Combinatorial conjectures on the range of Young diagrams appearing in plethysms

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Introduction

"Plethysm" $\{\lambda\}\otimes\{\mu\}$ is a sort of composition of Schur functions (or partitions of positive integers), which was first introduced by D.E.Littlewood [20]. From the first time, it is an important problem to decompose it into irreducible factors, and several efforts are achieved concerning this problem. (See the references at the end of this paper.) But, at present, general decomposition formulas (or rules) are not yet known except some special cases. In view of the table in our previous paper [5], however, it seems that there exist some hidden combinatorial properties on $\{\lambda\}\otimes\{\mu\}$, and the purpose of this paper is to give a series of conjectures expressing combinatorial formulas on the first and the last terms of $\{\lambda\}\otimes\{\mu\}$ with respect to some natural order on partitions, which gives a partial "answer" to the above problem.

Unfortunately, at present, we do not know the proof of these conjectures. However, these conjectures give the correct results up to total degree ≤ 16 (resp. ≤ 18) for the first (resp. last) term of $\{\lambda\}\otimes\{\mu\}$ in view of the table in [5]. In addition, concerning the last term, several types of conjectures give the same results up to total degree ≤ 27 (cf. § 5), and it seems that all conjectures stated in this paper actually hold. If these conjectures are correct, we can determine the range of partitions appearing in $\{\lambda\}\otimes\{\mu\}$ for large total degree without calculating explicit decompositions.

Concerning this problem, we already calculated plethysms $\{\lambda\}\otimes\{\mu\}$ in [5] up to total degree ≤ 16 (and total degree ≤ 18 for last terms only). And in view of this table, we stated some conjectures on the first and the last terms of $\{\lambda\}\otimes\{\mu\}$ in [5; §4]. The first term can be expressed in a quite simple closed form (cf. § 1, CONJECTURE 1.2). But, the conjectures on the last term stated in [5; § 4] was unnecessarily complicated, and we simplify them to a relatively compact form in § 2 of this paper. In addition, we add other two types of conjectures on the last term of $\{\lambda\}\otimes\{\mu\}$ in § 3 and § 4. Roughly speaking, the first type conjecture in § 2 indicates the recursive (or inductive) structure on the last term, and the second type conjecture in § 3 indicates that the last term of $\{\lambda\}\otimes\{\mu\}$ can be determined essentially by μ and the latter part of λ (cf. CONJECTURE 3.1). In § 4, we give the third type conjecture, giving the explicit formula on the last term (CONJECTURE 4.1), which may be considered as a principal conjecture in this paper. But in general, it is not a closed formula, and requires a case by case calculations depending on the combinatorial property of Young diagrams. As a special case, if this conjecture is correct, we obtain a relatively simple formula in the range of $\{m\}\otimes[n]$ and $\{1^m\}\otimes[n]$, where

$$[n] = \begin{cases} \{1^n\} & \text{if } m \text{ is even} \\ \{n\} & \text{if } m \text{ is odd} \end{cases}$$

(CONJECTURE 4.2). For example, from this formula, we have the decompositions

$$\{5\}\otimes\{10.\} = \{50.\} + \cdots\cdot + \{7^4 6 5^3 1\},$$

$$\{5\}\otimes\{100.\} = \{500.\} + \cdots\cdot + \{9^{44} 7^{11} 5^5 2\},$$

$$\{5\}\otimes\{1000.\} = \{5000.\} + \cdots\cdot + \{13.^{208} 12.^{122} 10.^{82} 75\},$$

etc. (For other examples, see § 5 and § 6.) To calculate these last terms, we need the following expansions of positive integers by binomial coefficients:

$$10 = \binom{6}{5} + \binom{4}{4} + \binom{3}{3} + \binom{2}{2} + \binom{1}{1},$$

$$100 = \binom{8}{5} + \binom{7}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1},$$

$$1000 = \binom{12}{5} + \binom{9}{4} + \binom{8}{3} + \binom{7}{2} + \binom{5}{1}.$$

(Note that for these plethysms, if the parity of m is reversed, then in contrast to the above, we can express the range in simple forms, i.e., we have the following conjectures:

$$\{m\}\otimes\{n\} = \{mn\} + \cdots\cdot + \{m^n\} \quad (m = \text{even}),$$

$$\{m\}\otimes\{1^n\} = \{mn-n+1, 1^{n-1}\} + \cdots\cdot + \{m^n\} \quad (m = \text{odd}),$$

$$\{1^m\}\otimes\{n\} = \{n^m\} + \cdots\cdot + \{1^{mn}\} \quad (m = \text{even}),$$

$$\{1^m\}\otimes\{1^n\} = \{n^{m-1}, 1^n\} + \cdots\cdot + \{1^{mn}\} \quad (m = \text{odd}).$$

cf. CONJECTURES 1.3, 2.1 and the explanation following it.) In § 4, in order to express the last term of $\{\lambda\}\otimes\{\mu\}$ for general λ, we define the expansion of integers by some polynomials which is naturally associated with the Young diagram λ. This generalizes the well-known expansion of integers by binomial coefficients which we indicated above.

In the latter part of this paper (§ 5 and § 6), we give a table of the first and the last term of $\{\lambda\}\otimes\{\mu\}$ up to total degree ≤ 27 as a reference data. Up to total degree ≤ 16, all results are obtained by actual decompositions of $\{\lambda\}\otimes\{\mu\}$, and for higher degree cases, most terms are calculated by using our conjectures in § 1 ~ § 4. (For details, see the explanations in § 5 and § 6. We check these results by making programs of computers.) We believe that they give the correct results. But, there may be a counter-example to our conjectures in higher degree case which we did not yet calculate, and the author is grateful to the readers if they teache a counter-example (or a "proof") to the author in case they find it.

Finally, it is interesting to note that in order to express the last term of plethysms, the expression of type

$$\dim\{\lambda\}_k \quad \text{or} \quad \frac{|\lambda|}{k}\cdot\dim\{\lambda\}_k$$

necessarily appears for any formulas in § 2 ~ § 4, where $\dim\{\lambda\}_k$ denotes the dimension of the representation space of $GL(k,\mathbf{C})$ corresponding to the partition λ, and $|\lambda| = \Sigma\,\lambda_i$. ($\dim\{\lambda\}_k$ is a polynomial of k with degree $|\lambda|$.) These values satisfy some interesting recursive identities which depends on the combinatorial property of Young diagrams (see § 2). Perhaps, in considering plethysms, these values are fundamental quantities naturally associated with Young diagrams. And it seems to the author that there exist some further deep and interesting combinatorial structures in "plethysms". We desire that someone will investigate these properties.

1. Fundamental range and the first term

We first explain our problem precisely. We introduce the natural lexicographic order in the set of partitions of non-negative integers as follows: For $\nu = \{\nu_1, \nu_2, \cdots\}$ $(\nu_1 \geq \nu_2 \geq \cdots \geq 0)$ and $\eta = \{\eta_1, \eta_2, \cdots\}$ $(\eta_1 \geq \eta_2 \geq \cdots \geq 0)$, we set $\nu > \eta$ if and only if

$$\nu_1 = \eta_1, \cdots\cdot, \nu_p = \eta_p \quad \text{and} \quad \nu_{p+1} > \eta_{p+1} \quad \text{for some } p.$$

And we express the decomposition of the plethysm $\{\lambda\}\otimes\{\mu\} = \Sigma\, a_\nu\{\nu\}$ $(a_\nu \in \mathbb{N})$ with respect to this order from large to small. (For the definition of plethysm, see [19], [23], [26], [35], etc. The table in [5] is arranged in this order.) Then, it is a natural problem to determine which partition is the first or the last term of $\{\lambda\}\otimes\{\mu\}$. In this paper, we give several combinatorial conjectures concerning this problem. (As far as the author calculate, he cannot find a counter-example to these conjectures, and it seems that they are all "correct".)

Now, we put $|\lambda| = \Sigma\, \lambda_i$, and $|\mu| = \Sigma\, \mu_i$, and call the number $|\lambda||\mu|$ the total degree of $\{\lambda\}\otimes\{\mu\}$. We first state a conjecture on the fundamental range of partitions appearing in $\{\lambda\}\otimes\{\mu\}$.

