On conformally flat twisted product manifolds

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Abstract: We study the conformally flat twisted product manifolds $M = B \times f F$ of Riemannian manifolds and investigate the conditions of $M$ to be a warped product space.

Key words: twisted product manifolds, conformally flat, Einstein

Introduction

The warped product manifolds were defined by Bishop and O'Neill [2] to study the manifolds with negative curvature. The properties of warped products have studied from many points of view and this notion is used to construct a lot of examples for studying geometry ([1],[2],[4],[6], etc.).

Generalizing the idea of warped products, B.Y. Chen [3] defined the twisted products to construct a family of totally umbilical submanifolds with various properties. Machida and Sato [5], Ponge and Reckziegel [7] studied twisted products and twistor spaces in the case of pseudo-Riemannian manifolds. However the mathematical properties of the twisted product manifold are not studied enough yet.

In this paper, we are to study the conformally flat twisted product spaces $M = B \times f F$ of Riemannian manifolds and investigate the conditions to be a warped product space, and other geometrical properties of each space.

1. Preliminaries

Let $(B, g)$ and $(F, \tilde{g})$ be $n$-dimensional and $p$-dimensional Riemannian manifolds and $f$ a positive smooth function on $B \times F$. Consider the product manifold $B \times F$ with projections

$$\pi : B \times F \rightarrow B \quad \text{and} \quad \sigma : B \times F \rightarrow F.$$ 

The twisted product manifold $M = B \times f F$ is by definition the manifold $B \times F$ with the Riemannian structure given by (see [3])

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(1.1) \[ \|X\|^2 = \|\pi_*X\|^2 + (f(b, p))^2 \|\sigma_*X\|^2 \]

for any vector \(X\) tangent to \(M\) at \((b, p)\). If \(f\) depends only on \(B\), this is the warped product of \(B\) and \(F\) in the sense of Bishop and O’Neill [2]. Let \(G\) be the Riemannian metric on \(M\) and \(\dim M = m = n + p\).

For a local coordinate system \((u^a)\) of \(B\), the metric tensor \(g\) has the components \((g_{ba})\). Similarly, for a local coordinate system \((u^a)\) of \(F\), \(g\) has the components \((\bar{g}_{yx})\). Then, with respect to the local coordinate system \((u^a, u^b)\) of \(M\), \(G\) has the components

\[
(G_{ji}) = \begin{pmatrix}
g_{ba} & 0 \\
0 & f^2 \bar{g}_{yx}
\end{pmatrix}.
\]

Throughout this paper, the ranges of indices are as follows:

\[
i, j, k, \ldots : 1, 2, \ldots, n + p = m \\
a, b, c, \ldots : 1, 2, \ldots, n \\
x, y, z, \ldots : n + 1, \ldots, n + p
\]

unless otherwise stated.

Let \(\nabla_b\) (resp. \(\nabla_x\)) be the components of the covariant derivative with respect to \(g\) (resp. \(\bar{g}\)) and \(\left\{a_{bc}\right\}\) (resp. \(\left\{x_{yz}\right\}\)) the Christoffel symbol of \(g\) (resp. \(\bar{g}\)). Then the Christoffel symbols \(\left\{a_{ji}\right\}\) of \(G\) on \(M\) are given as follows

\[
(1.3) \quad \left\{a_{bc}\right\} = \left\{a_{bc}\right\},
\]

\[
(1.4) \quad \left\{x_{yz}\right\} = \left\{x_{yz}\right\} + \frac{1}{f} \left(f_y \delta^x_z + f_z \delta^x_y - f^x \bar{g}_{yz}\right),
\]

\[
(1.5) \quad \left\{x_{ya}\right\} = \frac{1}{f} f_a \delta^x_y,
\]

\[
(1.6) \quad \left\{a_{xy}\right\} = -f f^a \bar{g}_{xy}
\]

and the others are zero, where \(f_a = \frac{\partial f}{\partial u^a}\) and \(f_y = \frac{\partial f}{\partial u^y}\).

