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On Doubly Warped Product Manifolds

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Abstract : We study the doubly warped product manifold $M = B_h \times_f F$ of Riemannian manifolds related to critical Riemannian metrics and conharmonic transformation, and investigate the conditions of M to be a warped product space or a Riemannian product.

1. Introduction

To study the manifolds with negative curvature, R. L. Bishop and B. O'Neill [2] introduced a generalization of the product of two Riemannian manifolds in 1969, and called it the warped products of Riemannian manifolds. In fact, the warped products appears in the mathematical and physical literature long before that time; $R^n - \{0\}$ is naturally isometric to the warped product $R^+ \times_r S^{n-1}$, and a surface of revolution is a warped product with leaves the different positions of the rotated curve and fibres the circles of revolution. Moreover, the Minkowski space time and the Schwartzschild space time are all warped products. The study of relativity theory demands a wider class of manifold and the idea of doubly warped products was introduced and studied by many authors. But the mathematical property of the doubly warped products has not been studied enough yet. In this paper, we are going to study the doubly warped product manifold in connection with the conharmonic transformation and critical Riemannian metrics.

Throughout this paper, a manifold is assumed to be a connected paracompact manifold of class C^∞ or analytic. In addition, concerning Riemannian manifolds we often write M instead of (M, G).

2. Preliminaries

Let (B, g) and $(F, \bar{g})$ be the Riemannian manifolds with real dimension n and p respectively, and let h (resp. f) be a positive smooth function on B (resp. F). Then, the doubly warped product manifold $M = B_h \times {}_f F$ is defined by the Riemannian metric G

(2. 1) $$G = (f \circ \sigma)^2 \pi^*(g) + (h \circ \pi)^2 \sigma^*(\bar{g})$$

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on the product manifold $B\times F$, where $\pi : B\times F \longrightarrow B$ and $\sigma : B\times F \longrightarrow F$ are the natural projections [3]. If the warping function h or f is constant, then the doubly warped products reduce to a warped product [2,5,6,8,10].

For a local coordinate system (u^a) of B, the metric tensor g has the components (g_{ab}). Similarly, for a local coordinate system (u^x) of F, $\bar{g}$ has the components $(\bar{g}_{xy})$. Then, with respect to the local coordinate system (u^a, u^x) of M, the Riemannian metric G on M has the components

$$(2.\ 2) \qquad G=(G_{ij})=\begin{pmatrix} f^2 g_{ab} & 0 \\ 0 & h^2\bar{g}_{xy} \end{pmatrix}.$$

Throughout this paper, we used the following ranges of indices system:

$$i, j, k, \cdots : 1,2, \cdots , n+p=m$$

$$a, b, c, \cdots : 1,2, \cdots , n$$

$$x, y, z, \cdots : n+1, n+2, \cdots, n+p$$

unless otherwise stated.

Let ∇_b (resp. $\bar{\nabla}_x$) be the components of the covariant derivative with respect to g (resp. $\bar{g}$), and $\{{}^{a}_{bc}\}$ (resp. $\{\overline{{}^{x}_{yz}}\}$) be the Christoffel symbol of g (resp. $\bar{g}$). Then, the Christoffel symbols $\{\widetilde{{}^{h}_{ij}}\}$ of G on M are given as follows :

$$(2.\ 3) \qquad \{\widetilde{{}^{a}_{bc}}\} = \{{}^{a}_{bc}\},$$

$$(2.\ 4) \qquad \{\widetilde{{}^{a}_{bx}}\} = \tfrac{1}{f} f_x \delta^a_b,$$

$$(2.\ 5) \qquad \{\widetilde{{}^{a}_{xy}}\} = -\tfrac{h}{f^2} h^a \bar{g}_{xy},$$

$$(2.\ 6) \qquad \{\widetilde{{}^{x}_{ab}}\} = -\tfrac{f}{h^2} f^x g_{ab},$$

$$(2.\ 7) \qquad \{\widetilde{{}^{x}_{by}}\} = \tfrac{1}{h} h_b \delta^x_y,$$

$$(2.\ 8) \qquad \{\widetilde{{}^{x}_{yz}}\} = \{\overline{{}^{x}_{yz}}\},$$

where $h_d = \frac{\partial h}{\partial u^d}$, $f_x = \frac{\partial f}{\partial u^x}$, $h^a = h_b g^{ba}$, $f^y = f_x \bar{g}^{xy}$, $(g^{ba}) = (g_{ba})^{-1}$ and $(\bar{g}^{yx}) = (\bar{g}_{yx})^{-1}$.

