \in C$.

**Proof.** Let $f = (V_f, E_f) \in \mathcal{T}(G)$ be any tree such that $q \subseteq V_f$ for some $q \in C$. By the definition of $\mathcal{H}_C^v(C)$, there exists a tree $\alpha \in \mathcal{H}_C^v(C)$ such that $g \subseteq f$. Then by the definition of $\mathcal{H}_C^v(C)$, there exists a tree $\alpha \in \mathcal{H}_C^v(C)$ such that $h \subseteq g \subseteq f$. 

Note that $h \subseteq f$ implies both $V_h \subseteq V_f$ and $E_h \subseteq E_f$.

**Theorem 1.** Let $G=(V, E)$ and $C$ be a graph and a coterie under $V$, respectively. $C$ is $G$-dominated if and only if there exists an $f = (V_f, E_f) \in \mathcal{T}(G)$ satisfying the following formula:

$$\text{For any } h = (V_h, E_h) \in \mathcal{H}_C^v(C), h \subseteq f \text{ and } V_h \cap V_f \neq \emptyset \text{ hold. (1)}$$

**Proof.** If part: Let $f \in \mathcal{T}(G)$ be any tree satisfying (1). Fix an $h \in \mathcal{H}_C^v(C)$. Since $h$ is connected, $V_h \subseteq V_f$ implies $E_h \subseteq E_f$. Thus $h \subseteq f$ if and only if $E_h \subseteq E_f$.

We first show $V_h \subseteq V_f$. Suppose otherwise that $V_h \nsubseteq V_f$. Since $h \in \mathcal{H}_C^v(C)$, there is a $q \in C$ such that $q \subseteq V_h \subseteq V_f$. By Lemma 2, there exists an $h' \in \mathcal{H}_C^v(C)$ such that $h' \subseteq f$, a contradiction. Hence, we have $V_h \subseteq V_f$, and therefore, $E_h \subseteq E_f$, for any $h \in \mathcal{H}_C^v(C)$.
We need to check all the trees in $\mathcal{T}(G)$ to test the G-nondominatedness of a given coterie $C$ based on Theorem 1. If $C$ is ND, we can test its G-nondominatedness by checking a smaller number of trees as we will see next. A tree $f \in \mathcal{T}(G)$ is called a cut-tree if the removal of $f$ from $G$ disconnects $G$, or more formally:

**DEFINITION 6.** A tree $f = (V_f, E_f) \in \mathcal{T}(G)$ is called a cut-tree of $G$ if there is no tree in $\mathcal{T}(G)$ with vertex set $\overline{V_f} = V - V_f$.

The following lemma is by Ibaraki and Kameda [10].

**LEMMA 3** [10]. Let $C$ be a coterie under $V$. Then $C$ is ND if and only if for any $x \subseteq V$, there exists a quorum $q \in C$ such that

$$(q \subseteq x) \oplus (q \subseteq \overline{x}),$$

where $\oplus$ denotes the exclusive OR, and $\overline{x}$ is the complement of $x$ (i.e., $\overline{x} = V - x$).

**THEOREM 2.** Let $G = (V, E)$ and $C$ be a graph and an ND coterie under $V$, respectively. Let $f = (V_f, E_f)$ be any tree in $\mathcal{T}(G)$. If $f$ satisfies (1), then $f$ is a cut-tree of $G$.

**PROOF.** Suppose that $f$ is not a cut-tree and derive a contradiction. Since $f$ is not a cut-tree, there is a $g = (V_g, E_g) \in \mathcal{T}(G)$ such that $V_g = \overline{V_f}$. Since $C$ is ND, there exists a $q \in C$ such that exactly one of $q \subseteq V_f$ or $q \subseteq V_g$ holds by Lemma 3.

Suppose that $q \subseteq V_f$ and $q \nsubseteq V_g$ hold. By Lemma 2, $h \nsubseteq f$ for some $h \in \mathcal{H}^*_C(C)$, a contradiction. Hence, $q \nsubseteq V_f$ and $q \subseteq V_g$ hold. Again by Lemma 2, there exists an $h \in \mathcal{H}^*_C(C)$ such that $h \nsubseteq g$ holds, which implies $V_h \cap V_f = \emptyset$ because $V_g = \overline{V_f}$, a contradiction.

By Theorem 2, checking all the cut-trees is sufficient to test whether or not a given ND coterie $C$ is G-ND; we do not need to check all the trees in $\mathcal{T}(G)$. This definitely improves the time complexity of the decision procedure, but the determination of its time complexity is still open.

**EXAMPLE 3.** Consider graph $G$ in Fig. 1 and coterie $C$ given in Example 1. $G$ has three cut-trees $\{\{(b, e), (b, c), (a, c, e), (b, c, (b, e))\}\}$ and $\{(b, d, e), (b, e), (d, e))\}$ and $\{(b, d, e), (b, e), (d, e))\}$. The first two satisfy (1), and the last one belongs to $\mathcal{H}^*_C(C)$ (see Fig. 2). Since $C$ is nondominated but is G-dominated, there are cut-trees satisfying (1).

