Dehn surgeries on knots which yield lens spaces and genera of knots

BY HIROSHI GODA
The Graduate School of Science and Technology, Kobe University,
1-1 Rokkodai, Nada, Kobe 657-8501, Japan
e-mail: goda@math.kobe-u.ac.jp

AND MASAKAZU TERAGAITO
Department of Mathematics and Mathematics Education, Hiroshima University,
1-1-1 Kagamigama, Higashihiroshima 739-8524, Japan
e-mail: mteraga@sed.hiroshima-u.ac.jp

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1. Introduction

It is an interesting open question when Dehn surgery on a knot in the 3-sphere $S^3$ can produce a lens space (see [10, 12]). Some studies have been made for special knots; in particular, the question is completely solved for torus knots [21] and satellite knots [3, 29, 31]. It is known that there are many examples of hyperbolic knots which admit Dehn surgeries yielding lens spaces. For example, Fintushel and Stern [8] have shown that 18- and 19-surgeries on the $(-2, 3, 7)$-pretzel knot give lens spaces $L(18, 5)$ and $L(19, 7)$, respectively. However, there seems to be no essential progress on hyperbolic knots. It might be a reason that some famous classes of hyperbolic knots, such as 2-bridge knots [26], alternating knots [5], admit no surgery yielding lens spaces.

In this paper we focus on the genera of knots to treat the present condition methodically and show that there is a constraint on the order of the fundamental group of the resulting lens space obtained by Dehn surgery on a hyperbolic knot. Also, this new standpoint enables us to present a conjecture concerning such a constraint, which holds for all known examples.

Let $K$ be a hyperbolic knot in $S^3$. The exterior of $K$, denoted by $E(K)$, is the complement of an open tubular neighbourhood of $K$. Let $r$ be a slope on $\partial E(K)$, that is, the isotopy class of an essential simple closed curve in $\partial E(K)$, and let $K(r)$ be the closed 3-manifold obtained by $r$-Dehn surgery on $K$. Thus $K(r) = E(K) \cup V_r$, where $V_r$ is a solid torus attached to $\partial E(K)$ along their boundaries in such a way that $r$ bounds a meridian disc in $V_r$. Slopes on $\partial E(K)$ are parameterized as $m/n \in \mathbb{Q} \cup \{1/0\}$ in the usual way [23].

If $K(r)$ is a lens space, then $r$ is an integer by the Cyclic Surgery Theorem [4] and $\pi_1 K(r)$ has the order $|r|$. Furthermore, there are at most two such slopes and if there are two then they are consecutive.

**Theorem 1.1.** Let $K$ be a hyperbolic knot in $S^3$. If $K(r)$ is a lens space, then $|r| \leq 12g - 7$, where $g$ is the genus of $K$. 
In [2], Berge introduced ‘double-primitive’ knots and showed that such knots admit integral surgeries which yield lens spaces. Furthermore, he gave a list of double-primitive knots, including all known knots with surgeries yielding lens spaces. As he wrote there, there is still a possibility that his list of double-primitive knots is not complete, but he has suggested that the list is complete and if a knot has a surgery yielding a lens space then the knot appears in his list (see [10, 12, 20]).

All knots in Berge’s list can be expressed as closed positive (or negative) braids and therefore they are fibred [25]. Then it is easy to calculate their genera, since Seifert’s algorithm gives fibre surfaces, that is, minimal genus Seifert surfaces for such knots.

On the basis of the above remark and the calculation for Berge’s knots, we suggest the following.

**Conjecture.** Let \( K \) be a hyperbolic knot in \( S^3 \). If \( K(r) \) is a lens space, then \( K \) is fibred and \( 2g + 8 \leq |r| \leq 4g - 1 \), where \( g \) is the genus of \( K \).

Since the \((-2, 3, 7)\)-pretzel knot has genus 5, it is expected that this estimate would be best possible. Surprisingly, the \((-2, 3, 7)\)-pretzel knot is the only hyperbolic knot that has genus 5 in Berge’s list.

In [3], it is conjectured that for a hyperbolic knot \( K \), if \( K(r) \) is a lens space then \( |r| \geq 18 \). This implies that no lens space \( L \) with \( |\pi_1(L)| < 5 \) can arise from a non-trivial knot [3, 10, 30]. An affirmative answer to the above conjecture would imply that a hyperbolic knot, admitting a surgery which yields a lens space, has genus at least 5 and therefore that 18 is the minimal order of the fundamental groups of lens spaces obtained by surgery on hyperbolic knots.

For the case of genus one we have the complete answer.

**Theorem 1.2.** No Dehn surgery on a genus one, hyperbolic knot in \( S^3 \) gives a lens space.

Combining this with known facts, we can completely determine Dehn surgeries on genus one knots which yield lens spaces.

**Theorem 1.3.** A genus one knot \( K \) in \( S^3 \) admits Dehn surgery yielding a lens space if and only if \( K \) is the \((\pm 3, 2)\)-torus knot and the surgery slope is \((\pm 6n + \varepsilon)/n \) for \( n \neq 0 \), \( \varepsilon = \pm 1 \).

As in earlier results, we have proved that no Dehn surgery on a genus one knot gives \( L(2, 1) \) [27] (see also [6]) or \( L(4k, 2k \pm 1) \) for \( k \geq 1 \) [28]. It was also known that if a genus one knot has a non-trivial Alexander polynomial, then the knot has no cyclic surgery of even order [22, corollary 2]. Recently, [19] showed that the lens space \( L(2k, 1) \) cannot be obtained by surgery on a strongly invertible knot.

To prove Theorems 1.1 and 1.2, we analyze the graphs of the intersection of surfaces properly embedded in a knot exterior. One comes from a Heegaard torus of a lens space and the other is a minimal genus Seifert surface for the knot. By virtue of the use of a Seifert surface, instead of a level sphere in a thin position of the knot, the graphs can include the information on the order of the fundamental group of the resulting lens space after Dehn surgery. In Section 2, it is found out that there are some constraints on Scharlemann cycles. The proof of Theorem 1.1 is divided into two cases according to the number \( t \) of points of intersection between the Heegaard
torus and the core of the attached solid torus. In Section 3, the case that $t \geq 4$ is dealt with, and the special case that $t = 2$ is discussed in Section 4 and the proof of Theorem 1.1 is completed. Finally in Sections 5 and 6, we specialize to the case that $K$ has genus one and prove Theorems 1.2 and 1.3.