Let $\tilde{R}$, R and $\bar{R}$ be the curvature tensors of M, B and F, respectively. Then we get

$$(2.\ 9) \qquad \tilde{R}_{dcb}{}^a = R_{dcb}{}^a - \tfrac{\|f_u\|^2}{h^2}(\delta^a_d g_{cb} - \delta^a_c g_{db}),$$

$$\tilde{R}_{wzy}{}^{x} = \bar{R}_{wzy}{}^{x} - \frac{\|h_e\|^2}{f^2}(\delta^x_w \bar{g}_{zy} - \delta^x_z \bar{g}_{wy}), \tag{2.10}$$

$$\tilde{R}_{cxd}{}^{a} = \frac{1}{fh} f_x (h_d \delta^a_c - h^a g_{cd}), \tag{2.11}$$

$$\tilde{R}_{ydz}{}^{x} = \frac{1}{fh} h_d (f_z \delta^x_y - f^x \bar{g}_{yz}) , \tag{2.12}$$

$$\tilde{R}_{xdy}{}^{a} = \frac{1}{f}(\bar{\nabla}_x f_y)\delta^a_d + \frac{h}{f^2}(\nabla_d h^a)\bar{g}_{xy}, \tag{2.13}$$

$$\tilde{R}_{cyd}{}^{x} = \frac{1}{h}(\nabla_c h_d)\delta^x_y + \frac{f}{h^2}(\bar{\nabla}_y f^x) g_{cd}, \tag{2.14}$$

$$\tilde{R}_{zyx}{}^{a} = \frac{hh^a}{f^3}(f_z \bar{g}_{yx} - f_y \bar{g}_{zx}), \tag{2.15}$$

$$\tilde{R}_{dcb}{}^{x} = \frac{ff^x}{h^3}(h_d g_{cb} - h_c g_{db}) \tag{2.16}$$

and the rests are zero. From the relations (2. 2)-(2. 16), the Ricci curvatures $\tilde{S}$, S and $\bar{S}$ of M, B and F are given by

$$\tilde{S}_{ba} = S_{ba} - \frac{(n-1)}{h^2}\|f_u\|^2 g_{ba} - \frac{p}{h}(\nabla_b h_a) - \frac{1}{h^2} f(\bar{\Delta} f) g_{ba}, \tag{2.17}$$

$$\tilde{S}_b{}^c = \frac{1}{f^2} S_b{}^c - \frac{(n-1)}{f^2h^2}\|f_u\|^2 \delta^c_b - \frac{p}{f^2h}(\nabla_b h^c) - \frac{1}{fh^2}(\bar{\Delta} f)\delta^c_b, \tag{2.18}$$

$$\tilde{S}_{by} = \frac{(m-2)}{fh} f_y h_b, \tag{2.19}$$

$$\tilde{S}_{yx} = \bar{S}_{yx} - \frac{(p-1)}{f^2}\|h_d\|^2 \bar{g}_{yx} - \frac{n}{f}(\bar{\nabla}_y f_x) - \frac{1}{f^2} h(\Delta h)\bar{g}_{yx}, \tag{2.20}$$

where $\triangle h$ (resp. $\bar{\triangle} f$) is the Laplacian of h (resp. f) for g (resp. $\bar{g}$).

Hence the scalar curvatures $\tilde{K}$, K and $\bar{K}$ and of M, B and F respectively satisfy

$$\begin{aligned}\tilde{K} = \frac{1}{f^2} K + \frac{1}{h^2}\bar{K} - \frac{n(n-1)}{f^2h^2}\|f_u\|^2 - \frac{p(p-1)}{f^2h^2}\|h_d\|^2 \\ -\frac{2p}{f^2h}(\Delta h) - \frac{2n}{fh^2}(\bar{\Delta} f).\end{aligned} \tag{2.21}$$