CONJECTURE 1.1 (cf. [5; p.126]). With respect to the above order, all terms of $\{\lambda\}\otimes\{\mu\}$ are contained between the range

$$\{|\mu|\lambda_1, \cdots\cdot, |\mu|\lambda_m\} \sim \{\lambda_1{}^{|\mu|}, \cdots\cdot, \lambda_m{}^{|\mu|}\},$$

where $\{\lambda\} = \{\lambda_1, \cdots, \lambda_m\}$ $(\lambda_1 \geq \cdots \geq \lambda_m > 0)$. In addition, the first term of $\{\lambda\}\otimes\{\mu\}$ is equal to $\{|\mu|\lambda_1, \cdots, |\mu|\lambda_m\}$ if and only if $\mu = \{\mu_1\}$, i.e., the depth of μ is 1. The last term of $\{\lambda\}\otimes\{\mu\}$ is equal to $\{\lambda_1{}^{|\mu|}, \cdots, \lambda_m{}^{|\mu|}\}$ if and only if

$$\{\mu\} = \begin{cases} \{n\} & \text{in the case } |\lambda| = \text{even} \\ \{1^n\} & \text{in the case } |\lambda| = \text{odd} \end{cases}$$

for some n.

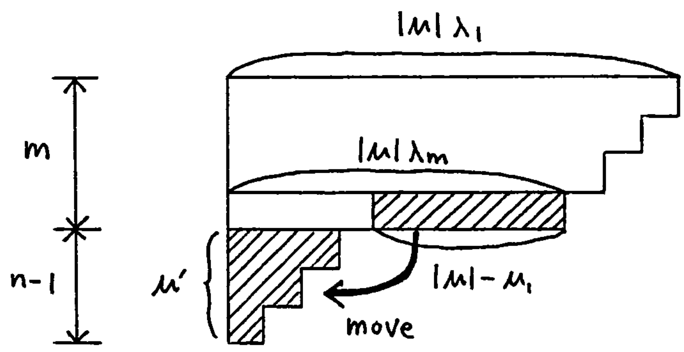
The precise first term of $\{\lambda\}\otimes\{\mu\}$ is given by the following conjecture.

CONJECTURE 1.2 (cf. [5; p.126]). The first term of the plethysm $\{\lambda\}\otimes\{\mu\}$ is equal to

$$\{|\mu|\lambda_1, |\mu|\lambda_2, \cdots, |\mu|\lambda_{m-1}, |\mu|(\lambda_m - 1) + \mu_1, \mu_2, \cdots, \mu_n\},$$

where $\lambda = \{\lambda_1, \cdots, \lambda_m\}$, $\mu = \{\mu_1, \cdots, \mu_n\}$ $(\lambda_m, \mu_n > 0)$. The coefficient of the first term is always 1.

We can explain the above formula by considering the following figure



where $\mu' = \{\mu_2, \cdots, \mu_n\}$. Namely, in the diagram $\{|\mu|\lambda_1, |\mu|\lambda_2, \cdots, |\mu|\lambda_m\}$, we have only to move $|\mu'|$ boxes from the bottom row to the left bottom in the shape of μ'.

For example, the first term of $\{21\}\otimes\{21\}$ is given by $\{3\cdot 2,\ 3\cdot(1-1)+2,\ 1\} = \{621\}$. In the case $\{21^2\}\otimes\{31\}$, we have $\{4\cdot 2,\ 4\cdot 1,\ 4\cdot(1-1)+3,\ 1\} = \{8431\}$. Up to total degree ≤ 16, this conjecture is actually correct in view of the table in [5].

As a special case, we have the following formula on the first term.

CONJECTURE 1.3. We have the decompositions

$$\begin{aligned}
\{m\}\otimes\{n\} &= \{mn\} + \cdots\cdot , \\
\{m\}\otimes\{1^n\} &= \{mn-n+1, 1^{n-1}\} + \cdots\cdot , \\
\{1^m\}\otimes\{n\} &= \{n^m\} + \cdots\cdot , \\
\{1^m\}\otimes\{1^n\} &= \{n^{m-1}, 1^n\} + \cdots\cdot .
\end{aligned}$$

In the case $m = 2$ or $n = 2$, we already know the explicit decomposition of these four plethysms, and for these cases, the above conjectures actually hold. (cf. [26; p.138, p.140].) The conjecture for the first plethysm $\{m\}\otimes\{n\}$ is essentially proved in [40; p.110], and in the same place it is showed that the plethysm $\{m\}\otimes\{1^n\}$ contains the term $\{mn-n+1, 1^{n-1}\}$ with multiplicity 1. See also the table in [12]. The last terms of these four plethysms are a little more complicated according as the parity of m. (See § 2 ~ § 4.)

In the case $\{\lambda\} = \{m\}$ or $\{1^m\}$, and $|\mu| = 3$, the closed formula of $\{\lambda\}\otimes\{\mu\}$ is also known (cf. [26; p.141] and the conjugate formula [19; p.220]). For these cases, CONJECTURE 1.2 is also correct. (But, the formula for $\{m\}\otimes\{1^3\}$ in [26; p.141] seems to be incorrect. In terms of the notations in [26], the correct formula is given by

$$e_3 \circ h_n = \sum_{\mu} ([1/6 \cdot m(\mu)] + \varepsilon'(\mu)) s_{\mu} ,$$

where $\varepsilon'(\mu) = 1$ if $m(\mu) =$ even and $\mu_2 = \underline{\text{odd}}$, or $m(\mu) \equiv 3$ or 5 (mod 6), and $\varepsilon'(\mu) = 0$ otherwise.) As far as the author knows, the known closed formulas of $\{\lambda\}\otimes\{\mu\}$ are exhausted by these examples, if we add the following formulas stated in [27] (and their conjugate):

$$\{m-1,1\}\otimes\{2\}, \quad \{m-1,1\}\otimes\{1^2\}, \quad \{2,1^{m-2}\}\otimes\{2\}, \quad \{2,1^{m-2}\}\otimes\{1^2\},$$
$$\{m-2,2\}\otimes\{2\}, \quad \{m-2,2\}\otimes\{1^2\}, \quad \{2^2,1^{m-4}\}\otimes\{2\}, \quad \{2^2,1^{m-4}\}\otimes\{1^2\}.$$

2. Conjecture on the last term (part I)

At present, we have three types of conjectures on the last term of $\{\lambda\}\otimes\{\mu\}$, which give the same correct results for the cases calculated in [5]. In this section, we give the first type conjecture, expressing the last term of $\{\lambda\}\otimes\{\mu\}$ as a function of μ for each fixed λ, which is essentially same as the one stated in our previous paper [5]. We simplify the quite complicated arguments in [5] to the following relatively compact form. But, still it requires some unfamiliar inductive arguments. The essence of this inductive process is summarized in CONJECTURE 2.4. By these conjectures, we can make a table on the last term of $\{\lambda\}\otimes\{\mu\}$ for large $|\mu|$, by which we can find the explicit formula of the last term stated in § 4.

Before stating the inductive process, we first remark that the coefficient of the last term of $\{\lambda\}\otimes\{\mu\}$ is not necessary equal to 1 in contrast to the first term. For example, the last term of $\{21\}\otimes\{4\}$ is $2\{3^2 2^2 1^2\}$ and the last term of $\{21\}\otimes\{4^2 1\}$ is

$3\{3^4 2^5 1^5\}$. But, at present, we do not know a formula (or even a conjecture) of this coefficient. And in § 2 ~ § 4, we ignore the coefficient of the last term of $\{\lambda\}\otimes\{\mu\}$.

Now, for a partition λ and a positive integer p, we put

$$[p] = \begin{cases} \{1^p\} & \text{if } |\lambda| = \text{even} \\ \{p\} & \text{if } |\lambda| = \text{odd} \end{cases},$$

and consider the last term of $\{\lambda\}\otimes[p]$. We say that $\{\lambda\}\otimes[p]$ is a fundamental plethysm because the last term of general $\{\lambda\}\otimes\{\mu\}$ can be determined by that of $\{\lambda\}\otimes[p]$, as the following conjecture shows.

CONJECTURE 2.1. (1) The case $|\lambda|$ = even: We express the transpose $\{{}^t\mu\}$ as $\{a,b,\cdots,c\}$ $(a \geq b \geq \cdots \geq c > 0)$, and calculate the last terms of fundamental plethysms $\{\lambda\}\otimes\{1^a\}$, $\{\lambda\}\otimes\{1^b\}$, $\cdots$, $\{\lambda\}\otimes\{1^c\}$. Then, by arranging the numbers appearing in these last terms from large to small, we obtain the last term of $\{\lambda\}\otimes\{\mu\}$.