**THEOREM 3.** Let $G = (V, E)$ and $C$ be a graph and a coterie under $V$, respectively. $C$ is G-dominated if there exists a cut-tree $f = (V_f, E_f)$ of $G$ satisfying the following formula:

For some $q \in C$, $V_f = \overline{q}$ holds.

**PROOF.** We show that every cut-tree $f$ satisfying (2) also satisfies (1). Let $f$ be a cut-tree of $G$ satisfying $V_f = \overline{q}$ for some $q \in C$.

We first show $V_h \cap V_f \neq \emptyset$ for any $h \in \mathcal{H}^*_C(C)$. For any quorum $p \in C - \{q\}$, $p \cap \overline{q} \neq \emptyset$ by the minimality property of $C$, which implies that $V_h \cap V_f \neq \emptyset$ for any tree $h = (V_h, E_h) \in \mathcal{H}^*_C(C - \{q\})$, since $V_h$ contains a quorum, $p \in C$, as a subset. Since $f$ is a cut-tree, there is no tree in $\mathcal{T}(G)$ with vertex set $\overline{V_f} = q$. Thus, $q \subseteq V_h$ for any $h \in \mathcal{H}^*_C(C)$, which implies $V_h \cap V_f \neq \emptyset$.

Hence, $V_h \cap V_f \neq \emptyset$ for any $h \in \mathcal{H}^*_C(C)$. Suppose otherwise that there exists an $h \in \mathcal{H}^*_C(C)$ satisfying $V_h \subseteq V_f$. This implies that there is a $p \in C$ satisfying $p \subseteq \overline{q}$. However, it is a contradiction since $p \cap \overline{q} \neq \emptyset$ contradicts the intersection property of $C$. Hence, $V_h \nsubseteq V_f$ for any $h \in \mathcal{H}^*_C(C)$, and $g \nsubseteq f$.

As a final remark in this section, since the connectivity of a given graph can be tested in time $O(m + n)$, the sufficient condition of Theorem 3 can be tested in time $O((m + n) | C|)$, where $m$ and $n$ are the sizes of vertex and edge sets, respectively.

### 4 The Majority Coterie on Graphs

The majority coterie is one of the most well-studied coteries. This section discusses on which graphs the majority coterie becomes G-ND.

**DEFINITION 7.** The majority coterie $C$ under $V$ is defined as follows:

1. When $|V|$ is odd, $C$ is the set of all subsets of $V$ whose cardinality is exactly $(|V| + 1)/2$.
2. When $|V|$ is even, let $v$ be an arbitrary fixed vertex in $V$. Then $C = C_1 \cup C_2$, where $C_1$ is the set of all subsets of $V$, containing $v$, whose cardinality is exactly $|V|/2$, and $C_2$ is the set of all subsets of $V$, not containing $v$, whose cardinality is exactly $|V|/2 + 1$.

We call $v$ the semiprimary vertex.

We start with two simple lemmas.

**LEMMA 4.** Let $G = (V, E)$ and $f = (V_f, E_f)$ be a connected graph and a tree in $\mathcal{T}(G)$, respectively. For any nonnegative integer $k \leq |V| - |V_f|$, there exists a tree $g \in \mathcal{T}(G)$ such that $V_g \subseteq V_f$ and $|V_g| = |V_f| + k$. 
PROOF. Since G is connected, there exists a spanning tree h of G such that \( f \subseteq h \). The existence of a tree g satisfying the condition of this lemma is clear from this fact. 

**Lemma 5.** Let \( G = (V, E) \) and \( f = (V_f, E_f) \) be a biconnected graph and a tree in \( \mathcal{T}(G) \), respectively. For any nonnegative integer \( k \leq |V| - |V_f| - 1 \) and vertex \( v \in V - V_f \), there exists a tree \( g \in \mathcal{T}(G) \) such that \( v \not\in V_g \), \( V_f \subseteq V_g \) and \( |V_g| = |V_f| + k \) hold.

**Proof.** Let \( v \in V - V_f \) be any vertex. Since G is biconnected, \( H = G - [v] \), i.e., the subgraph of G induced by vertex set \( V - [v] \), is connected. By applying Lemma 4 to \( H \), the proof completes.

In the last section, we showed that the existence of a cut-tree \( f \) satisfying (2) is a sufficient condition for a coterie on a graph to be \( G \)-dominated. The condition, however, is not necessary. Here, we show that the condition is necessary. Hence, without loss of generality, we assume that \( G \) is biconnected in the next theorem.

**Theorem 4.** Let \( G = (V, E) \) and C be a biconnected graph and the majority coterie under V, respectively. C is \( G \)-dominated if and only if there exists a cut-tree \( f = (V_f, E_f) \) of G satisfying (2).