2. Preliminaries

Throughout this paper, $K$ will be assumed to be a hyperbolic knot in $S^3$. For a slope $r$, suppose that $K(r) = E(K) \cup V_r$ is a lens space. Since $K$ is not a torus knot, the Cyclic Surgery Theorem [4, corollary 1] implies that the slope $r$ must be integral. We may assume that $r > 1$. Thus $\pi_1 K(r)$ has the order $r$. For simplicity, we denote $V_r$ by $V$. Let $K^*$ be the core of $V$.

Let $\hat{T}$ be a Heegaard torus in $K(r)$. Then $K(r) = U \cup W$, where $U$ and $W$ are solid tori. We can assume that $\hat{T}$ meets $K^*$ transversely in $t$ points and that $\hat{T} \cap V$ consists of $t$ mutually disjoint meridian discs of $V$. Then $T = \hat{T} \cap E(K)$ is a punctured torus with $t$ boundary components, each having slope $r$ on $\partial E(K)$.

Let $S \subset E(K)$ be a minimal genus Seifert surface of $K$. Then $S$ is incompressible and boundary-incompressible in $E(K)$.

By an isotopy of $S$, we may assume that $S$ and $T$ intersect transversely and $\partial S$ meets each component of $\partial T$ in exactly $r$ points. We choose $\hat{T}$ so that the next condition (*) is satisfied:

$$ (*) \hat{T} \cap K^* \neq \emptyset$$

and each arc component of $S \cap T$ is essential in $S$ and in $T$.

This can be achieved if $K^*$ is put in thin position with respect to $\hat{T}$ [9, 11, 14]. (Note that if $K^*$ can be isotoped to lie on $\hat{T}$, then $K$ would be a torus knot.) Furthermore, we may assume that $\hat{T}$ is chosen so that $t$ is minimal over all Heegaard tori in $K(r)$ satisfying (*). This minimality of $\hat{T}$ will be crucial in this paper.

Since $S$ is incompressible in $E(K)$ and $E(K)$ is irreducible, it can be assumed that no circle component of $S \cap T$ bounds a disc in $T$. But it does not hold for $S$ in general. We further assume that the number of loop components of $S \cap T$ is minimal up to an isotopy of $S$.

The arc components of $S \cap T$ define graphs $G_S$ in $\hat{S}$ and $G_T$ in $\hat{T}$ as follows [4, 16], where $\hat{S}$ is the closed surface obtained by capping $\partial S$ off by a disc. Let $G_S$ be the graph in $\hat{S}$ obtained by taking as the (fat) vertex the disc $\hat{S} - \text{Int } S$ and as edges the arc components of $S \cap T$ in $\hat{S}$. Similarly, $G_T$ is the graph in $\hat{T}$ whose vertices are the discs $\hat{T} - \text{Int } T$ and whose edges are the arc components of $S \cap T$ in $\hat{T}$. Number the components of $\partial T$, $1, 2, \ldots, t$ in sequence along $\partial E(K)$. Let $\partial_i T$ denote the component of $\partial T$ with label $i$. This induces a numbering of the vertices of $G_T$.

Let $u_i$ be the vertex of $G_T$ with the label $i$ for $i = 1, 2, \ldots, t$. Let $H_{x,x+1}$ be the part of $V$ between consecutive fat vertices $u_x$ and $u_{x+1}$ of $G_T$. When $t = 2$, $V$ is considered to be the union $H_{1,2} \cup H_{2,1}$. Each endpoint of an edge in $G_S$ at the unique vertex $v$ has a label, namely the label of the corresponding component of $\partial T$. Thus the labels $1, 2, \ldots, t$ appear in order around $v$ repeated $r$ times.

The graphs $G_S$ and $G_T$ satisfy the parity rule [4] which can be expressed as the following: the labels at the endpoints of an edge of $G_S$ have distinct parities.

A trivial loop in a graph is a length one cycle which bounds a disc face. By (*), neither $G_S$ nor $G_T$ contains trivial loops.

A family of edges $\{e_1, e_2, \ldots, e_p\}$ in $G_S$ is a Scharemann cycle (of length $p$) if it
assumption that $p_K$ gives a punctured lens space. Since a lens space parities. This observation implies that the edges of $G_T$, and let $\delta$ be the disc bounded by $\delta$ in $\mathring{T}$, then all vertices of $G_T$ must lie in the disc bounded by $\delta$ in $\mathring{T}$.

**Lemma 2.2.** Let $\xi$ be a loop in $S \cap T$. Suppose that $\xi$ bounds a disc $\delta$ in $S$ with $\Int \delta \cap \mathring{T} = \emptyset$. If $\xi$ is inessential in $\mathring{T}$, then all vertices of $G_T$ must lie in the disc bounded by $\xi$ in $\mathring{T}$.

**Proof.** Let $\delta'$ be the disc bounded by $\xi$ in $\mathring{T}$, then $\delta' \cap V \neq \emptyset$, since $\xi$ is essential in $T$ by the assumption on $S \cap T$. If both sides of $\xi$ on $\mathring{T}$ meet $V$, replace $\mathring{T}$ by $\mathring{T}' = (\mathring{T} - \delta') \cup \delta$. Then $\mathring{T}'$ gives a new Heegaard torus of $K(r)$ satisfying $(\ast)$. However this contradicts the choice of $\mathring{T}$, since $|\mathring{T}' \cap K^*| < |\mathring{T} \cap K^*|$. Hence all vertices of $G_{\mathring{T}}$ lie in $\delta'$.

**Lemma 2.3.** Let $\sigma$ be a Scharlemann cycle in $G_S$ of length $p$ with label pair $\{x, x+1\}$ and let $f$ be the face of $G_S$ bounded by $\sigma$. Suppose that $p \neq r$. Then the edges of $\sigma$ cannot lie in a disc in $\mathring{T}$ and $\Int f \cap \mathring{T} = \emptyset$.

**Proof.** Assume for contradiction that the edges of $\sigma$ lie in a disc $D$ in $\mathring{T}$. Let $\Gamma$ be the subgraph of $G_T$ consisting of two vertices $u_x$ and $u_{x+1}$ along with the edges of $\sigma$.

First, suppose that $\Int f \cap D \neq \emptyset$. Then all components in $\Int f \cap D$ are parallel to $\partial D$ in $D - \Gamma$ by the minimality of $S \cap T$. Thus we can replace $D$ by a subdisc which does not meet $\Int f$. We may now assume that $\Int f \cap D = \emptyset$. Then $N(D \cup H_{x,x+1} \cup f)$ gives a punctured lens space. Since a lens space $K(r)$ is irreducible, this means that $K(r)$ is a lens space whose fundamental group has order $p$. This contradicts the assumption that $p \neq r$. Thus the edges of $\sigma$ cannot lie in a disc in $\mathring{T}$.