(2) The case $|\lambda|$ = odd: We express $\{\mu\}$ as $\{a,b,\cdots,c\}$ $(a \geq b \geq \cdots \geq c > 0)$, and calculate the last terms of fundamental plethysms $\{\lambda\}\otimes\{a\}$, $\{\lambda\}\otimes\{b\}$, $\cdots$, $\{\lambda\}\otimes\{c\}$. Then, by arranging the numbers appearing in these last terms from large to small, we obtain the last term of $\{\lambda\}\otimes\{\mu\}$.

Example. The case $\{31\}\otimes\{21^2\}$: Since ${}^t\{21^2\} = \{31\}$ and the last terms of $\{31\}\otimes\{1^3\}$ and $\{31\}\otimes\{1\}$ are $\{3^4\}$ and $\{31\}$, respectively, the last term of $\{31\}\otimes\{21^2\}$ is equal to $\{3^5 1\}$.

The case $\{21\}\otimes\{32\}$: Since the last terms of $\{21\}\otimes\{3\}$ and $\{21\}\otimes\{2\}$ are $\{32^2 1^2\}$ and $\{2^3\}$, respectively, the last term of $\{21\}\otimes\{32\}$ is equal to $\{32^5 1^2\}$.

Similarly, as we stated in Introduction, we have

$$\{m\}\otimes\{n\} = \cdots + \{m^n\} \quad (m = \text{even}),$$

$$\{m\}\otimes\{1^n\} = \cdots + \{m^n\} \quad (m = \text{odd}),$$

$$\{1^m\}\otimes\{n\} = \cdots + \{1^{mn}\} \quad (m = \text{even}),$$

$$\{1^m\}\otimes\{1^n\} = \cdots + \{1^{mn}\} \quad (m = \text{odd}).$$

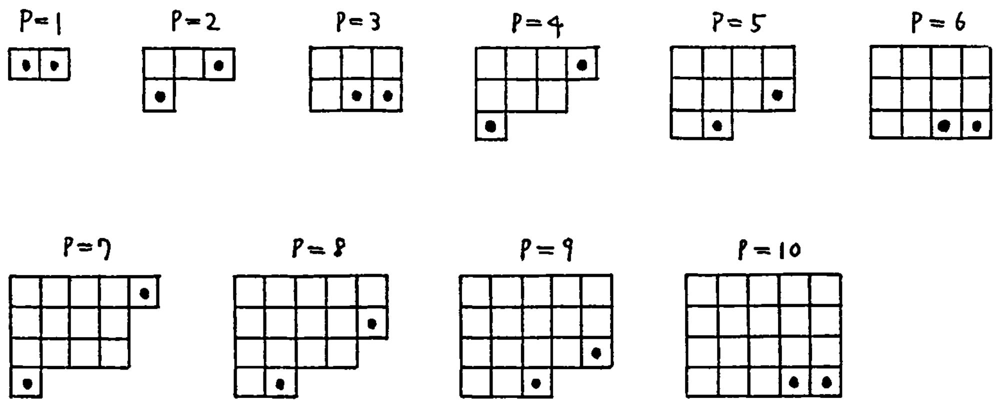
For the cases where m has the reversed parity, see CONJECTURE 4.2 in § 4.

Remark. We stated in [5; p.150, CONJECTURE 11] the essentially same conjecture. But, the explanation in [5] was unnecessarily complicated. Perhaps, this conjecture can be proved by using the determinantal formula of plethysms stated [19; p.222] (or its conjugate version), and the Littlewood-Richardson rule. But, it seems that some additional property on $\{\lambda\}\otimes[p]$ is required to complete the proof.

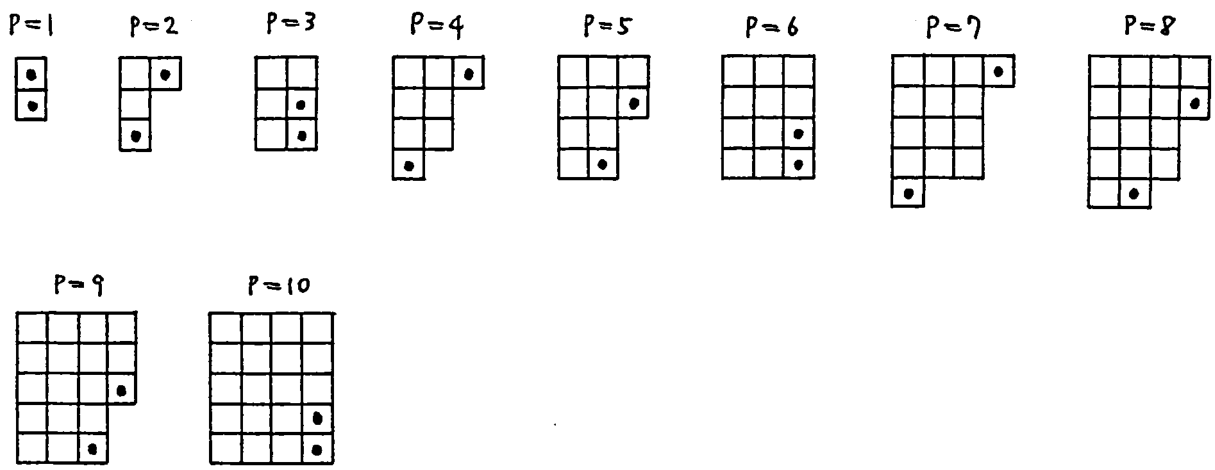
In the table in § 5, we marked the symbol * (or **) for fundamental plethysms. (The meaning of the symbol ** is explained in § 3.)

Now, we explain the inductive process which appears in expressing the last term of $\{\lambda\}\otimes[p]$. To understand this inductive structure, it is helpful to present several examples on the last term of $\{\lambda\}\otimes[p]$ in advance. In the following, we exhibit the examples for the cases $|\lambda| \leq 4$. (See also § 6.) They form mysterious sequences of Young diagrams, possessing some hidden symmetric rules. We advise to the readers that they check each property stated in the following conjectures for these sequences.

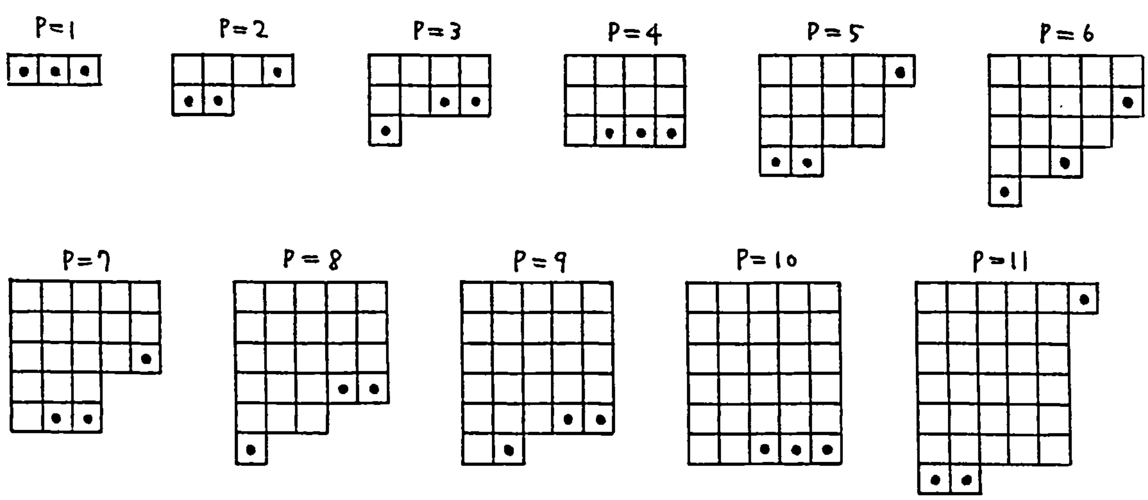
$\lambda = \{2\}$:

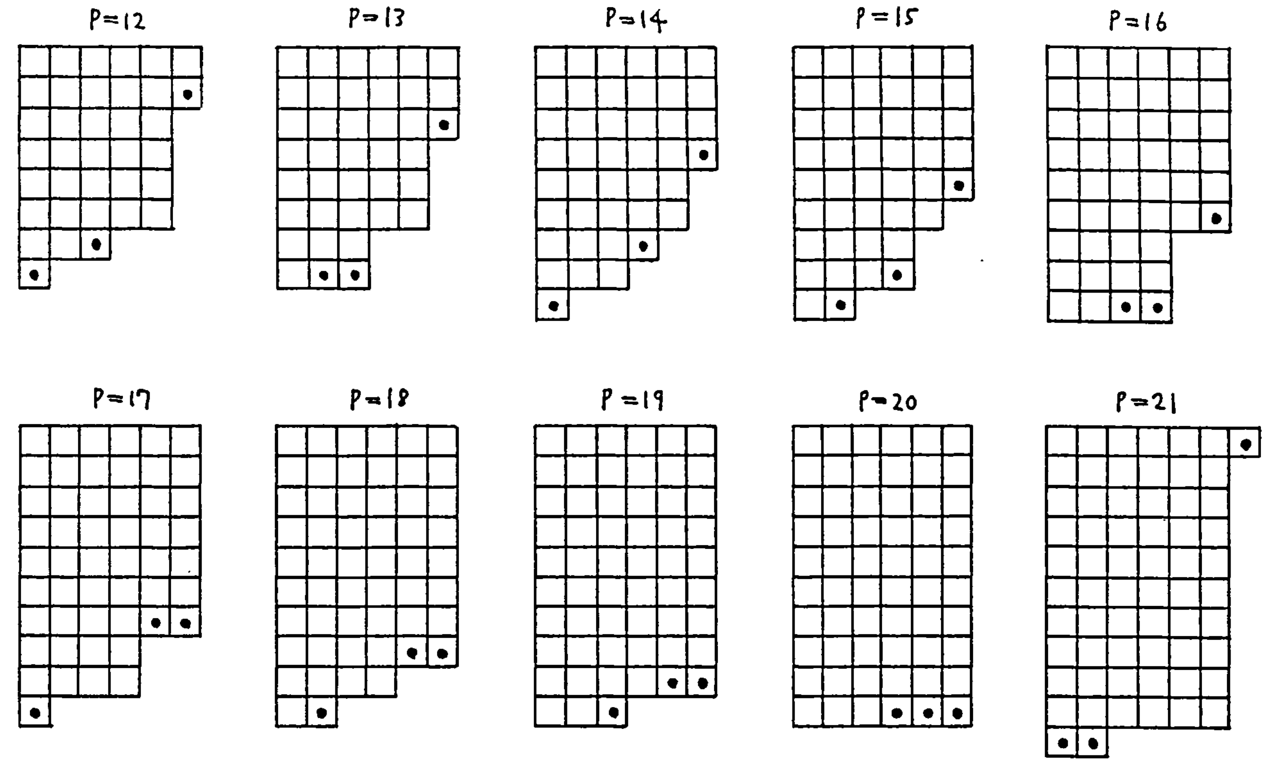


$\lambda = \{1^2\}$:

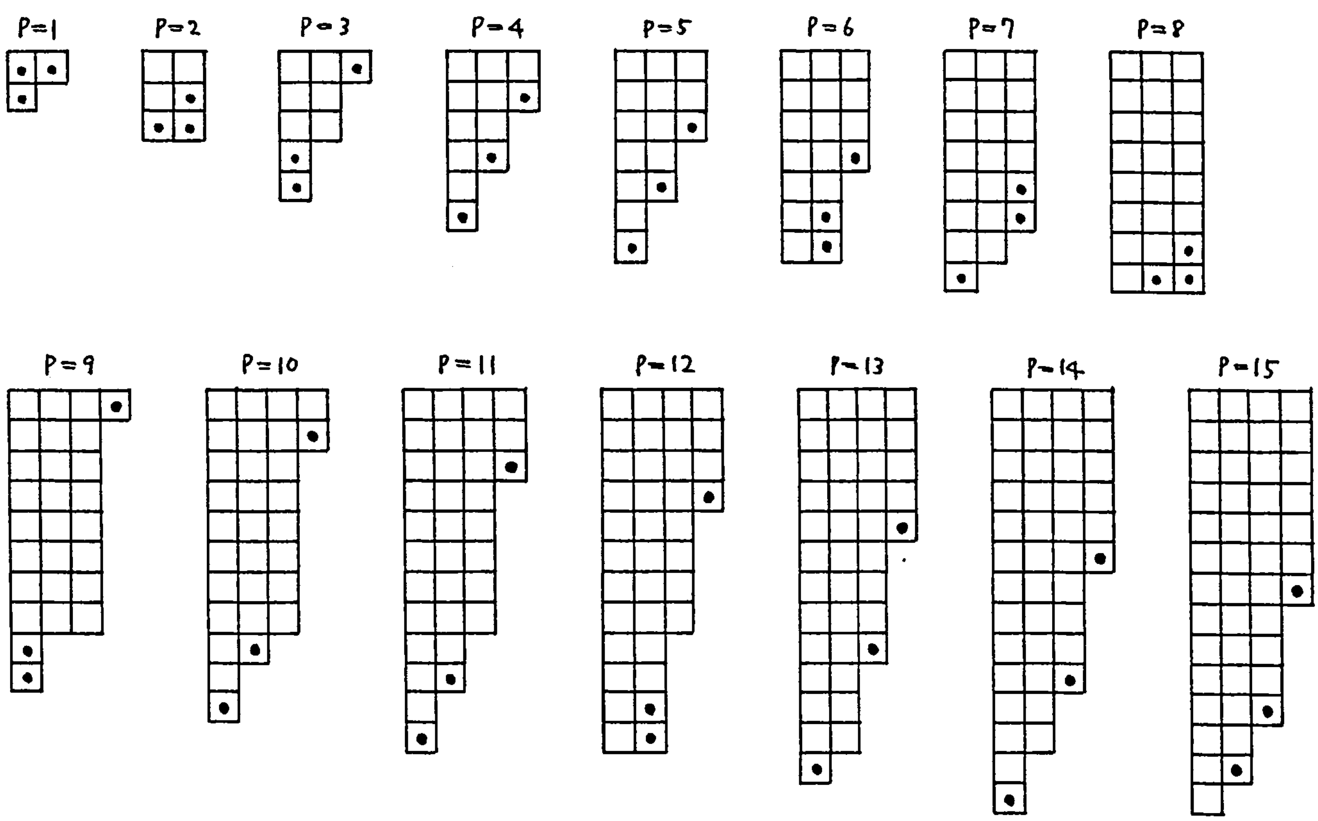


$\lambda = \{3\}$:

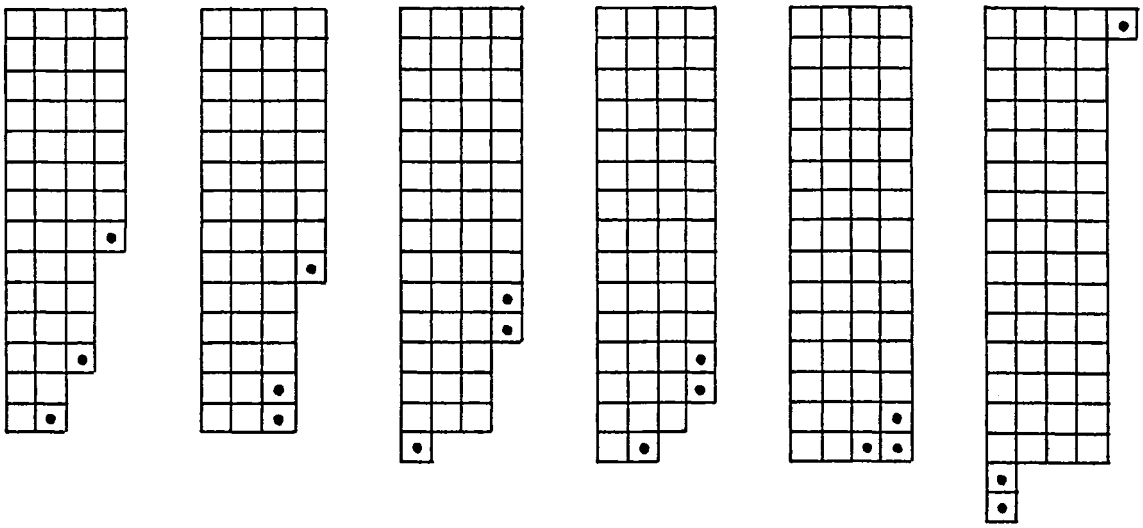



P=12
P=13
P=14
P=15
P=16
P=17
P=18
P=19
P=20
P=21

$\lambda = \{21\}$:

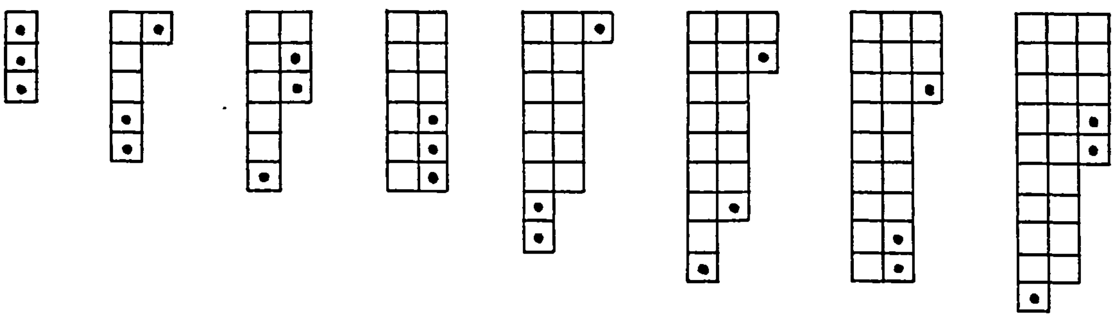

P=1
P=2
P=3
P=4
P=5
P=6
P=7
P=8
P=9
P=10
P=11
P=12
P=13
P=14
P=15

P=16 P=17 P=18 P=19 P=20 P=21

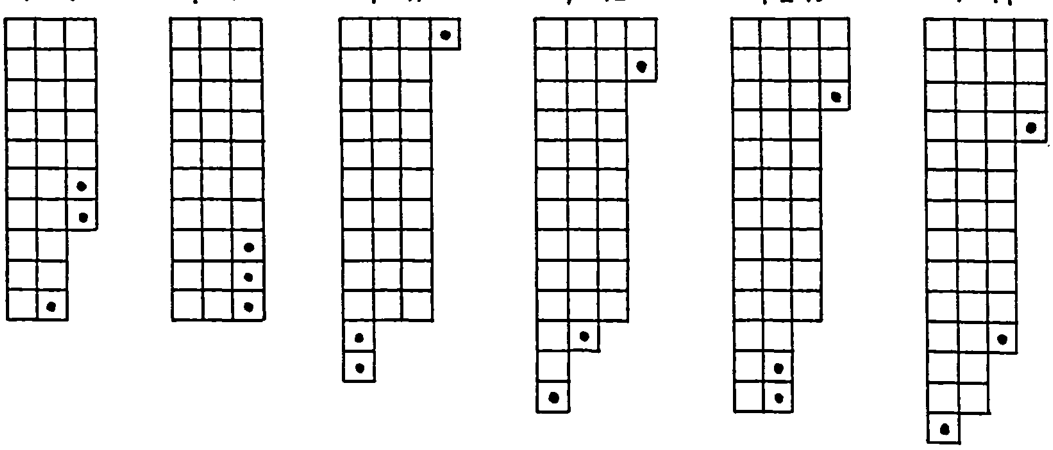


$\lambda = \{1^3\}$:

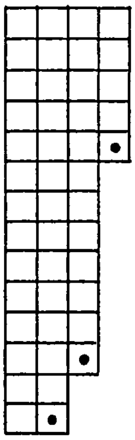
P=1 P=2 P=3 P=4 P=5 P=6 P=7 P=8



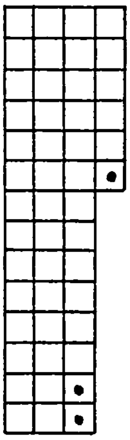
P=9 P=10 P=11 P=12 P=13 P=14



P=15



P=16



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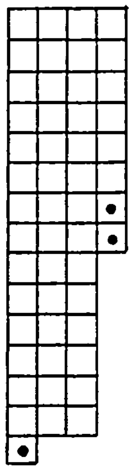
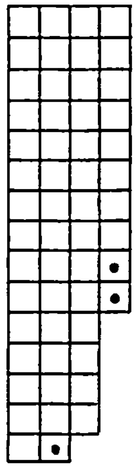
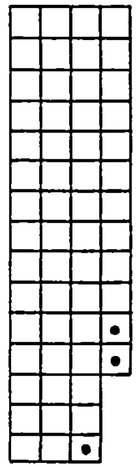


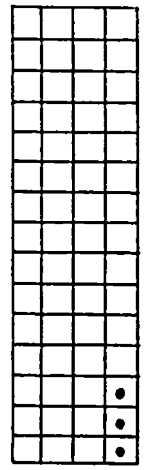
P=18



P=19



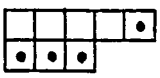
P=20



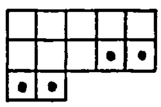
$\lambda = \{4\}$:

P=1

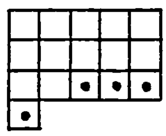
P=2



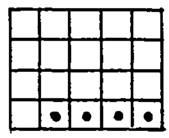
P=3



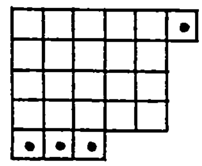
P=4



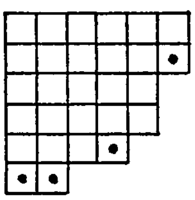
P=5



P=6



P=7



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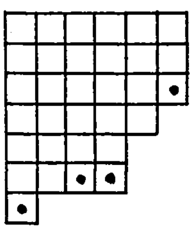

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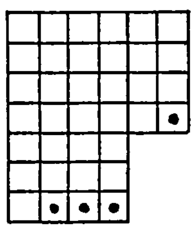
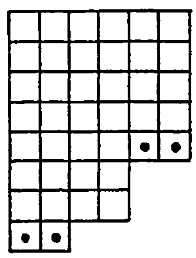
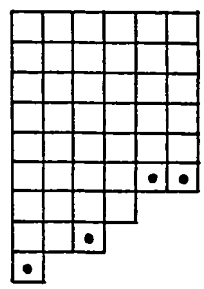


P=10



P=11



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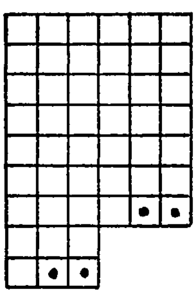

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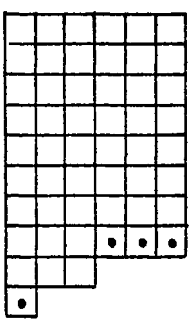

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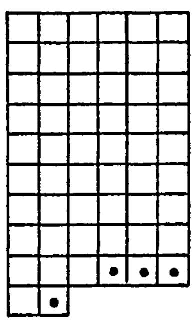
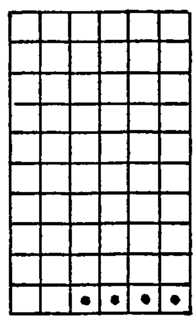
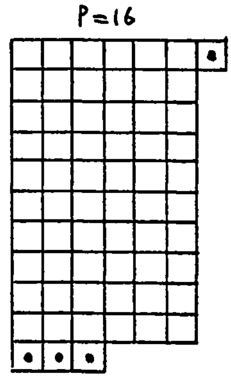


