ON CONFORMAL TRANSFORMATIONS
AND FIBRED RIEMANNIAN SPACES

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Abstract

We prove that a conformal transformation \( \phi : (M, g') \rightarrow (M, g) \) with \( \text{Ric}_g = \text{Ric}_g \) preserves Riemannian curvature tensors. Moreover, in a fibred Riemannian space, if any horizontal mapping covering is a Ricci-invariant conformal transformation and the total space is Einstein, then each fibre is a totally geodesic submanifold of the total space.

1. Introduction

Let \( M \) be an \( m \)-dimensional Riemannian manifold with metric tensor \( g \). A diffeomorphism \( \phi : M \rightarrow M \) is called a conformal transformation if there is a positive function \( \rho \) on \( M \) such that \( \phi^* g = \rho^2 g \). In this case, we express \( \phi \) as \( \phi : (M, g') \rightarrow (M, g) \), where \( g' = \rho^2 g \). If \( \rho \) is constant, then \( \phi \) is called a homothety [3,4,6]. A classical theorem of Liouville determines all possible conformal transformations between the Euclidean space. As a generalization, we call a conformal transformation \( \phi : (M, g') \rightarrow (M, g) \) a Liouville transformation if \( \text{Ric}_g = \text{Ric}_g \) [3], where \( \text{Ric}_g \) is the Ricci curvature with respect to \( g \). In [2,5], they proved that a globally defined Liouville transformation of a complete Riemannian manifold is a homothety.

In this paper, we study the local properties of Liouville transformations by use of adapted coordinates and fibred Riemannian spaces. We prove that

**Theorem 1.** Let \( \phi : (M, g') \rightarrow (M, g) \) \( (g' = \rho^2 g) \) be a conformal transformation with \( \text{Ric}_g = \text{Ric}_g \). Then \( \phi \) preserves Riemannian curvature tensors. That is \( R_{g'} = R_g \), where \( R_g \) is the Riemannian curvature tensor with respect to \( g \).

On the other hand, in a fibred Riemannian space \( (M, B, G, \pi) \), the horizontal mapping covering is a transformation between fibres. If any local horizontal mapping covering in a fibred Riemannian space is an isometry (resp. conformal mapping), then we call it a fibred Riemannian space with isometric fibre (resp. conformal fibre). It is well known that [2] a necessary and sufficient condition for a fibred Riemannian space have isometric fibre (resp. conformal fibre) is

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\[
(\mathcal{L}_X G^\nu)^\gamma = 0 \quad \text{(resp. } \rho G^\nu)\]

for any vector field \(X\) in \(B\), \(\rho\) a function on \(M\) and \(X^\nu\) is a lift of \(X\). In a fibred Riemannian space with isometric fibre (resp. conformal fibre), each fibre is a totally geodesic (resp. totally umbilical) submanifold of the total space and vice versa [2]. From these facts, we have

**Theorem 2.** In a fibred Riemannian space, if any local horizontal mapping covering is a Ricci-invariant conformal transformation and the total space is a non-Euclidean Einstein space, then each fibre is a totally geodesic submanifold of the total space.

Throughout this paper, the ranges of indices are as follows:

\[
i, j, k, \ldots : 1, 2, \ldots, n + p = m,\]
\[
a, b, c, \ldots : 1, 2, \ldots, n,\]
\[
\alpha, \beta, \gamma, \ldots : n + 1, \ldots, m,\]

unless otherwise stated.