3 . Doubly warped product manifolds with critical Riemannian metrics

Let M be a compact orientable Riemannian manifold and let $\mathcal{A}(M)$ be the space of C^∞ Riemannian metrics G on M satisfying $\int_M dV_G = 1$, where dV_G is the volume element measured by G. For an element G in $\mathcal{A}(M)$, we assume that $f(k)$ is a scalar field on M determined by G as the contraction of a tensor product of the curvature tensor. Then $H_M(G) = \int_M f(k) dV_G$ defines a mapping $H_M : \mathcal{A}(M) \longrightarrow R$. In this case, a critical point of H is called a critical Riemannian metric with respect to $f(k)$ [1,6,9]. In this paper, we consider the two types of critical Riemannian metrics G_A and G_B of the functionals

$$A_M(G) = \int_M \tilde{K} dV_G \tag{3. 1}$$

and

$$B_M(G) = \int_M \tilde{K}^2 dV_G . \tag{3. 2}$$

It is well known that G is critical Riemannian metrics for the functional A defined by (3.1) if and only if G is an Einstein metric. From this fact, we have

Proposition 3.1. *Let M be a doubly warped product manifold and G be a critical Riemannian metric of a functional A. Then M is a warped product.*

Proof. Since M is an Einstein manifold, it is easy to get $(l-2)f_y\, h_b = 0$, from the equation (2. 19), which completes the proof.

Relating the critical Riemannian metric of the functional $B_M(G)$ defined by (3. 2), B. H. Kim[6] obtained the following lemma.

Lemma 3.2. *The Riemannian metric G on M is a critical Riemannian metric for $B_M(G)$ if and only if*

$$m\tilde{\nabla}_j\tilde{\nabla}_i\tilde{K} - m\tilde{K}\tilde{S}_{ji} - (\tilde{\Delta}\tilde{K})G_{ji} + \tilde{K}^2 G_{ji} = 0 , \tag{3. 3}$$

where $\tilde{\Delta}\tilde{K}$ is the Laplacian of $\tilde{K}$ with respect to G.

Corollary 3.3. *If the scalar curvature vanishes identically on (M, G), then G is a critical Riemannian metric of the functional B.*

The proof of the above result is easily obtained by use of Lemma 3.2. If the Riemannian metric G on M is a critical Riemannian metric of the functional B and the scalar curvature on M is non-zero constant, then we have $h_b f_y = 0$, using the equations (2. 2), (2. 19) and (3. 3). Now we have

Proposition 3.4. *If the Riemannian metric G on M is a critical Riemannian metric for the functional B and the scalar curvature is non-zero constant on M, then M becomes a warped product.*

4 . Conharmonically flat doubly warped product manifolds

A harmonic function ω is defined as a function whose Laplacian vanishes. It is easy to see that a harmonic function is not in general transformed into a harmonic function by the conformal transformation. Y. Ishi [4] studied the condition ρ in order that the function defined by

$$\bar{\omega} = e^{2\alpha\rho}\omega \tag{4. 1}$$

may become a harmonic function to the Riemannian metric

$$\bar{G} = e^{2\rho}G \tag{4. 2}$$

for a suitable constant α , where ρ is a positive function on M.

Let $\triangle_G\omega$ (respectively $\triangle_{\bar{G}}\omega$) denote the Laplacian of ω for G (resp. $\bar{G}$). Then from [4,7], we have

$$\begin{aligned}\triangle_{\bar{G}}\bar{\omega} = e^{2(\alpha-1)\rho}[&\triangle_G\omega + 2\alpha(\triangle_G\rho)\\ &+(4\alpha+m-2)\rho^i(\partial_i\omega) + 2\alpha(2\alpha+m-2)\|\rho_i\|^2\omega].\end{aligned} \tag{4. 3}$$

Immediately we have [4,7]

Proposition 4.1. *Let ω be a harmonic function on (M, G) and $\alpha = (2-m)/4$. Then the function $\bar{\omega}$ defined by (4. 1) is harmonic for $\bar{G}$ if and only if*

$$\triangle_G\rho + \tfrac{m-2}{2}\|\rho_k\|^2 = 0. \tag{4. 4}$$

In this point of view, Y. Ishi [4] introduced a conharmonic transformation which is the conformal transformation Ø : $(M,G) \longrightarrow (M,G)$ satisfying the equation (4. 4).