Let \(\bar{R}, R\) and \(\bar{R}\) be the curvature tensors of \(M\), \(B\) and \(F\) respectively, then we have

\[
(1.7) \quad \bar{R}_{dbc}^a = R_{dbc}^a,
\]

\[
(1.8) \quad \bar{R}_{dx}^c = \frac{1}{f} \left(\partial d f_y\right) \delta^c_z - \frac{1}{f} \left(\partial d f^z\right) \bar{g}_{xy} = \frac{1}{f^2} f_a f_y \delta^c_z + \frac{1}{f^2} f_a f^c \bar{g}_{xy},
\]

\[
(1.9) \quad \bar{R}_{dxb}^c = \frac{1}{f} \left(\nabla_d f_b\right) \delta^c_z, \quad \bar{R}_{dx}^a = -f \left(\nabla_d f^a\right) \bar{g}_{xy},
\]

\[
(1.10) \quad \bar{R}_{xyz}^a = -f \left(\partial_x f^a\right) \bar{g}_{yz} + f \left(\partial_y f^a\right) \bar{g}_{xz} - f^a f_y \bar{g}_{xz} + f^a f_x \bar{g}_{yz},
\]
(1.11) 
\[ \tilde{R}^w_{xyz} = \tilde{R}^w_{xyz} + \frac{1}{f} \left[ (\nabla_x f_y) \delta^w_y + (\nabla_y f_x) \bar{\varphi}^w_{xz} - (\nabla_x f_y) \bar{\varphi}^w_{yz} - (\nabla_y f_x) \delta^w_z \right] \\
- \frac{2}{f^2} \left[ f_x f_z \delta^w_x + f_y f^w_x \bar{\varphi}^w_{xz} - f_x f^w_y \bar{\varphi}^w_{yz} - f_y f_z \delta^w_x \right] \\
+ \|f_x\|^2 \left[ \tilde{\varphi}_{xz} \delta^w_x - \tilde{\varphi}_{yz} \delta^w_y \right] - \frac{|f_x|^2}{f^2} \left[ \tilde{\varphi}_{xz} \delta^w_x - \tilde{\varphi}_{yz} \delta^w_y \right]. \]

and the others are zero.

The components of Ricci tensors are given by

(1.12) 
\[ \tilde{S}_{ab} = S_{ab} - \frac{p}{f} (\nabla_a f_b), \]

(1.13) 
\[ \tilde{S}_{ax} = -(p-1) \left( \frac{1}{f} \partial_a f_x - \frac{1}{f^2} f_a f_x \right), \]

(1.14) 
\[ \tilde{S}_{xx} = \bar{S}_{xx} - f(\Delta f) \bar{\varphi}_{xx} - \frac{1}{f} (\Delta f) \bar{\varphi}_{xx} - \frac{(p-2)}{f^3} \nabla_x f_x \\
+ \frac{2(p-2)}{f^2} f_x f_x - (p-1) \|f_x\|^2 \bar{\varphi}_{xx} - \frac{(p-3)}{f^2} \|f_x\|^2 \bar{\varphi}_{xx}, \]

where \( \Delta f = \nabla_x f^x, \bar{\Delta} f = \nabla_x f^x \) and \( \tilde{S}, S \) and \( \bar{S} \) are the Ricci tensors of \( M, B \) and \( F \) respectively.

Let \( \tilde{K}, K \) and \( \bar{K} \) be the scalar curvatures of \( M, B \) and \( F \) respectively, then we have

(1.15) 
\[ \tilde{K} = K + \frac{1}{f^2} K - \frac{2p}{f} (\Delta f) - \frac{2(p-1)}{f^3} (\bar{\Delta} f) - \frac{p(p-1)}{f^4} \|f_x\|^2 - \frac{(p-1)(p-4)}{f^4} \|f_x\|^2 f_x. \]

