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REMARKS ON CONFORMAL TRANSFORMATIONS

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Abstract

We study conformal transformations between spaces of constant curvature. We give a necessary and sufficient condition that such a conformal transformation preserves Ricci tensors. And we show that it becomes a homothety in this case.

1. Introduction

Let M and M^* be m-dimensional connected Riemannian manifolds with metric tensor fields g and g^* respectively, and we consider a conformal diffeomorphism ϕ of M^* into M. Then the metric tensor fields are related by

$$g^* = \frac{1}{\rho^2} g,$$

where ρ is a positive valued scalar field on M and said to be associated with ϕ. In case $M = M^*$, ϕ is called a conformal transformation. If ρ is constant, then ϕ is called a homothety [1, 2, 3]. A classical theorem of Liouville determines all possible conformal diffeomorphisms between Euclidean metrics. As a generalization, a conformal transformation ϕ is called a Liouville transformation if $\mathrm{Ric}(g) = \mathrm{Ric}(g^*)$ [3]. A concircular transformation is by definition a conformal transformation preserving Riemannian circles. It is well known that a conformal transformation between Einstein spaces is a concircular transformation [4, 5]. Hence it is natural to investigate conformal transformations between spaces of constant curvature. The purpose of this paper is to discuss conformal transformations between spaces of constant curvature and to prove the following theorem :

Theorem. (1) *A conformal transformation between spaces of constant curvature with* $\dim M > 2$ *is a Liouville transformation if and only if* $2\rho(\nabla_l \rho^l) = m\rho_l \rho^l$. *In this case, it reduces to a homothety.*

(2) *Let M be a space of constant curvature with* $\dim M > 2$ *and ϕ is a Liouville transformation. Then M^* is also a space of constant curvature and ϕ is a homothety.*

Key words: Conformal transformation, concircular transformation, space of constant curvature.
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2. Conformal transformation

Let $\phi:(M^*,g^*)\to(M,g)$ be a conformal transformation. The geometric objects $\{R, S, K\}$ are the Riemannian curvature, the Ricci curvature and the scalar curvature respectively. $\{R^*,S^*,K^*\}$ are the corresponding objects of (M^*,g^*). Then we have [1, 2]

$$R^{*\ h}_{kji} = R_{kji}{}^h + \rho^{-1}(\delta_k^h\nabla_j\rho_i - \delta_j^h\nabla_k\rho_i + g_{ji}\nabla_k\rho^h - g_{ki}\nabla_j\rho^h) - \rho^{-2}\rho_l\rho^l(\delta_k^h g_{ji} - \delta_j^h g_{ki}), \tag{2.1}$$

$$S^*_{ji} = S_{ji} + (m-2)\rho^{-1}\nabla_j\rho_i + \rho^{-1}g_{ji}(\nabla_l\rho^l) - (m-1)\rho^{-2}(\rho_l\rho^l)g_{ji}, \tag{2.2}$$

$$K^* = \rho^2K + 2(m-1)\rho(\nabla_l\rho^l) - m(m-1)\rho_l\rho^l \tag{2.3}$$

where $\rho_h = \partial_h\rho$ and the range of indices i, j, k, l, is 1, 2, 3, ..., m. Assume that M and M^* are spaces of constant curvature. Then, by use of (2.1) and (2.3), we get

$$\begin{aligned}(2.4)\qquad &\left(\frac{2\nabla_l\rho^l}{m\rho} - \frac{\rho_l\rho^l}{\rho^2}\right)(g_{ji}\delta_k^h - g_{ki}\delta_j^h)\\ &= \rho^{-1}(\delta_k^h\nabla_j\rho_i - \delta_j^h\nabla_k\rho_i + g_{ji}\nabla_k\rho^h - g_{ki}\nabla_j\rho^h) - \rho^{-2}\rho_l\rho^l(\delta_k^h g_{ji} - \delta_j^h g_{ki}).\end{aligned}$$

Contracting i and j in (2.4), we obtain

$$\begin{aligned}(2.5)\qquad &\left(\frac{2\nabla_l\rho^l}{m\rho} - \frac{\rho_l\rho^l}{\rho^2}\right)(m-1)\delta_k^h\\ &= \rho^{-1}(\delta_k^h\nabla_l\rho^l + (m-2)\nabla_k\rho^h) - \rho^{-2}(m-1)\rho_l\rho^l\delta_k^h.\end{aligned}$$

If $m > 2$, this implies

$$\nabla_j\rho_i = \alpha g_{ji}, \tag{2.6}$$

where $\alpha = \dfrac{\nabla_l\rho^l}{m}$.

3. Proof of Theorem

Assume that M and M^* are spaces of constant curvature and that ϕ is a conformal transformation. Then from (2.2), $S^* = S$ if and only if

$$(m-2)\rho\nabla_j\rho_i + \rho(\nabla_l\rho^l)g_{ji} - (m-1)(\rho_l\rho^l)g_{ji} = 0. \tag{3.1}$$

Hence, we can see that

$$2\rho(\nabla_l\rho^l) = m\rho_l\rho^l. \tag{3.2}$$

Conversely, conditions (3.2) and (2.6) imply (3.1). In addition, from (2.3) we have $K^* = \rho^2K$ in this case. Since K and K^* are both constants, so is ρ, and we have Theorem (1).

Next, assume that ϕ is a Liouville transformation, that is $S = S^*$. In [1], present authors proved that a conformal transformation with $S = S^*$ preserves Riemannian curvature tensor, that is $R = R^*$. Hence (2.1) induces

$$\rho(\delta_k^h \nabla_j \rho_i - \delta_j^h \nabla_k \rho_i + g_{ji} \nabla_k \rho^h - g_{ki} \nabla_j \rho^h) - \rho_l \rho^l (\delta_k^h g_{ji} - \delta_j^h g_{ki}) = 0. \tag{3.3}$$

If we contract (3.3) with respect to k and h, we obtain

$$\rho\{(m-2)\nabla_j \rho_i + g_{ji} \nabla_l \rho^l\} - (m-1)(\rho_l \rho^l) g_{ji} = 0, \tag{3.4}$$

and that

$$2(m-1)\rho(\nabla_l \rho^l) - m(m-1)\rho_l \rho^l = 0. \tag{3.5}$$

This implies

$$2\rho(\nabla_l \rho^l) = m\rho_l \rho^l. \tag{3.6}$$

Then we see that $K^* = \rho^2 K$ from (2.3) and (3.6). Hence if we assume that M is a space of constant curvature, then we can see that

$$\begin{aligned} R^{*}{}_{kji}{}^{h} &= R_{kji}{}^{h} \\ &= \frac{K^*}{m(m-1)}(g^*_{ji}\delta_k^h - g^*_{ki}\delta_j^h), \end{aligned}$$

that is M^* is also a space of constant curvature. Since K^* and K are both constants that are related by $K^* = \rho^2 K$, it follows that ρ is also a constant. This implies that ϕ is a homothety, and we complete the proof of Theorem (2).

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