So far we have proved that the (1,3)-tensor

$$T_{kji}{}^h = \tilde{R}_{kji}{}^h + \tfrac{1}{m-2}[\tilde{S}_{ik}\delta_j^h + \tilde{S}_j{}^hG_{ik} - \tilde{S}_{ij}\delta_k^h - \tilde{S}_k{}^hG_{ij}], \tag{4. 5}$$

is invariant under the conformal transformation Ø if and only if Ø is a conharmonic transformation [4,7]. If the tensor T vanishes on M, then we call M a conharmonically flat manifold.

From now, we assume that the doubly warped product manifold M is conharmonically flat. Then the identities (2. 9) - (2.20) and (4. 5) lead to

$$(4.6)\qquad (m-2)R_{dcb}{}^{a} = S_{bc}\delta^a_d + S_d{}^a g_{bc} - S_{bd}\delta^a_c - S_c{}^a g_{bd} + \tfrac{(n-p)}{h^2}\|f_u\|^2(g_{bd}\delta^a_c - g_{bc}\delta^a_d) + \tfrac{2f}{h^2}(\bar{\Delta} f)(g_{bd}\delta^a_c - g_{bc}\delta^a_d) + \tfrac{p}{h}\{(\nabla_c h^a)g_{bd} - (\nabla_d h^a)g_{bc} + (\nabla_b h_d)\delta^a_c - (\nabla_b h_c)\delta^a_d\}$$

$$(4.7)\qquad \tfrac{(m-2)}{f}(\bar{\nabla}_x f_y)\delta^a_d + \tfrac{(m-2)}{f^2}h(\nabla_d h^a)\bar{g}_{xy} = -\bar{S}_{yx}\delta^a_d + \tfrac{(p-1)}{f^2}\|h_d\|^2\bar{g}_{yx}\delta^a_d + \tfrac{n}{f}(\bar{\nabla}_y f_x)\delta^a_d + \tfrac{h}{f^2}(\Delta h)\bar{g}_{yx}\delta^a_d - \tfrac{h^2}{f^2}S_d{}^a\bar{g}_{yx} + \tfrac{(n-1)}{f^2}\|f_u\|^2\bar{g}_{yx}\delta^a_d + \tfrac{p}{f^2}h(\nabla_d h^a)\bar{g}_{yx} + \tfrac{1}{f}(\bar{\Delta} f)\bar{g}_{yx}\delta^a_d.$$

If we transvect G^{ji} and contract with respect to k and h successively in the equation (4. 5) with T=0, then

$$(4.8)\qquad 0 = \tilde{K} = \tfrac{1}{f^2}K + \tfrac{1}{h^2}\bar{K} - \tfrac{n(n-1)}{f^2h^2}\|f_u\|^2 - \tfrac{p(p-1)}{f^2h^2}\|h_e\|^2 - \tfrac{2p}{f^2h}(\Delta h) - \tfrac{2n}{fh^2}(\bar{\Delta} f).$$

From this result and Corollary 3.3, we can state

Proposition 4.2. *If (M, G) is a conharmonically flat doubly warped product manifold, then G is a critical Riemannian metric for the functional B.*

Assume that the dimensions of B and F are the same, that is n=p, then (4. 6) lead to

$$(4.9)\qquad 2(n-1)R_{dcb}{}^{a} = S_{bc}\delta^a_d + S_d{}^a g_{bc} - S_{bd}\delta^a_c - S_c{}^a g_{bd} + \tfrac{2f}{h^2}(\bar{\Delta} f)(g_{bd}\delta^a_c - g_{bc}\delta^a_d) + \tfrac{n}{h}\{(\nabla_c h^a)g_{bd} - (\nabla_d h^a)g_{bc} + (\nabla_b h_d)\delta^a_c - (\nabla_b h_c)\delta^a_d\}.$$

Using the identity (4. 9), it is easy to see that

$$(4.10)\qquad S_{bc} = \{\tfrac{1}{n}K - \tfrac{2(n-1)}{nh^2}f(\bar{\Delta} f) - \tfrac{1}{h}(\Delta h)\}g_{bc} - \tfrac{(n-2)}{h}(\nabla_b h_c)$$

and

$$(4.11)\qquad 2(n-1)\{f(\bar{\Delta} f) + h(\Delta h)\} = 0.$$

Thus we have

Lemma 4.3. *Let M be a doubly warped product manifold. If M is conharmonically flat and n =p (>1), then the equation*

$$f(\bar{\Delta} f) + h(\Delta h) = 0 \tag{4.12}$$

hold.