2. Conformally flat twisted product manifold

Now we put

\[ \psi_{dy} = \frac{1}{f} \partial_d f_y - \frac{f_d f_y}{f^2} \]

and

\[ \psi_{xy} = \frac{1}{f} \nabla_y f_x - \frac{2f_y f_x}{f^2} + \frac{1}{2} \left( \|f_x\|^2 + \|f_y\|^2 \right) \bar{\varphi}_{xy}, \]
then the identities (1.8), (1.11), (1.13), (1.14) and (1.15) are reduced to

\[(2.3)\quad \tilde{R}_{dz}^x = \psi_d^x \delta_z^y - \psi_d^x \tilde{g}_{zy},\]

\[(2.4)\quad \tilde{R}_{wz}^x = \tilde{R}_{wz}^x + \psi_w^x \delta_z^y - \psi_z^x \delta_w^y + \psi_z^x \tilde{g}_{wy} - \psi_w^x \tilde{g}_{zy},\]

\[(2.5)\quad \tilde{S}_{dy} = -(p-1)\psi_d^y,\]

\[(2.6)\quad \tilde{S}_{zy} = \tilde{S}_{zy} - f(\Delta f)\tilde{g}_{zy} - (p-2)\psi_{zy} - \psi_w^w \tilde{g}_{zy},\]

\[(2.7)\quad \tilde{K} = K + \frac{1}{f^2} \tilde{K} - 2p \frac{\Delta f}{f} - 2(p-1)\frac{\psi_w^w}{f^2},\]

where we have put

\[\psi_w^w = \frac{1}{f} \Delta f + \frac{1}{2} \|f_c\|^2 + \frac{p-4}{2f^2} \|f_c\|^2.\]

Assume that the twisted product manifold is conformally flat, that is, \(\tilde{R}_{ijkl}^h\) of \(M = B \times_f F\) satisfies (see[1])

\[(2.8)\quad \tilde{R}_{ijkl}^h = \frac{1}{m-2} \left( \tilde{S}_{ij}^h \delta_k^r - \tilde{S}_{ik}^h \delta_j^r + \tilde{S}_{ik}^h G_{ji} - \tilde{S}_{ij}^h G_{ki} \right) - \frac{1}{(m-1)(m-2)} \tilde{K} \left( \delta_k^h G_{ij} - G_{ki} \delta_j^h \right).\]

Using (1.9) and (2.8), we have

\[(2.9)\quad \frac{1}{f} \left( \nabla_d f_c \right) \delta_z^y = -\frac{1}{(m-2)} \left( S_{dc} - \frac{p}{f} \left( \nabla_d f_c \right) \right) \delta_z^y - \frac{1}{(m-2)} \tilde{S}_y^x g_{dc} + \frac{1}{(m-1)(m-2)} \tilde{K} g_{dc} \delta_z^y.\]

If we contract (2.9) with respect to \(x\) and \(y\), then we get

\[(2.10)\quad \frac{2p}{f} \left( \Delta f \right) = pK + \frac{n}{f^2} \left[ \tilde{K} - \frac{2(p-1)}{f} \left( \Delta f \right) - p(p-1)\|f_c\|^2 \right.\]

\[\left. - \frac{(p-1)(p-4)}{f^2} \|f_c\|^2 \right] - \frac{np}{(m-1)} \tilde{K},\]

and that

\[(2.11)\quad \frac{2p(p-1)(n-1)}{(m-1)f} \left( \Delta f \right) = \left( \frac{p(1-p)}{(m-1)} \right) K + \frac{n(1-n)}{(m-1)f^2} \tilde{K} + \frac{2n(n-1)(p-1)}{(m-1)f^3} \left( \Delta f \right)\]

\[+ \frac{pn(n-1)(p-1)}{(m-1)f^5} \|f_c\|^2 + \frac{n(n-1)(p-4)}{(m-1)f^4} \|f_c\|^2.\]
with the help of (1.15), or equivalently
\[
p(p-1)K = \frac{n(1-n)}{f^2} \bar{K} - \frac{2p(p-1)(n-1)}{f} (\Delta f) + \frac{2n(n-1)(p-1)}{f^3} (\bar{\Delta} f) \\
+ \frac{pn(n-1)(p-1)}{f^2} \| f_x \|^2 + \frac{n(n-1)(p-1)(p-4)}{f^4} \| f_x \|^2.
\]

If we use (1.10) and (2.8), then we get
\[
(2.12) \quad - f^2 \psi^a_z \bar{g}_{xa} + f^2 \psi^a_x \bar{g}_{za} = \frac{f^2}{m-2} (\bar{S}_x^a \bar{g}_{xa} - \bar{S}_y^a \bar{g}_{za}).
\]