Assume that $\Int f \cap \mathring{T} \neq \emptyset$. Let $\mu$ be an innermost component of $\Int f \cap \mathring{T}$ on $f$. 

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bounds a disc face of $G_S$ and all the edges have the same pair of labels $\{x, x+1\}$, for some $x$, at their two endpoints, which is called the label pair of the Scharlemann cycle. Note that each edge $e_i$ connects the vertex $u_x$ with $u_{x+1}$ in $G_T$. A Scharlemann cycle of length two is called an $S$-cycle for short. Remark that the interior of the face bounded by a Scharlemann cycle may meet $\mathring{T}$, since $T$ is not necessarily incompressible in $E(K)$.

Let $\sigma$ be a Scharlemann cycle in $G_S$ with label pair $\{x, x+1\}$. If the edges of $\sigma$ (and vertices $u_x$ and $u_{x+1}$) are contained in an annulus in $\mathring{T}$, and if they do not lie in a disc in $\mathring{T}$, then we say that the edges of $\sigma$ lie in an essential annulus in $\mathring{T}$.

**Lemma 2.1.** Let $\sigma$ be a Scharlemann cycle in $G_S$ of length $p$ with label pair $\{x, x+1\}$, where $p$ is 2 or 3. Let $f$ be the face of $G_S$ bounded by $\sigma$. If the edges of $\sigma$ do not lie in a disc in $\mathring{T}$, then they lie in an essential annulus $A$ in $\mathring{T}$. Furthermore, if $\Int f \cap \mathring{T} = \emptyset$, then $M = N(A \cup H_{x,x+1} \cup f)$ is a solid torus such that the core of $A$ runs $p$ times in the longitudinal direction of $M$.

**Proof.** If $p = 2$, then it is obvious that the edges of $\sigma$ lie in an essential annulus in $\mathring{T}$.

Assume $p = 3$. Let $\sigma = \{e_1, e_2, e_3\}$. If the endpoints of $e_1, e_2, e_3$ appear in this order when one travels around $u_x$ clockwise, say, then those of $e_1, e_2, e_3$ appear in the same order when one travels around $u_{x+1}$ anticlockwise, since $u_x$ and $u_{x+1}$ have distinct parities. This observation implies that the edges of $\sigma$ lie in an essential annulus in $\mathring{T}$.

Consider the genus two handlebody $N(A \cup H_{x,x+1})$. Then $M$ is obtained by attaching a 2-handle $N(f)$. Since there is a meridian disc of $N(A)$ which intersects $\partial f$ once, $\partial f$ is primitive and therefore $M$ is a solid torus. It is not hard to see that the core of $A$ runs $p$ times in the longitudinal direction of $M$. (See also [17, lemma 3-7]).
Since the edges of $\sigma$ do not lie in a disc in $\hat{T}$, it follows from Lemma 2.2 that $\mu$ is essential in $\hat{T}$. Then it can be assumed that the disc $\delta$ bounded by $\mu$ on $f$ is contained in $\hat{W}$, say, one of the solid tori bounded by $\hat{T}$ in $K(r)$. Thus $\delta$ is a meridian disc of $W$.

In $\hat{W}$, compress $\hat{T}$ along $\delta$ to obtain a 2-sphere $Q$. There is a disc $E$ in $Q$ which contains the edges of $\sigma$ and two vertices $u_x$ and $u_{x+1}$. Even if $\text{Int} f \cap E \neq \emptyset$, the cut-and-paste operation gives a new $f$ with $\text{Int} f \cap E = \emptyset$. Thus $N(E \cup H_{x,x+1} \cup f)$ gives a punctured lens space whose fundamental group has order $p$, which contradicts the assumption again.

When there exist two Scharlemann cycles with disjoint label pairs, the assumption on the length in the statement of Lemma 2.3 is not necessary.

**Lemma 2.4.** Let $\sigma_1$ and $\sigma_2$ be Scharlemann cycles in $G_S$ with disjoint label pairs and let $f_1$ and $f_2$ be the faces of $G_S$ bounded by $\sigma_1$ and $\sigma_2$ respectively. Then the edges of $\sigma_i$ lie in an essential annulus $A_i$ in $\hat{T}$ with $A_1 \cap A_2 = \emptyset$ and $\text{Int} f_i \cap \hat{T} = \emptyset$ for $i = 1, 2$.

**Proof.** Let $\{x_i, x_i + 1\}$ be the label pair of $\sigma_i$. Assume that the edges of $\sigma_i$ lie in a disc $D_i$ in $\hat{T}$ for contradiction. By the same argument in the proof of Lemma 2.3, we may assume that $\text{Int} f_i \cap D_i = \emptyset$. If $\text{Int} f_i \cap \hat{T} = \emptyset$, then $N(D_i \cup H_{x_i,x_i+1} \cup f_i)$ gives a punctured lens space in a solid torus, which is impossible. Therefore $\text{Int} f_i \cap \hat{T} \neq \emptyset$.

Choose an innermost component $\xi$ of $\text{Int} f_i \cap \hat{T}$ on $f_i$. Let $\delta$ be the disc bounded by $\xi$ on $f_i$.

Assume that $\xi$ is inessential in $\hat{T}$. By Lemma 2.2, $G_T$ lies in the disc bounded by $\xi$. Then the edges of $\sigma_2$ also lie in a disc $D_2$ in $\hat{T}$. We remark that one of $D_1$ and $D_2$ may be contained in the other, possibly. As above, we can assume that $\text{Int} f_2 \cap D_2 = \emptyset$.

If $D_1 \cap D_2 = \emptyset$, then we can assume that $\text{Int} f_i \cap D_j = \emptyset$ for $i, j \in \{1, 2\}$ by the cut-and-paste operation of $f_i$.

Otherwise, $D_2 \subset D_1$, say. Clearly, $\text{Int} f_1 \cap D_j = \emptyset$ for $j = 1, 2$. If $\text{Int} f_2 \cap D_1 = \emptyset$, then it can be assumed that each component of $\text{Int} f_2 \cap D_1$ is parallel to $\partial D_2$ in $D_1 - \Gamma$, where $\Gamma$ is the subgraph of $G_T$, consisting of the vertices $u_{x_i}, u_{x_i+1}$ along with the edges of $\sigma_i$ for $i = 1, 2$. But this contradicts Lemma 2.2. Therefore, we can assume that $\text{Int} f_i \cap D_j = \emptyset$ for $i, j \in \{1, 2\}$ in either case.