Corollary 4.4. *Under the same assumptions of Lemma 4.3, the warping function f is harmonic if and only if h is harmonic.*

Let B and F be compact manifolds.
Then

$$\bar{\Delta} f^2 = 2(||f_x||^2 + f\bar{\Delta} f)$$

and that

$$\int_B f\bar{\Delta} f dv_B = \frac{1}{2}\int_B \bar{\Delta} f^2 dv_B - \int_B ||f_x||^2 dv_B.$$

Applying Green's Theorem to the equation (4. 12), we have

$$\begin{aligned} 0 &= \int_{B\times F}(f\bar{\Delta} f + h\Delta h)dv_G \\ &= \int_B h(\Delta h)dv_B + \int_F f(\bar{\Delta} f)dv_F \\ &= \frac{1}{2}\int_B \Delta h^2 dv_B - \int_B ||h_a||^2 dv_B + \frac{1}{2}\int_F \bar{\Delta} f^2 dv_F - \int_F ||f_x||^2 dv_F, \end{aligned}$$

which is equivalent to

$$\int_B ||h_a||^2 dv_B + \int_F ||f_x||^2 dv_F = 0.$$

Thus we have

Theorem 4.5. *If M is a conharmonically flat doubly warped product manifold and n=p (>1), then M is a Riemannian product.*

Let f be harmonic and $\nabla_b h_c = 0$. By use of the equation (4. 10) and Corollary 4.4, B becomes an Einstein space. Moreover, B is also a space of constant curvature by the aid of (4. 9).

Conversely, if f is harmonic and B is a space of constant curvature, that is

$$R_{abcd} = \frac{K}{n(n-1)}(g_{bc}g_{ad} - g_{ac}g_{bd}), \tag{4.13}$$

then it reduces

$$(4.14)\qquad (\nabla_c h^a)g_{bd} + (\nabla_b h_d)\delta_c^a - (\nabla_d h^a)g_{bc} - (\nabla_b h_c)\delta_d^a = 0$$

by the substitution (4. 13) to the equation (4. 9).

Contracting (4. 14) with respect to a and d, we obtain $\nabla_b h_c=0$. Hence we have

Theorem 4.6. *Let M be a conharmonically flat doubly warped product B and F with n = p (>1), and f (resp. h) be a harmonic function. Then B (resp. F) is a space of constant curvature if and only if* $\nabla_b h_c=0$ *(resp.* $\bar{\nabla}_x f_y=0$*).*

From now, we consider the case of $n \neq p$. If we contract (4. 7) with respect to a and d, then we get

$$(4.15)\qquad \frac{n(m-2)}{f}(\bar{\nabla}_x f_y) + \frac{(m-2)}{f^2}h(\Delta h)\bar{g}_{xy}$$
$$= -n\bar{S}_{yx} + \frac{n(p-1)}{f^2}\|h_e\|^2\bar{g}_{yx} + \frac{n^2}{f}(\bar{\nabla}_y f_x) + \frac{n}{f^2}h(\Delta h)\bar{g}_{yx}$$
$$-\frac{h^2}{f^2}K\bar{g}_{yx} + \frac{n(n-1)}{f^2}\|f_u\|^2\bar{g}_{yx} + \frac{p}{f^2}h(\Delta h)\bar{g}_{yx} + \frac{n}{f}(\bar{\Delta} f)\bar{g}_{yx}\,.$$

Transvecting $\bar{g}^{yx}$ to (4. 15), it is reduced to

$$(4.16)\qquad 0 = \frac{p}{f^2}K + \frac{n}{h^2}\bar{K} - \frac{np(n-1)}{f^2h^2}\|f_u\|^2 - \frac{np(p-1)}{f^2h^2}\|h_e\|^2$$
$$-\frac{2p}{f^2h}(\Delta h) - \frac{2n}{fh^2}(\bar{\Delta} f).$$

If we consider (4. 8) and (4. 16), we obtain

$$(4.17)\qquad (p-n)K - \frac{n(n-1)(p-n)}{h^2}\|f_u\|^2 + \frac{2p(n-1)}{n}\Delta h + \frac{2n(n-1)}{h^2}f(\bar{\Delta} f) = 0.$$