## 2. Fibred Riemannian space

Let \((M, B, G, \pi)\) be a fibred Riemannian space, where \((M, G)\) is the total space with a projectable Riemannian metric \(G\), \(B\) the base space and \(\pi\) the projection \(M \to B\) of maximum rank everywhere. We suppose that the dimensions of \(M\) and \(B\) are \(m\) and \(n\) respectively. The fibre at any point \(Q\) in \(M\) is of dimension \(p = m - n\) and denoted by \(F(Q)\) or simply by \(F\). We suppose that each fibre is connected. The horizontal and vertical parts of a tensor field \(T\) are denoted by \(T^h\) and \(T^v\) respectively. Let \(h = (h_{\beta}^\alpha)\) and \(L = \{L_{a}^\beta\}\) be the components of the second fundamental tensor and the normal connection of each fibre respectively. For a basis \(\{E_a, C_b\}\) of the tangent space of \(M\) and \(\{E^a, C^b\}\) dual to \(\{E_a, C_b\}\), it is well known that [2,5]

\[
(2.1) \quad \nabla_i E^a_{\beta} = \Gamma_{i\beta}^\gamma E^\gamma_{\alpha} C_{\alpha}^\gamma + L_{a}^\beta C^\gamma_{\alpha} E^\gamma_{\alpha} - h_{i\beta}^\gamma C^\gamma_{\alpha} E^\alpha_{\alpha},
\]
\[
(2.2) \quad \mathcal{L}_X E^\nu = 0, \quad \mathcal{L}_X C^\alpha = 2 L_{a}^\beta E^\beta - P_{\nu}^\alpha C^\beta,
\]

where \(\Gamma\) is the Christoffel symbol of the Riemannian metric of \(B\), \(\mathcal{L}_X\) the Lie derivation with respect to \(E_x\) and \(P_{\nu}^\alpha\) is defined by

\[
[E_a, C_b] = P_{a}^\beta C^\beta_b
\]

for the bracket operator in \([\cdot, \cdot]\).

## 3. Proof of Theorem 1

Let \(\phi : (M, g^\prime) \to (M, g)\) \((g^\prime = \rho^{-2} g)\) be a conformal transformation. The geometric objects \(\{R, S, K, \Gamma\}\)
are the Riemannian curvature, Ricci curvature, scalar curvature, and the Christoffel symbol of \((M, g)\) respectively. \(\{R^i, S^i, K^i, \Gamma^i\}\) are the corresponding objects of \((M, g')\). Then we have [1]

\[
R^i_{\rho} = g^i_{\rho} + \rho^{-1}(\delta^i_{\nu} \nabla_{\nu} \rho - \delta^i_{\nu} \rho \nabla_{\nu} \rho + g^i_{\nu} \nabla_{\nu} \rho - g^i_{\nu} \nabla_{\nu} \rho) - \rho^{-2}\rho \rho' (\delta^i_{\nu} g^\nu - \delta^i_{\nu} g^\nu).
\]

\[
S^i_{\rho} = S^i_{\rho} + (m-2)\rho^{-1} \nabla_{\rho} \rho' + \rho^{-1} g^i_{\nu} \nabla_{\nu} \rho' - (m-1)\rho^{-2}\rho \rho' g^i_{\nu}.
\]

\[
K^i = K^i + 2m^{-1} \rho \nabla_{\rho} \rho' - \rho \rho'.
\]

If we assume that \(S^i_{\rho} = S^i_{\rho}\), then we get

\[
\nabla_{\rho} \rho' = \frac{1}{2\rho} \rho \rho' g^i_{\nu}.
\]

For an arc length \(u\) of a \(\rho\)-curve, if we take an adapted coordinate system \((u, u')\), then the metric \(ds^2\) of \(M\) is given by

\[
ds^2 = du^2 + [\rho'(u)]^2 d\bar{s}^2
\]

where \(d\bar{s}^2 = f_{\rho} du' d\bar{u}'\) is the metric form of the \(\rho\)-hypersurface \(M\) of \(M\). Along the \(\rho\)-curve, from (3.4), we get

\[
2\rho \rho'' = (\rho')^2.
\]

The general solution of (3.6) is given by

\[
\rho = (Au + B)^2,
\]

where \(A\) and \(B\) are constants. If \(M\) is complete, then \(\rho = B^2\), that is \(\phi\) is a homothety. Hence it has a meaning only in the case of local version. For an adapted coordinate, the metric of \(M\) is given by

\[
ds^2 = du^2 + 4A^2(Au + B)^2 d\bar{s}^2.
\]