Then $N(D_i \cup H_{x_i,x_i+1} \cup f_i)$ and $N(D_2 \cup H_{x_i,x_i+1} \cup f_2)$ give two disjoint punctured lens spaces in $K(r)$, which is impossible. (When $D_2 \subset D_1$, say, we have to push $D_2$ into a suitable direction away from $D_1$.)

Therefore $\xi$ is essential in $\hat{T}$. Then $\delta$ is a meridian disc of the solid torus $W$, say. Compressing $\hat{T}$ along $\delta$ gives a 2-sphere $Q$ on which there are two disjoint discs $E_1, E_2$ each containing the edges of $\sigma_1, \sigma_2$, respectively. Then the same argument as above gives a contradiction.

Therefore the edges of $\sigma_i$ cannot lie in a disc in $\hat{T}$ for $i = 1, 2$, and then there are disjoint essential annuli $A_i$ in $\hat{T}$ in which the edges of $\sigma_i$ lie for $i = 1, 2$, respectively.

Suppose that $\text{Int} f_1 \cap \hat{T} = \emptyset$. Consider an innermost component $\eta$ of $\text{Int} f_1 \cap \hat{T}$ in $f_1$. By Lemma 2.2, $\eta$ is essential in $\hat{T}$. As above, there are two disjoint punctured lens spaces in $K(r)$, which is impossible again. Similarly for $f_2$. Therefore $\text{Int} f_i \cap \hat{T} = \emptyset$ for $i = 1, 2$.

Let $f$ be a face of $G_S$. Although $\text{Int} f \cap \hat{T} \neq \emptyset$ in general, a small collar neigh-
bourhood of $\partial f$ in $f$ is contained in one side of $\hat{T}$. Then we say that $f$ lies on that side of $\hat{T}$.

The next two lemmas deal with the situation where $G_S$ has two Scharlemann cycles of length two and three simultaneously.

**Lemma 2.5.** Let $\sigma$ be an $S$-cycle in $G_S$ and let $\tau$ be a Scharlemann cycle in $G_S$ of length three. Let $f$ and $g$ be the faces of $G_S$ bounded by $\sigma$ and $\tau$ respectively. If $\sigma$ and $\tau$ have disjoint label pairs, then $\sigma$ and $\tau$ lie on opposite sides of $\hat{T}$ and $r \equiv 0 \pmod{3}$.

**Proof.** Let $\{x, x+1\}$, $\{y, y+1\}$ be the label pairs of $\sigma$ and $\tau$, respectively. By Lemma 2.4, the edges of $\sigma$ give an essential cycle in $\hat{T}$ after shrinking two fat vertices $u_x$ and $u_{x+1}$ to points and $\text{Int} f \cap \hat{T} = \emptyset$. Then $f$ is contained in the solid torus, $W$ say, and the union $H_{x,x+1} \cup f$ gives a Mobius band $B$ properly embedded in $W$, after shrinking $H_{x,x+1}$ to its core radially. (See Fig. 1.)

Similarly, by Lemma 2.4, the edges of $\tau$ lie in an essential annulus $A$ in $\hat{T}$ which is disjoint from the edges of $\sigma$ and $\text{Int} g \cap \hat{T} = \emptyset$.

Suppose that $g \subset W$. If a solid torus $J$ is attached to $W$ along their boundaries so that the slope of $\partial B$ bounds a meridian disc of $J$, then the resulting manifold $N = J \cup W$ contains a projective plane and therefore $N = L(2, 1)$. However, in $N$, the edges of $\tau$ are contained in a disc $D$ obtained by capping a boundary component of $A$ off by a meridian disc of $J$. Then $N(D \cup H_{y,y+1} \cup g)$ gives a punctured lens space whose fundamental group has order three in $N$, which is impossible. Thus $f$ and $g$ lie on opposite sides of $\hat{T}$.

Next, assume that $r \equiv 0 \pmod{3}$ for contradiction. We may assume that $f \subset W$ and $g \subset U$.

By Lemma 2.4, $M = N(\partial \text{U} H_{y,y+1} \cup g)$ is a solid torus, and $A$ runs three times in the longitudinal direction on $\partial M$. The annulus $A' = \text{cl}(\partial M - A)$ is properly embedded in $U$ and so $A'$ is parallel to $\text{cl}(\hat{T} - A)$. Therefore, $A$ runs three times in the longitudinal direction of $U$.

The slope determined by $\partial B$ on $\partial W$ meets a meridian of $W$ twice. On $\partial U$, the slope can be expressed $a/3$ and the meridian of $W$ defines a slope $b/r$ for some integers $a, b$. Then $\Delta(a/3, b/r) = |ar - 3b| = 3|ar/3 - b| \neq 2$, which is a contradiction.

**Lemma 2.6.** Let $\sigma, \tau, f, g$ be as in Lemma 2.5. Suppose that $\sigma$ and $\tau$ lie on opposite
sides of $\hat{T}$ and have the same label pair and that $r \neq 2, 3$. If there is an essential annulus $A$ in $\hat{T}$ in which the edges of $\sigma$ and $\tau$ lie, then $r \neq 0 \mod 3$.

**Proof.** By Lemma 2.3, $\text{Int } f \cap \hat{T} = \emptyset$ and $\text{Int } g \cap \hat{T} = \emptyset$. We remark that $t = 2$. Hence $\sigma$ and $\tau$ have the label pair $\{1, 2\}$.

We may assume that $H_{i, 2} \subset W$ and $f \subset W$. Then $M_1 = N(A \cup H_{i, 2} \cup f)$ is a solid torus and $A$ runs twice in the longitudinal direction on $\partial M_1$ by Lemma 2.4. Furthermore, the annulus $A'_1 = \text{cl}(\partial M_1 - A)$ is parallel to $\text{cl}(\hat{T} - A)$ in $W$. Similarly, $M_2 = N(A \cup H_{i, 2} \cup g)$ is a solid torus and $A$ runs three times in the longitudinal direction on $\partial M_2$ by Lemma 2.4. The annulus $A'_2 = \text{cl}(\partial M_2 - A)$ is also parallel to $\text{cl}(\hat{T} - A)$ in $U$. Then the same argument as in the proof of Lemma 2.5 gives the desired result.

3. The generic case

In this section we prove Theorem 1.1 under the hypothesis $t \geq 4$. The case $t = 2$ will be dealt with separately in the next section.