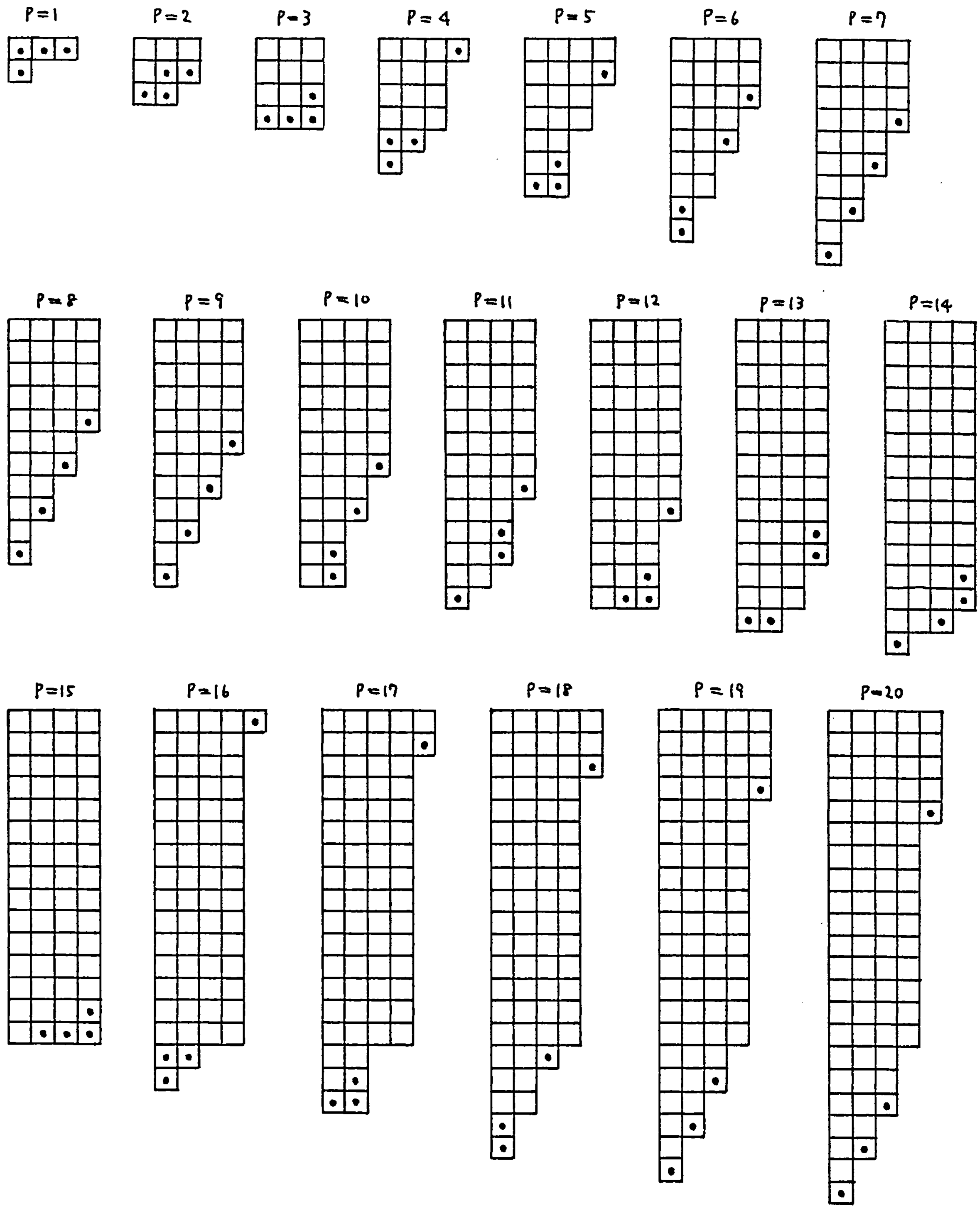
P=15



P=16



$\lambda = \{31\}$: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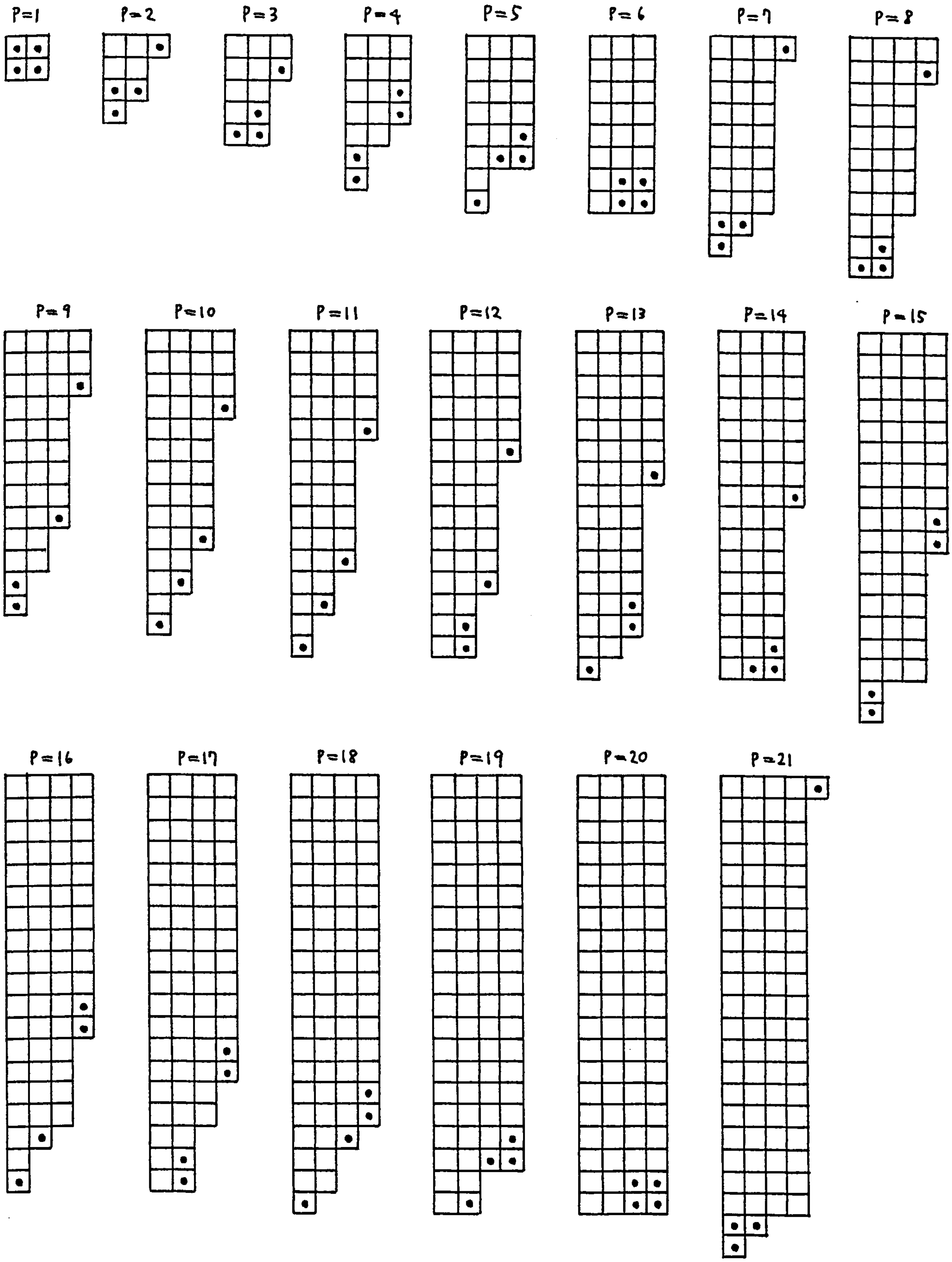