Contracting (2.12) with respect to \( x \) and \( z \), we obtain
\[
(2.13) \quad \bar{S}_{ya} + (m-2) \psi_{ya} = 0
\]
if \( p \neq 1 \). Using (2.5) and (2.13),
\[
(2.14) \quad \psi_{ya} = 0
\]
if \( n \neq 1 \), and that \( \bar{S}_{ya} = 0 \). Since
\[
\psi_{xy} = \frac{1}{f} \partial_a f_y - \frac{f_d f_y}{f^2} = \partial_a \partial_y \log f,
\]
\( \psi_{ya} = 0 \) means that \( f \) is the product of certain functions \( f^* \) on \( B \) and \( \tilde{f} \) on \( F \). Therefore, if we denote the fibre with the metric \( g^*_y = \bar{f}^2 \bar{g}_{xy} \) by \( F^* \), then we have

**Theorem 2.1.** If the twisted product manifold \( M = B \times_f F \) of the Riemannian manifolds \( B \) and \( F \) is conformally flat and \( p \neq 1 \), \( n \neq 1 \), then \( M \) is the warped product space \( B \times_f F^* \) of \( B \) and \( F^* \).

If we assume that \( M = B \times_f F \) is conformally flat and \( n \neq 1 \), then the identities (1.7), (2.8) and (2.11) give rise to
\[
R_{dcba} = \frac{1}{(m-2)} \left( S_{cb} \delta^a_d - S_{db} \delta^a_c + S^a_d g_{cb} - S^a_d g_{db} \right)
\]
\[
(2.15) \quad - \frac{p}{(m-2)} \left( \delta^a_d \nabla f_b - \delta^a_b \nabla f_d + g_{cb} \nabla f^a - g_{db} \nabla f^a \right)
\]
\[
- \frac{1}{(m-1)(m-2)} \bar{K} \left( g_{cb} \delta^a_d - g_{db} \delta^a_c \right)
\]
and
(2.16) \[ \tilde{K} = \frac{f^2}{n(1-n)} \left\{ \frac{2p(p-1)(n-1)}{f} \Delta f + p(p-1)K - \frac{pn(n-1)(p-1)}{f^2} \right\} \]

If we substitute (2.16) into (1.15), then the scalar curvature \( \tilde{K} \) of \( M \) is reduced to

(2.17) \[ \tilde{K} = \frac{(m-1)(n-p)}{n(n-1)} K - \frac{2p(m-1)}{nf} (\Delta f). \]

So, contracting (2.15) with respect to \( b \) and \( c \), and using (2.17), it follows that

(2.18) \[ S_{cb} = \frac{1}{n} K g_{cb} + \frac{(n-2)}{nf} (\Delta f) g_{cb} - \frac{(n-2)}{f} (\nabla_c f_b). \]

Hence, if \( n = 2 \), then \( S_{cb} = \frac{1}{n} K g_{cb} \), that is, \( B \) becomes an Einstein space if \( K \) is constant. Since 2-dimensional Einstein space is a space of constant curvature, we have

**Theorem 2.2.** Let \( M = B \times_f F \) be the conformally flat twisted product manifold. If the scalar curvature \( K \) of \( B \) is constant and \( \dim B = 2 \), then \( B \) is the space of constant curvature.

Assume that \( B \) is a space of constant curvature and \( n \geq 3 \). Then \( B \) becomes an Einstein space and that (2.18) is reduced to

(2.19) \[ \nabla_c f_b = \frac{1}{n} (\Delta f) g_{cb}. \]

This means that \( f \) is the concircular function.

Conversely, if \( f \) is the concircular function, then the equations (2.18) and (2.19) imply that \( B \) is an Einstein space. Therefore if we substitute \( S_{cb} = \frac{1}{n} K g_{cb} \), (2.17) and (2.19) into (2.15), then we get

(2.20) \[ R_{deb} = \frac{1}{n(n-1)} K (g_{cb} \delta_d^b - g_{db} \delta_c^b). \]

because \( m = n + p \). Hence we have

**Theorem 2.3.** Let \( B \) be a Riemannian manifold with dimension \( n \geq 3 \) and the twisted product manifold \( M = B \times_f F \) is conformally flat. Then \( B \) is the space of constant curvature if and only if \( f \) is the concircular function.

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