**Lemma 3.1.** Let $\{e_1, e_2, \ldots, e_t\}$ be mutually parallel edges in $G_S$ numbered successively. Then $\{e_{t/2}, e_{t/2 + 1}\}$ is an $S$-cycle.

**Proof.** We may assume that $e_i$ has the label $i$ at one endpoint for $1 \leq i \leq t$. If $e_t$ has the label $1$ at the other endpoint, then $\{e_{t/2}, e_{t/2 + 1}\}$ is an $S$-cycle. Therefore we may suppose that $e_{j+1}$ has the label $1$ at the other endpoint for some $j < t/2$ by the parity rule. Then $\sigma_1 = \{e_j, e_{j+1}\}$ and $\sigma_2 = \{e_{t/2+j}, e_{t/2+j+1}\}$ form $S$-cycles with disjoint label pairs.

Let $f_i$ be the face of $G_S$ bounded by $\sigma_i$. By Lemma 2.4, the edges of $\sigma_1$ lie in an essential annulus in $\hat{T}$ and $\text{Int } f_i \cap \hat{T} = \emptyset$. Then, as in the proof of Lemma 2.5, we obtain two disjoint Möbius band $B_1$ and $B_2$ from $H_{j,j+1} \cup f_j$ and $H_{t/2+j,t/2+j+1} \cup f_j$ by shrinking $H_{j,j+1}$ and $H_{t/2+j,t/2+j+1}$ to their cores radially.

Since $\partial B_1$ and $\partial B_2$ are parallel in $\hat{T}$, they divide $\hat{T}$ into two annuli $A_1$ and $A_2$.

In $G_T$, $u_k$ and $u_{2j-k+1}$ lie in the same annulus for $1 \leq k \leq j - 1$, since the edge $e_k$ connects the two vertices in $G_T$. Similarly, $u_{2j-\ell}$ and $u_{t+1-\ell}$ for $1 \leq \ell \leq t/2 - j - 1$ lie in the same annulus. Therefore we see that $\text{Int } A_i$ contains an even number of vertices for $i = 1, 2$. Let $F = B_1 \cup B_2 \cup A_1$. Then $F$ meets $K^+$ in an even number of points (after a perturbation). Then $F' = F \cap E(K)$ gives a punctured Klein bottle properly embedded in $E(K)$ having an even number of boundary components. By attaching suitable annuli in $\partial E(K)$ to $F'$ along boundaries, we have a closed non-orientable surface in $E(K)$, which is impossible.

**Lemma 3.2.** $G_S$ does not contain more than $t$ mutually parallel edges.

**Proof.** Let $e_1, e_2, \ldots, e_t, e_{t+1}$ be mutually parallel edges in $G_S$ numbered successively. By Lemma 3.1, $\{e_{t/2}, e_{t/2 + 1}\}$ is an $S$-cycle. Furthermore, $\{e_t, e_{t+1}\}$ forms another $S$-cycle and these two $S$-cycles have disjoint label pairs. Then the same argument as in the proof of Lemma 3.1 gives a contradiction.

The reduced graph $\overline{G}_S$ of $G_S$ is defined to be the graph obtained from $G_S$ by amalgamating each set of mutually parallel edges of $G_S$ to a single edge. If an edge $\overline{e}$ of $\overline{G}_S$ corresponds to $s$ mutually parallel edges of $G_S$, then the weight of $\overline{e}$ is defined to be $s$ and we denote by $w(\overline{e}) = s$. If $w(\overline{e}) = t$, then $e$ is called a full edge.
Proposition 3.3. If \( t \geq 4 \), then \( r \leq 12g - 7 \), where \( g \) is the genus of \( K \).

Proof. Since \( G_S \) does not contain trivial loops, the unique vertex \( v \) has valency at most \( 12g - 6 \) in \( G_S \) (see [15, lemma 0.2]). Therefore the edges of \( G_S \) are partitioned into at most \( 6g - 3 \) families of parallel edges.

By Lemma 3.2, \( w(\overline{\tau}) \leq t \) for any edge \( \overline{\tau} \) of \( G_S \). Recall that the vertex \( v \) has valency \( rt \) in \( G_S \). Then \( rt \leq (12g - 6)t \), hence \( r \leq 12g - 6 \).

Finally, suppose that \( r = 12g - 6 \). Then any edge of \( G_S \) is full and each face of \( G_S \) is a 3-sided disc. By Lemma 3.1, we may assume that \( G_S \) contains an \( S \)-cycle with label pair \( \{t/2, t/2 + 1\} \) and a Scharlemann cycle of length three with label pair \( \{t, 1\} \). Then \( r \equiv 0 \) (mod 3) by Lemma 2.5, which is a contradiction. Therefore \( r \leq 12g - 7 \).

4. The case that \( t = 2 \)

By the parity rule, each edge of \( G_T \) connects different vertices \( u_1 \) and \( u_2 \). Then there are four edge classes in \( G_T \), i.e. isotopy classes of edges of \( G_T \) in \( \hat{T} \) rel \( u_1 \cup u_2 \). They are called \( 1, \alpha, \beta, \alpha\beta \) as illustrated in Fig. 2 (see [17, fig. 7-1]).

We label an edge of \( e \) of \( G_S \) by the class of the corresponding edge of \( G_T \) and we call the label the edge class label of \( e \).

For a face \( f \) of \( G_S \), if a small collar neighbourhood of \( \partial f \) in \( f \) is contained in \( U \) (W), then \( f \) is said to be black (resp. white).

Lemma 4.1. Suppose that \( r \neq 2 \). Then any two black (white) bigons in \( G_S \) have the same pair of edge class labels.

Proof. By Lemma 2.3, the interior of a black (white) bigon is disjoint from \( \hat{T} \). Then the proof of [18, lemma 5-2] remains valid. Remark that a final contradiction comes from the fact that a Klein bottle will be found in a solid torus \( U \) or \( W \).

Lemma 4.2. Let \( e \) and \( e' \) be edges of \( G_S \). If \( e \) and \( e' \) are parallel in \( G_S \), then they have distinct edge class labels.

Proof. If \( e \) and \( e' \) are parallel in \( G_S \) and have the same edge class label, then they are also parallel in \( G_T \). Then \( E(K) \) contains a Möbius band by [13, lemma 2.1], which contradicts the fact that \( K \) is hyperbolic.

Lemma 4.3. If \( r \neq 2 \), then \( G_S \) cannot contain more than 3 mutually parallel edges.

Proof. Suppose that there are 4 mutually parallel edges. Then there are two bigons with the same colour among these 4 parallel edges. By Lemma 4.1, these two bigons have the same pair of edge class labels. This contradicts Lemma 4.2.
Lemma 4.4. Suppose that \( r \neq 2 \). If \( G_S \) contains a black bigon and a white bigon which have an edge in common, then the other faces of \( G_S \) are not bigons.