$\lambda = \{2^2\}$: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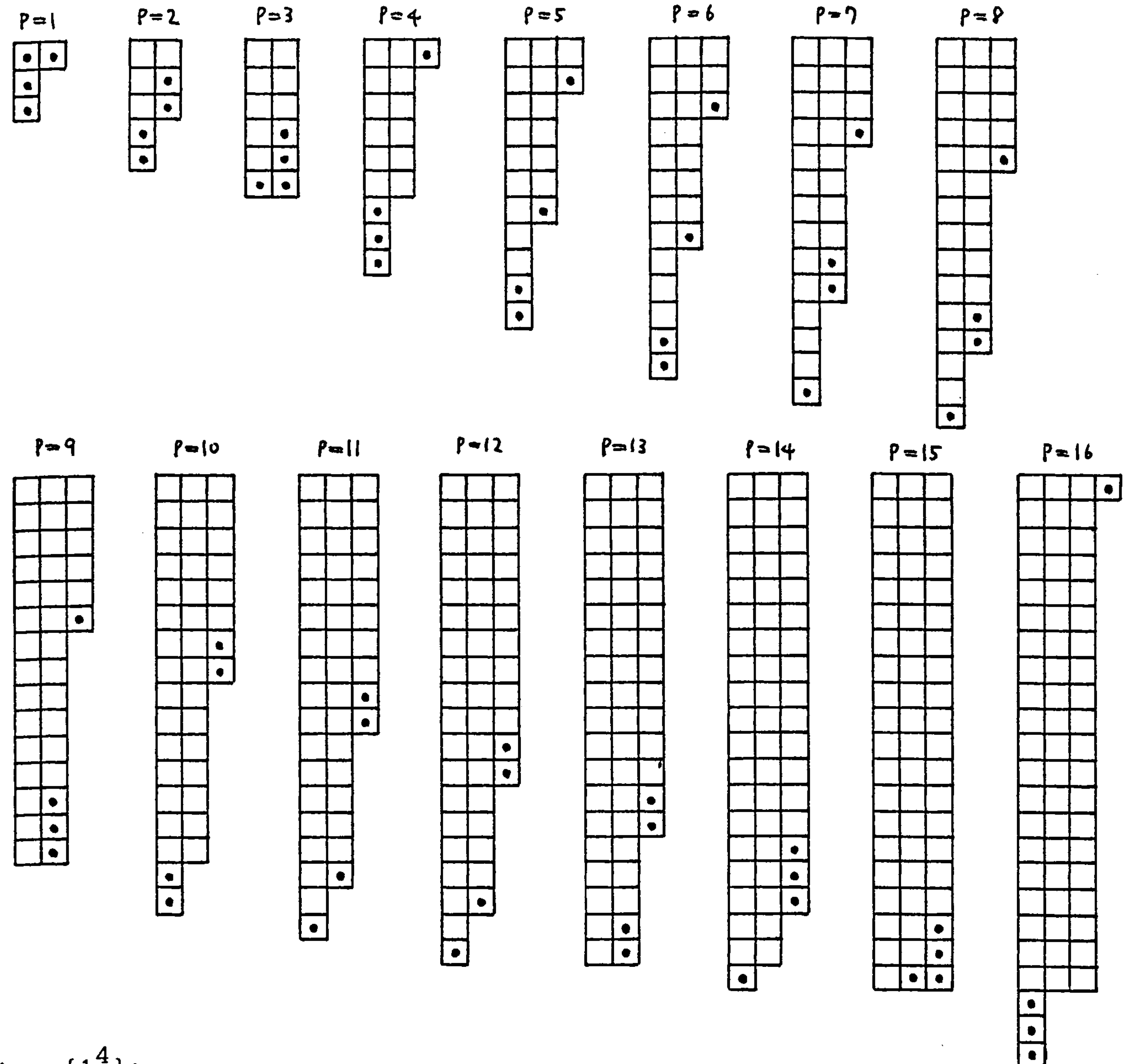
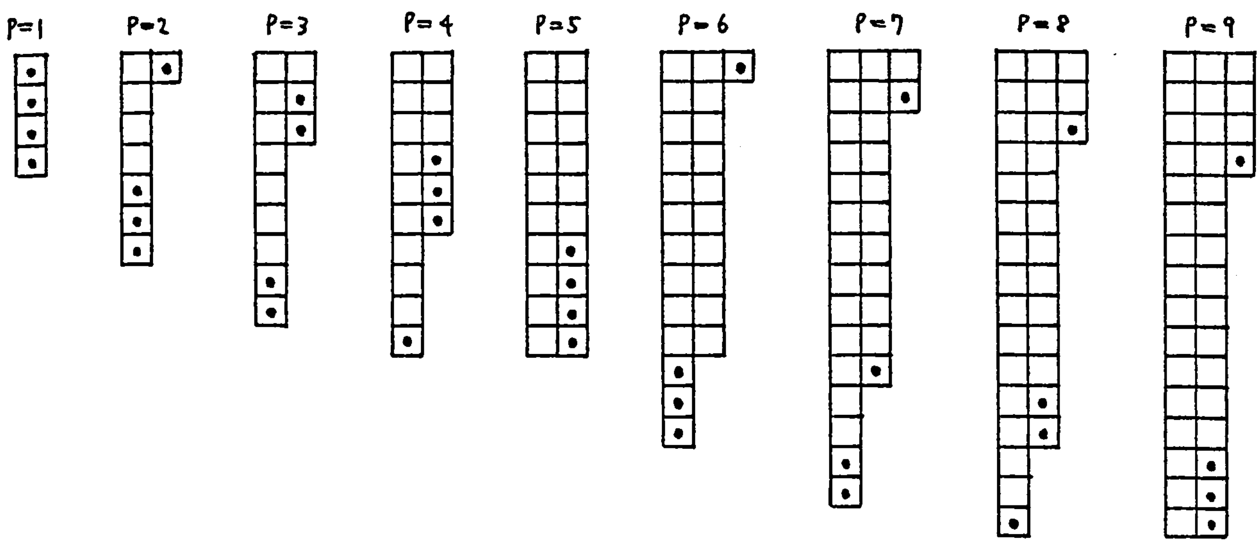
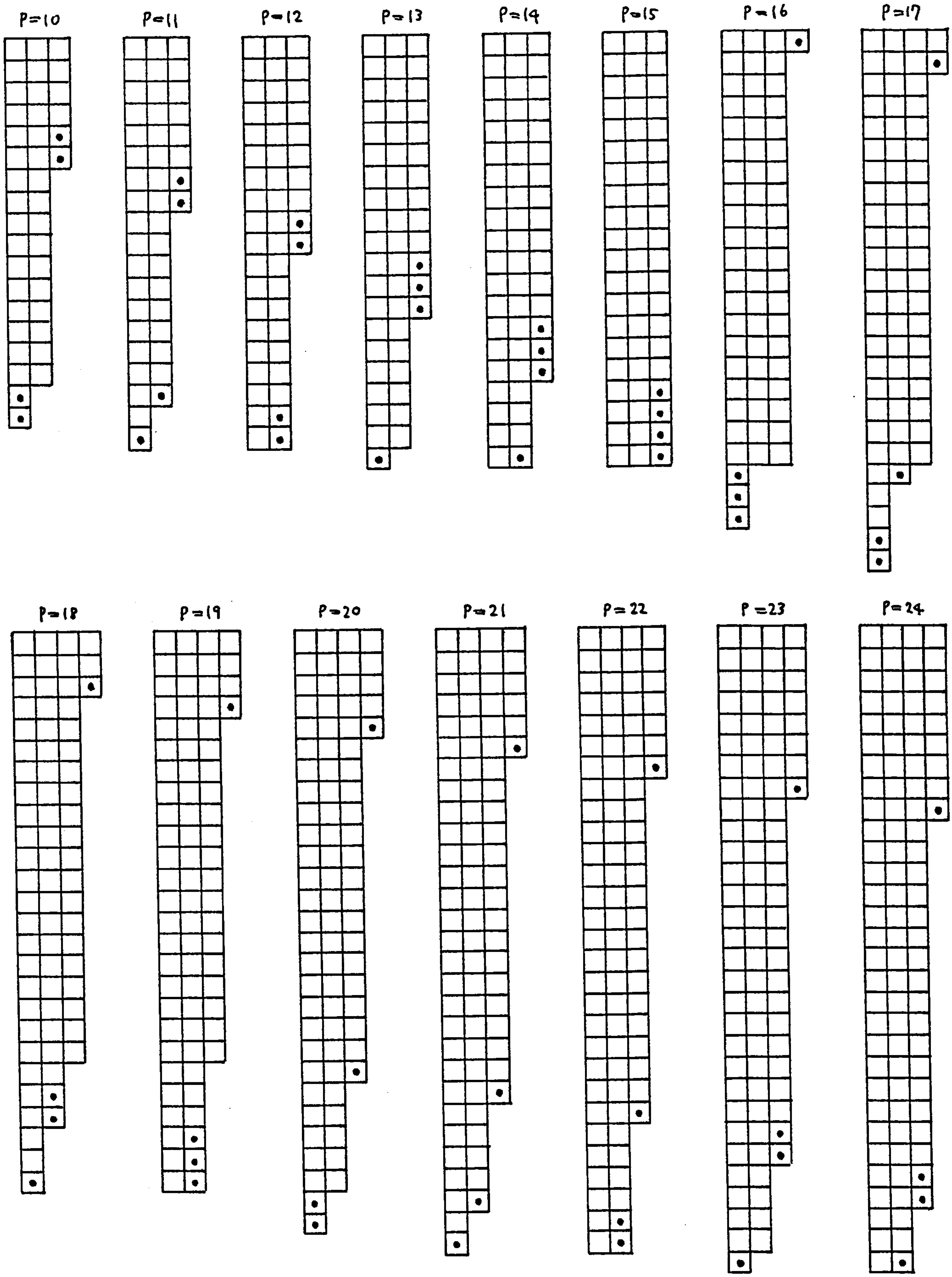
P=21