Contracting (4. 6) with respect to a and d ,

$$(4.18)\qquad 0 = Kg_{bc} - pS_{bc} + \frac{(n-p)}{h^2}\|f_u\|^2(g_{bc} - ng_{bc}) + \frac{2f}{h^2}(\bar{\Delta} f)(g_{bc} - ng_{bc})$$
$$+\frac{p}{h}\{(\nabla_c h_b) - (\Delta h)g_{bc} + (\nabla_b h_c) - n(\nabla_b h_c)\}$$

and it follows from (4. 18), we have

$$(4.19)\qquad (n-p)K = \frac{n(n-1)(n-p)}{h^2}\|f_u\|^2 + \frac{2n(n-1)}{h^2}f(\bar{\Delta} f) + \frac{2p(n-1)}{h}(\Delta h)\,.$$

Moreover the identities (4. 18) and (4. 19) can be rewritten as

$$S_{bc} = \{\tfrac{1}{p}K - \tfrac{(n-1)(n-p)}{h^2 p}\|f_u\|^2 - \tfrac{2(n-1)}{h^2 p}f(\bar{\triangle} f) - \tfrac{1}{h}(\triangle h)\}g_{bc} + \tfrac{(2-n)}{h}\nabla_b h_c \tag{4. 20}$$

and

$$K = \tfrac{n(n-1)}{h^2}\|f_u\|^2 + \tfrac{2n(n-1)}{h^2(n-p)}f(\bar{\triangle} f) + \tfrac{2p(n-1)}{h(n-p)}(\triangle h) \tag{4. 21}$$

respectively. Hence , if we substitute (4. 21) into (4. 20) , then we have

$$S_{bc} = \{\tfrac{(n-1)}{h^2}\|f_u\|^2 + \tfrac{2(n-1)}{h^2(n-p)}f(\bar{\triangle} f) + \tfrac{m-2}{h(n-p)}(\triangle h)\}g_{bc} + \tfrac{(2-n)}{h}\nabla_b h_c. \tag{4. 22}$$

By use of (4. 22), the identity (4. 6) is transformed into

$$\begin{aligned} R_{dcb}{}^a = {} & (\tfrac{1}{h^2}\|f_u\|^2 + \tfrac{2}{h(n-p)}(\triangle h) + \tfrac{2}{h^2(n-p)}f(\bar{\triangle} f))g_{bc}\delta^a_d \\ & -(\tfrac{1}{h^2}\|f_u\|^2 + \tfrac{2}{h(n-p)}(\triangle h) + \tfrac{2}{h^2(n-p)}f(\bar{\triangle} f))g_{bd}\delta^a_c \\ & -\tfrac{1}{h}(\nabla_b h_c)\delta^a_d - \tfrac{1}{h}(\nabla_d h^a)g_{bc} + \tfrac{1}{h}(\nabla_b h_d)\delta^a_c + \tfrac{1}{h}(\nabla_c h^a)g_{bd}\,. \end{aligned} \tag{4. 23}$$

Assume that the doubly warped product manifold *M* is conharmonically flat. Then, by use of the identities (4. 6), (4. 17) and (4. 23), we have

$$\begin{aligned} & R_{dcb}{}^a + \tfrac{1}{(n-2)}[S_{bd}\delta_c{}^a + S_c{}^a g_{bd} - S_{bc}\delta_d{}^a - S_d{}^a g_{bc}] \\ & = \tfrac{1}{(n-1)(n-2)}K(g_{bd}\delta_c{}^a - g_{bc}\delta_d{}^a), \end{aligned} \tag{4. 24}$$

that is, the base space *B* becomes conformally flat. Similarly, we also see that *F* is a conformally flat space. Thus we have the following

Theorem 4.7. *If M is a conharmonically flat doubly warped product of B and F with n ≠ p, then B and F are conformally flat.*

Since the scalar curvature on the conharmonically flat space vanishes identically, we easily see that due to (4. 24) and Theorem 4.7

Corollary 4.8. *If M is a conharmonically flat doubly warped product of B and F with n ≠ p, then B (resp. F) is conharmonically flat if and only if K=0 (resp.* $\bar{K}=0$*).*

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