Proof. Let \( e_1, e_2, e_3 \) be adjacent parallel edges of \( G_S \). By Lemma 4.2, these three edges have distinct edge class labels. Let \( \lambda, \mu, \nu \) be the edge class labels of \( e_1, e_2, e_3 \) respectively. Let us denote the endpoints of \( e_i \) by \( \partial_j e_i \) for \( j = 1, 2 \). (See Fig. 3.)

Note that \( \partial_1 e_1 \) and \( \partial_1 e_3 \) appear consecutively around the vertex \( u_1 \) in the order, when travelling around \( \partial u_1 \) anticlockwise, say. Then \( \partial_2 e_3 \) and \( \partial_2 e_1 \) appear consecutively around \( u_2 \) in the order, when traveling around \( \partial u_2 \) clockwise. These come from the facts that \( r \) is integral and that \( u_1 \) and \( u_2 \) have distinct parities. Then there is no other edge of edge class \( \lambda \) \((\nu)\) than \( e_1 \) \((e_3)\) in \( G_T \). The conclusion follows from Lemma 4.1.

Proposition 4.5. If \( t = 2 \), then \( r \leq 12g - 7 \).

Proof. The unique vertex \( v \) has valency at most \( 12g - 6 \) in \( G_S \) and the edges of \( G_S \) are partitioned into at most \( 6g - 3 \) families of parallel edges. Recall that \( v \) has valency \( 2r \) in \( G_S \).

By Lemma 4.3, \( G_S \) cannot contain 4 mutually parallel edges. If \( G_S \) contains 3 mutually parallel edges, then we have \( r \leq (6g - 3) + 2 = 6g - 1 \) by Lemma 4.4.

If \( G_S \) does not contain 3 mutually parallel edges, then each edge of \( G_S \) has weight 1 or 2. Hence \( r \leq 2(6g - 3) = 12g - 6 \).

Suppose that \( r = 12g - 6 \). Then any edge of \( G_S \) is full and hence \( G_S \) has \( 6g - 3 \) black, say, bigons and each white face of \( G_S \) is a 3-sided disc. Therefore there are an \( S \)-cycle \( \sigma \) and a Scharlemann cycle \( \tau \) of length three in \( G_S \) with the same label pair \( \{1, 2\} \). By Lemma 4.1, all black bigons have the same pair of edge class label \( \{\lambda, \mu\} \), say. Then the edges of \( \tau \) have the same edge class labels \( \lambda, \mu \) by Lemma 2.3. This means that there is an essential annulus \( A \) in \( \hat{T} \) which contains the edges of \( \sigma \) and \( \tau \). By Lemma 2.6, we have \( r \equiv 0 \pmod{3} \), which is a contradiction. Therefore \( r \leq 12g - 7 \).

Proof of Theorem 1.1. This follows immediately from Propositions 3.3 and 4.5.

5. Genus one case: the case \( t \geq 4 \)

In the remainder of this paper, \( K \) is assumed to be a genus one, hyperbolic knot in \( S^3 \) in order to prove Theorem 1.2. First, we deal with the case \( t \geq 4 \) in this section.
Theorem 5.1. If $K$ has genus one, then $K(r)$ is neither $L(2,1)$ nor $L(4,1)$.

Proof. This follows from [6, 27].

Lemma 5.2. If $r$ is odd, then $G_S$ cannot have more than $t/2$ mutually parallel edges.

Proof. The vertex $v$ has valency $rt$ in $G_S$. Recall that the edges of $G_S$ are partitioned into at most three families of mutually parallel edges. Let $A$ be a family of mutually parallel edges in $G_S$ and suppose that $A$ consists of more than $t/2$ edges, $a_1, a_2, \ldots, a_p$ numbered consecutively. Note that $p \leq t$ by Lemma 3.2. We may assume that $a_i$ has the label $i$ at one endpoint for $1 \leq i \leq p$. Then $a_p$ has the label $t/2 + 1$ at the other endpoint, since $r$ is odd. (See Fig. 4.)

By the parity rule, $p \neq t/2 + 1$. Thus $p > t/2 + 1$. Then $\{a_{(t/2)p/2}, a_{(t/2)p/2+1}\}$ forms an $S$-cycle. Furthermore, some edge between $a_2$ and $a_{t/2}$ has the label $1$ at the other endpoint. Therefore, there is another $S$-cycle whose label pair is disjoint from that of the above $S$-cycle. Then the same construction as in the proof of Lemma 3.1 gives a contradiction.

By Proposition 3.3, we have that $r \leq 5$. In fact, the cases that $r = 3$ and 5 remain by Theorem 5.1.

Lemma 5.3. The case that $r = 3$ is impossible.

Proof. The vertex $v$ has valency $3t$ in $G_S$. By Lemma 5.2, $G_S$ consists of three families of mutually parallel edges, each containing exactly $t/2$ edges. Then there is no $S$-cycle in $G_S$, but there are two Scharlemann cycles $\tau_1$ and $\tau_2$ of length three in $G_S$. Let $g_i$ be the face of $G_S$ bounded by $\tau_i$ for $i = 1, 2$. We may assume that $g_1$ has the label pair $\{t, 1\}$ and $g_2$ has $\{t/2, t/2 + 1\}$.

By Lemma 2.4, there are disjoint essential annuli $A_i$ in $\hat{T}$ in which the edges of $\tau_i$ lie and Int $g_i \cap \hat{T} = \emptyset$ for $i = 1, 2$.

Claim 5.4. The faces $g_1$ and $g_2$ lie on opposite sides of $\hat{T}$.
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Proof of Claim 5-4. Suppose that \( g_i \subset W \), say, for \( i = 1, 2 \). Let \( s \) be the slope on \( \partial W \) determined by the essential annuli \( A_i \). Performing \( s \)-Dehn filling on \( W \), that is, attaching a solid torus \( J \) to \( W \) along their boundaries so that \( s \) bounds a meridian disc of \( J \), we obtain a closed 3-manifold \( M \), which is either \( S^3 \), \( S^2 \times S^1 \) or a lens space.

However, there are two disjoint discs \( D_1 \) and \( D_2 \), which contain the edges of \( \tau_1 \) and \( \tau_2 \), respectively, on the 2-sphere \( Q \) obtained by compressing \( \hat{T} \) along \( s \) by a meridian disc of \( J \). Then \( N(D_1 \cup H_{1,1} \cup g_1) \) and \( N(D_2 \cup H_{t/2,1} \cup g_2) \) give two punctured lens spaces in \( M \), which is impossible.