**Proof.** By Theorem 3, it suffices to show the necessity. Suppose that the majority coterie C on a graph G is \( G \)-dominated. Since C is ND, by Theorems 1 and 2, there exists a cut-tree \( f = (V_f, E_f) \) of G satisfying (1). We consider the following two cases: the case where \( |V| \) is odd and the case where \( |V| \) is even.

1) Suppose that \( |V| \) is odd. Since \( h \not\subseteq f \) for any \( h = (V_h, E_h) \in \mathcal{H}_{\mathcal{C}}(C) \), \( |V_f| \leq (|V| - 1)/2 \) by Definition 7. Then, by Lemma 4, there exists an \( f' = (V_{f'}, E_{f'}) \in \mathcal{T}(G) \) such that \( V_f \subseteq V_{f'} \) and \( |V_{f'}| = (|V| - 1)/2 \), which implies that \( V_f = \emptyset \) for some \( q \in C \), since \( \nabla_{f'} = (|V| + 1)/2 \). It is sufficient to show that \( f' \) is a cut-tree of \( G \). Suppose otherwise that \( f' \) is not a cut-tree. Then there is a \( g \in \mathcal{T}(G) \) such that \( V_g = V_{f'} \in C \). By Lemma 2, there is a \( g' = (V_{g'}, E_{g'}) \in \mathcal{H}_{\mathcal{C}}(C) \) such that \( g' \subseteq g \). Since \( V_{g'} \cap V_f = \emptyset \), \( V_{g'} \subseteq V_g \), and \( f' \subseteq V_f \), we have \( V_{g'} \cap V_f = \emptyset \), which contradicts (1).

2) Suppose that \( |V| \) is even. Let \( v \) be the semiprimary vertex of C. Since \( h \not\subseteq f \) for each \( h \in \mathcal{H}_{\mathcal{C}}(C) \), by Definition 7, either \( v \in V_f \) and \( |V_f| \leq |V|/2 - 1 \), or \( v \not\in V_f \) and \( |V_f| \leq |V|/2/2 \) holds. It is sufficient to show that \( f' \) is a cut-tree in each case. First, suppose \( v \in V_f \) and \( |V_f| \leq |V|/2 - 1 \). By Lemma 4, there exists a \( f' = \mathcal{T}(G) \) such that \( V_f \subseteq V_f \) and \( |V_f| = |V|/2 - 1 \). Since \( v \not\in V_f \) and \( |V_f| = |V|/2 - 1 \), \( f' \) can be shown to be a cut-tree. Next, suppose that \( v \not\in V_f \) and \( |V_f| \leq |V|/2 \). Since \( v \not\in V_f \), \( f' \subseteq V_f \) and \( |V_f| = |V|/2 \). Since \( v \not\in V_f \), \( f' \subseteq V_f \) and \( |V_f| = |V|/2 \). Using the argument as in case 1 again, we can show that \( f' \) is a cut-tree.

**Example 4.** Consider graph G in Fig. 1 and the majority coterie C under the vertex set \{a, b, c, d, e\}. That is, \( C = \{[a, b, c,], [a, b, d], [a, c, d], [a, c, e], [a, d, e], [b, c, d], [b, c, e], [b, d, e], [c, d, e]\} \).

Since there is a cut-tree \( f = ([b, c], ([b, e]) \) such that \( [b, e] = \{a, c, d\} \in C \), C on G is \( G \)-dominated.

Next, another graph \( G' \) in Fig. 4, consider C. Since there is no cut-tree of size 2 in \( G' \), there is no cut-tree \( f \) satisfying (2). Hence, C on \( G' \) is \( G' \)-ND.

**Fig. 4.** A graph \( G' \) with five vertices.
degree at most $3|V|/4 - 1$, a contradiction. If $|V_1| < |V|/4$, then, since $|V_1 \cup V_2| < 3|V|/4$, every vertex in $V_1$ has a degree less than $3|V|/4$, a contradiction.

Suppose that $C$ on $G$ is $G$-dominated. Then there is a cut-tree $f = (V_f, E_f)$ such that $V_f = \overline{q}$ for some $q \in C$.

Since $\kappa(G) > |V|/2$, $|V_f| \geq \kappa(G) > |V|/2$. On the other hand, $|V_f| = |\overline{q}| \leq |V|/2$, a contradiction. Hence, $C$ on $G$ is $G$-ND.

5 Conclusion

The concept of $G$-domination is introduced to search for a coterie that maximizes the availability on a given graph. In this paper, we presented a necessary and sufficient condition for a coterie on a graph to be $G$-nondominated. We also presented a sufficient condition for a nondominated coterie on a graph to be $G$-nondominated. We then discussed the majority coterie, and derived a necessary and sufficient condition for the majority coterie on a graph to be $G$-nondominated. Finally, we derived an easy sufficient condition for the majority coterie on a graph to be $G$-nondominated.

References