Then the Christoffel symbols of \(M\) are given by

\[
\Gamma^i_{\rho \rho} = \Gamma^i_{\rho \rho} = \Gamma^i_{\rho \rho} = 0,
\]

\[
\Gamma^i_{\rho \rho} = -4A^2(Au + B)f_{\rho},
\]

\[
\Gamma^i_{\rho \rho} = \frac{A}{Au + B} \delta^i_{\nu},
\]

\[
\Gamma^i_{\rho \rho} = \Gamma^i_{\rho \rho}.
\]

Hence, the non-zero component of the curvature tensor \(R\) of \(M\) is

\[
R^i_{\rho} = \bar{R}_{\rho}^i - 4A^2(\delta^i_{\nu} f_{\rho} - \delta^i_{\nu} f_{\nu}).
\]

The Riemannian metric of \((M, g')\) is given by

\[
ds'^2 = \rho^{-2} ds'^2 = \frac{1}{(Au + B)^2} du^2 + \frac{4A^2}{(Au + B)^2} d\bar{s}^2.
\]

that is

\[
g^n_{\mu} = \frac{1}{(Au + B)^2}, \quad g^n_{\mu} = \frac{4A^2}{(Au + B)^2} f_{\rho}.
\]
So, the Christoffel symbols of $ds^2$ are given by

$$\Gamma^r_{ij} = \frac{2A}{Au + B}, \quad \Gamma_r^i = \frac{A}{Au + B} \delta_i^r,$$

(3.11) \hspace{1cm} \Gamma^r_{ij} = \Gamma^r_{ij} + \Gamma^r_{ij}, \quad \Gamma^r_{ij} = \Gamma^r_{ij}, \quad \Gamma^r_{ij} = \Gamma^r_{ij}.$$

Therefore, we can calculate the non-zero component of the curvature tensor of $(M, g^*)$ as

(3.12) \hspace{1cm} R^{ab}_{\mu \nu} = \hat{\mathcal{R}}^{ab}_{\mu \nu} + \Gamma^r_{ab} \Gamma^r_{\mu \nu} - \Gamma^r_{ab} \Gamma^r_{\mu \nu} + \Gamma^r_{ab} \Gamma^r_{\mu \nu} - \Gamma^r_{ab} \Gamma^r_{\mu \nu}.

Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

From (2.1), we have

(4.1) \hspace{1cm} \nabla_s X' = \nabla_s (X' s) - L_s E' + L_s E' - L_s E' + L_s E' - L_s E' - L_s E' = \delta_i^r \delta^r_{fn} - \delta^r_{fn}.$$

Let $\tilde{S}$ be the Ricci tensor of $(M, G)$. By use of (2.2) and (4.1), we obtain

(4.2) \hspace{1cm} \left( L_{x'}, \tilde{S} \right) = \left( L_{x'}, \tilde{S} \right)

= \left( L_{x'}, \tilde{S} \right) = \left( L_{x'}, \tilde{S} \right) = \left( L_{x'}, \tilde{S} \right).

Since

(4.3) \hspace{1cm} L_{x'} \nabla_{x'} \tilde{S}_{x'} = X' \nabla_{x'} \tilde{S}_{x'} + \tilde{S}_{x'} \nabla_{x'} X' + \tilde{S}_{x'} \nabla_{x'} X',

we have

(4.4) \hspace{1cm} \left( L_{x'}, \tilde{S}_{x'} \right) = \left( X' \nabla_{x'} \tilde{S}_{x'} \right) = \left( X' \nabla_{x'} \tilde{S}_{x'} \right) = \left( X' \nabla_{x'} \tilde{S}_{x'} \right).

by substituting (4.1) into (4.3) and taking account of (4.2). In a fibred Riemannian space, if any local horizontal mapping covering is a Ricci-invariant conformal transformation, then $\left( L_{x'}, \tilde{S} \right)^{\mu \nu} = 0$. Therefore the condition that the metric $G$ on $M$ is Einstein and the equations (4.2) and (4.4) imply Theorem 2.
References