Therefore, \( t/2 \) and \( t \) must have opposite parities, and so \( t/2 \) is odd. In particular, \( t \geq 6 \).

In \( G_S \), there are exactly three edges whose endpoints have the pair of labels \( \{j, t+1-j\} \) for \( j = 1, 2, \ldots, t/2 \). Therefore, \( G_T \) consists of \( t/2 \) components, each consisting two vertices \( u_j \) and \( u_{t+1-j} \) along with three edges connecting them.

Claim 5-5. Each component of \( G_T \) does not lie in a disc in \( \hat{T} \).

Proof of Claim 5-5. If there is a component of \( G_T \) which lie in a disc in \( \hat{T} \), then we can take an innermost one \( \Lambda \). That is, \( \Lambda \) lies in a disc \( D \) in \( \hat{T} \) and there is no other component of \( G_T \) in \( D \). Consider the intersection between \( D \) and \( S \). Then \( S \) is divided into two discs \( g_3 \) and \( g_4 \) by the edges of \( \Lambda \). By the cut-and-paste operation of \( g_3 \) or \( g_4 \), and taking \( D \) by a smaller one, if necessary, we can assume that Int \( g_3 \) and Int \( g_4 \) do not meet \( D \). Then \( N(D \cup V \cup g_3 \cup g_4) \), where \( V \) is the attached solid torus, gives a connected sum of two lens spaces minus an open 3-ball in \( K(r) \), which is impossible.

Thus, we may assume that \( A_i \) contains only the edges and vertices of \( \tau_i \) for \( i = 1, 2 \).

Assume that \( g_1 \subset W \) and \( g_2 \subset U \). Let \( M_1 = N(A_1 \cup H_{1,1} \cup g_1) \) and \( M_2 = N(A_2 \cup H_{t/2,1} \cup g_2) \). Let \( A'_i = \text{cl} (\partial M_i - A_i) \) for \( i = 1, 2 \). Then \( A'_i \) is a properly embedded annulus in \( W \) and \( A'_2 \) is a properly embedded annulus in \( U \). By Lemma 2-1, \( M_i \) is a solid torus such that the core of \( A_i \) runs three times in the longitudinal direction of \( M_i \) for \( i = 1, 2 \). Therefore, \( A'_i \) is parallel to the annulus \( \text{cl} (\hat{T} - A_i) \) in \( W \) and \( A'_2 \) is parallel to \( \text{cl} (\hat{T} - A_2) \) in \( U \). Let \( \hat{T}' = (\hat{T} - (A_1 \cup A_2)) \cup A'_1 \cup A'_2 \). Then it is easy to see that \( \hat{T}' \) is a new Heegaard torus in \( K(r) \) such that \( |\hat{T}' \cap V| = t - 4 (\geq 0) \). Furthermore, \( \hat{T}' \) satisfies (⋆), which contradicts the choice of \( \hat{T} \).

Lemma 5-6. The case that \( r = 5 \) is impossible.

Proof. Since the vertex \( v \) has valency \( 5t \) in \( G_S \), there are more than \( t/2 \) mutually parallel edges in \( G_S \), which contradicts Lemma 5-2.

Proof of Theorem 1-2 when \( t \geq 4 \). By Proposition 3-3, \( r \leq 5 \), and in fact, the remaining cases are \( r = 3, 5 \) by Theorem 5-1. But these cases are impossible by Lemmas 5-3 and 5-6.

6. Genus one case: the case \( t = 2 \)

In the case that \( t = 2 \), the following lemma plays a key role.

Recall that an unknotting tunnel \( \gamma \) for a knot or link \( K \) in \( S^3 \) is a simple arc properly embedded in the exterior \( E(K) \) such that \( \text{cl} (E(K) - N(\gamma)) \) is homeomorphic to a handlebody of genus two.
Lemma 6.1. Let $K$ be a genus one knot in $S^3$ and let $S$ be a minimal genus Seifert surface of $K$. If $K$ has an unknotting tunnel $\gamma$ such that $\gamma \subset S$, then $K$ is 2-bridge.

Proof. Take a regular neighbourhood $N$ of $\gamma$ in $S$. Let $F = \text{cl}(S - N)$. Then $F$ is an annulus whose boundary defines a link $L$ in $S^3$. Note that $F$ is incompressible in the exterior of $L$ and $L$ has an unknotting tunnel. (If $F$ is compressible, then $L$ is a trivial link. Since such a link has the unique unknotting tunnel [24], namely the obvious one, this means that $K$ is trivial.) Then $L$ is a 2-bridge torus link by [7, theorem 1]. Furthermore, an unknotting tunnel of such a link is determined by [1]. Then $S$ can be restored by taking the union of $F$ and $N$, showing that $K$ is 2-bridge.

Lemma 6.2. Suppose that $K$ has genus one. Then $G_S$ cannot have more than two mutually parallel edges.

Proof. If there are three mutually parallel edges in $G_S$, there are two $S$-cycles $\sigma_1$ and $\sigma_2$ whose faces $f_1$ and $f_2$ lie on opposite sides of $\tilde{T}$. Since $K(r) \neq L(2,1)$ by Theorem 5-1, we can assume that $\text{Int} \ f_i \cap \tilde{T} = \emptyset$ for $i = 1, 2$ by Lemma 2-3. Then we may also assume that $f_1, H_{1,2} \subset W$ and $f_2, H_{2,1} \subset U$. Note that $\text{cl}(W - H_{1,2})$ and $\text{cl}(U - H_{2,1})$ are handlebodies of genus two, since the core of $H_{1,2}$ ($H_{2,1}$) lies on a Möbius band which is obtained from $H_{1,2} \cup f_1$ ($H_{2,1} \cup f_2$) by shrinking $H_{1,2}$ (resp. $H_{2,1}$) to its core radially.

Let $\alpha$ and $\beta$ be the arc components of $f_i \cap H_{1,2}$. Let $\gamma$ be a simple arc in $f_i$ which connects a point in $\alpha$ with one in $\beta$. (See Fig. 5.)

Then it can be seen that $\text{cl}(W - H_{1,2} - N(\gamma))$ is homeomorphic to $T \times I$, where $I$ denotes an interval. Therefore, $\gamma$ gives an unknotting tunnel of $K$ which lies on $S$.

By Lemma 6-1, $K$ is 2-bridge, which contradicts the fact that a hyperbolic 2-bridge knot has no cyclic surgery [26].

The remaining cases are $r = 3, 5$ again by Proposition 4-5 and Theorem 5-1.

Lemma 6.3. The case $r = 3$ is impossible.

Proof. Recall that $\tilde{T}$ is separating in $K(r)$ and therefore the faces of $G_S$ are partitioned into black and and white ones. This implies that $G_S$ has no parallel edges, since $G_S$ has just three edges. Then there are two Scharlemann cycles $\tau_1$ and $\tau_2$ of length three in $G_S$. Let $g_i$ be the face of $G_S$ bounded by $\tau_i$ for $i = 1, 2$. Clearly, $g_1$ and $g_2$ lie on opposite sides of $\tilde{T}$. The edges of $\tau_i$ are all edges of $G_T$. In particular, $\tau_1$ and $\tau_2$ have their edges in common.
Claim 6-4. The edges of \( \tau_i \) cannot lie in a disc in \( \widehat{T} \) for \( i = 1, 2 \).

Proof of Claim 6-4. Suppose that the edges of \( \tau_1 \) (and therefore \( \tau_2 \)) lie in a disc \( D \) in \( \widehat{T} \). By the cut-and-paste operation of \( g_i \), we can assume that \( \text{Int} g_i \cap D = \emptyset \) for \( i = 1, 2 \). Then \( N(D \cup V \cup g_1 \cup g_2) \) gives a connected sum of two lens spaces minus an open 3-ball, in \( K(r) \), which is impossible.

Thus there is an essential annulus \( A \) in \( \widehat{T} \) which contains \( G_T \) by Lemma 2-1. In particular, \( G_T \) has exactly one pair of parallel edges.

Claim 6-5. \( \text{Int} g_i \cap \widehat{T} = \emptyset \) for \( i = 1, 2 \).

Proof of Claim 6-5. Suppose that \( \text{Int} g_i \cap \widehat{T} \neq \emptyset \). Let \( \xi \) be an innermost component of \( \text{Int} g_i \cap \widehat{T} \) in \( g_i \) and let \( \delta \) be the disc bounded by \( \xi \) on \( g_i \). By the assumption on the loops in \( S \hat{T} \) stated in Section 2, \( \xi \) is essential in \( T \) and then \( \xi \) is parallel to \( \partial A \). We may suppose that \( \delta \subset W \). Then \( \delta \) is a meridian disc of \( W \). Let \( H = V \cap W \) and let \( g_j \cap H \neq \emptyset \) for some \( j \in \{1, 2\} \).

Let \( Q \) be the 2-sphere obtained by compressing \( \partial W \) along \( \delta \) and let \( B \) be the 3-ball bounded by \( Q \) in \( W \). On \( Q \), there is a disc \( E \) which contains the edges of \( \tau_j \). After the components of \( \text{Int} g_j \cap Q \) are removed by the cut-and-paste operation of \( g_j \), \( N(E \cup H \cup g_j) \) gives a punctured lens space in \( B \), which is impossible. Therefore, \( \text{Int} g_i \cap \widehat{T} = \emptyset \). Similarly for \( g_2 \).

Now, we may assume that \( g_1 \subset W \) and \( g_2 \subset U \) and that \( H_{1,2} = V \cap W \) and \( H_{2,1} = V \cap U \). As in the last paragraph of the proof of Lemma 5-3, let \( M_1 = N(A \cup H_{1,2} \cup g_1) \) and \( M_2 = N(A \cup H_{2,1} \cup g_2) \). Then \( M_i \) is a solid torus and \( A \) runs three times on \( M_i \) in the longitudinal direction for \( i = 1, 2 \) by Lemma 2-1. Let \( A'_i = \text{cl}(\partial M_i - A) \), then \( A'_i \) is parallel to the annulus \( B = \text{cl}(\widehat{T} - A) \) in \( W \) if \( i = 1 \), or \( U \) if \( i = 2 \).

Claim 6-6. \( \text{cl}(U - H_{2,1}) \) is a handlebody of genus two.

Proof of Claim 6-6. The torus \( A'_i \cup B \) bounds a solid torus \( U' \) in \( U \), which represents the parallelism of \( A'_i \) and \( B \). Then it can be seen that \( \text{cl}(U - H_{2,1}) \) is obtained from \( U' \) by attaching a 1-handle \( N(g_2) \). Hence we have the desired result.

Claim 6-7. Let \( e \) be one of the parallel edges in \( G_T \). Then \( e \) is an unknotted tunnel of \( K \).

Proof of Claim 6-7. Note that \( \text{cl}(W - H_{1,2}) \) is homeomorphic to \( \text{cl}(M_1 - H_{1,2}) \). Let \( k = K^* \cap W \), where \( K^* \) is the core of \( V \). Then \( k \) is a properly embedded arc in \( M_1 \) and \( \text{cl}(M_1 - H_{1,2}) = \text{cl}(M_1 - N(k)) \). Push \( e \) into \( W \) slightly. It can be assumed that \( \partial e \subset \partial N(k) \). (See Fig. 6)

Then it is not hard to see that \( \text{cl}(M_1 - N(k) \cup N(e)) \) has a product structure \( T \times I \). Since \( \text{cl}(U - H_{2,1}) \) is a handlebody of genus two by Claim 6-6, \( \text{cl}(E(K) - N(e)) \) is a handlebody of genus two, which gives the desired conclusion.

By Lemma 6-1, \( K \) is 2-bridge, and this means that the case \( r = 3 \) is impossible [26].

Lemma 6-8. The case \( r = 5 \) is impossible.
Proof. $G_S$ has exactly five edges. By Lemma 6·2, these edges of $G_S$ are partitioned into three families, two pairs of parallel edges and one edge which is not parallel to the others. However, this configuration contradicts the fact that the faces of $G_S$ are divided into black and white sides.

Proof of Theorem 1·2 when $t = 2$. By Proposition 4·5 and Theorem 5·1, the remaining cases are $r = 3, 5$. These are impossible by Lemmas 6·3, 6·8. This completes the proof of Theorem 1·2.

Proof of Theorem 1·3. Let $K$ be a genus one knot in $S^3$ and suppose that $K(r)$ is a lens space. By Theorem 1·2, $K$ is not hyperbolic and therefore it is either a satellite knot or a torus knot. If a satellite knot admits cyclic surgery, then it is a cable knot of a torus knot [3, 29, 31]. In particular, its genus is greater than 1. Thus we have that $K$ is a torus knot and so $K$ is the trefoil. The constraint on the slopes follows from [21]. The converse is obvious.

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