$\lambda = \{21^2\}$:



$\lambda = \{1^4\}$:




P=10
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P=21
P=22
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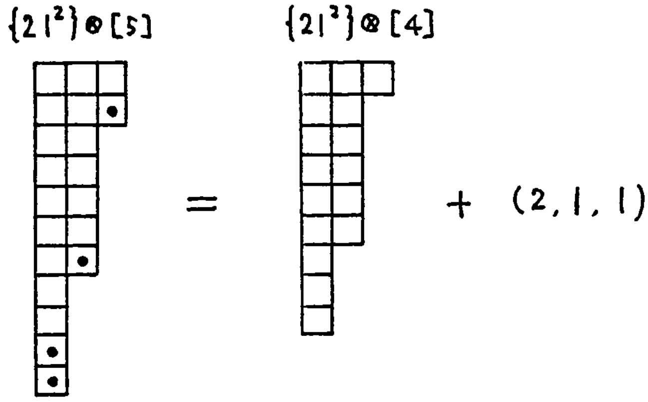
P=24

These sequences of Young diagrams possess the peculiar feature. For example, they constitute increasing sequences of Young diagrams. (The dotted boxes in the above table imply the new adding ones at the considering step p.) We express this process by the addition of row vectors as follows:



Then we obtain a beautiful sequences of row vectors possessing some recursive property. For example, we have the following sequences of row vectors:

p	λ = {3}	λ = {21}
1	(1,1,1)	(2,1)
2	(1,1,0,1)	(1,2)
3	(1,0,1,1)	(2,0,1)
4	(0,1,1,1)	(1,1,1)
5	(1,1,0,0,1)	(1,1,1)
6	(1,0,1,0,1)	(0,2,1)
7	(0,1,1,0,1)	(1,0,2)
8	(1,0,0,1,1)	(0,1,2)
9	(0,1,0,1,1)	(2,0,0,1)
10	(0,0,1,1,1)	(1,1,0,1)

p	$\lambda = \{3\}$	$\lambda = \{21\}$
11	(1,1,0,0,0,1)	(1,1,0,1)
12	(1,0,1,0,0,1)	(0,2,0,1)
13	(0,1,1,0,0,1)	(1,0,1,1)
14	(1,0,0,1,0,1)	(1,0,1,1)
15	(0,1,0,1,0,1)	(0,1,1,1)
16	(0,0,1,1,0,1)	(0,1,1,1)
17	(1,0,0,0,1,1)	(0,0,2,1)
18	(0,1,0,0,1,1)	(1,0,0,2)
19	(0,0,1,0,1,1)	(0,1,0,2)
20	(0,0,0,1,1,1)	(0,0,1,2)
21	(1,1,0,0,0,0,1)	(2,0,0,0,1)

p	$\lambda = \{1^3\}$	$\lambda = \{31\}$
1	(3)	(2,1,1)
2	(2,1)	(1,2,1)
3	(1,2)	(1,1,2)
4	(0,3)	(2,1,0,1)
5	(2,0,1)	(1,2,0,1)
6	(1,1,1)	(2,0,1,1)
7	(0,2,1)	(1,1,1,1)
8	(1,0,2)	(1,1,1,1)
9	(0,1,2)	(1,1,1,1)
10	(0,0,3)	(0,2,1,1)
11	(2,0,0,1)	(1,0,2,1)
12	(1,1,0,1)	(0,1,2,1)
13	(0,2,0,1)	(1,1,0,2)

p	$\lambda = \{1^3\}$	$\lambda = \{31\}$
14	(1,0,1,1)	(1,0,1,2)
15	(0,1,1,1)	(0,1,1,2)
16	(0,0,2,1)	(2,1,0,0,1)
17	(1,0,0,2)	(1,2,0,0,1)
18	(0,1,0,2)	(2,0,1,0,1)
19	(0,0,1,2)	(1,1,1,0,1)
20	(0,0,0,3)	(1,1,1,0,1)
21	(2,0,0,0,1)	(1,1,1,0,1)

Of course, the essence of the inductive process is contained in these sequences, and in the following we will clear up these properties.

As another peculiar feature on the last term of fundamental plethysms $\{\lambda\}\otimes[p]$, several rectangles appear repeatedly after some steps, which give a nice break on the inductive process. We first explain these rectangular cases.

We denote by $\dim \{\nu\}_k$ the dimension of the representation space of $GL(k,\mathbb{C})$ corresponding to the partition $\{\nu\} = \{\nu_1, \cdots, \nu_k\}$ $(\nu_1 \geq \cdots \geq \nu_k \geq 0)$. Explicitly, it is equal to

$$\frac{D(\nu_1+k-1, \nu_2+k-2, \cdots\cdot, \nu_{k-1}+1, \nu_k)}{D(k-1, k-2, \cdots\cdot, 1, 0)},$$

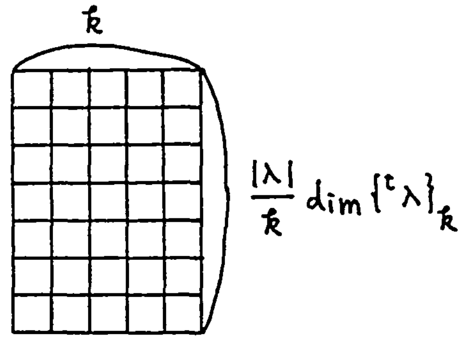
where $D(i_1, i_2, \cdots\cdot, i_k)$ is the difference product of $\{i_1, \cdots, i_k\}$ $(i_1 > \cdots\cdot > i_k)$ (cf. [18; p.115]). For example, we have

$\dim \{2\}_k = 1/2\cdot k(k+1)$, $\quad\dim \{1^2\}_k = 1/2\cdot k(k-1)$,

$\dim \{21\}_k = 1/3\cdot k(k-1)(k+1)$, $\quad\dim \{2^2\}_k = 1/12\cdot k^2(k-1)(k+1)$,

etc. Then, we have the following conjecture.

CONJECTURE 2.2. For a positive integer k ($\geq \lambda_1$), we put $p_k = \dim \{{}^t\lambda\}_k$. Then the last term of the fundamental plethysm $\{\lambda\}\otimes[p_k]$ is the rectangular Young diagram $\{k^{|\lambda|p_k/k}\}$.



In addition, among fundamental plethysms $\{\lambda\}\otimes[p]$, the case where the last term is a rectangle is exhausted by this case.

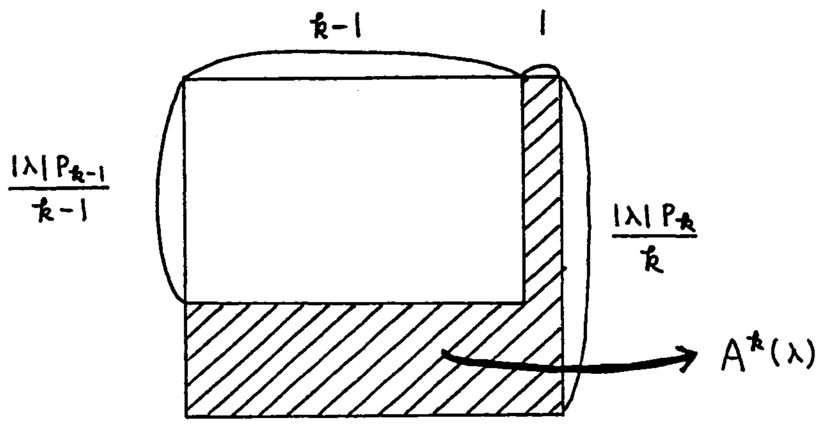
(Note that "rectangle" corresponds to the "relative invariant" in the viewpoint of the representation theory of general linear group $GL(k,\mathbf{C})$.)

Example. Using the closed formula of $\{2\}\otimes\{1^p\}$ and $\{1^2\}\otimes\{1^p\}$ stated in [26; p.138], we can see that this conjecture actually holds for these cases. For these partitions, we have $p_k = 1/2\cdot k(k-1)$ and $1/2\cdot k(k+1)$, respectively, and hence the sequence of the last term of $\{\lambda\}\otimes[p] = \{\lambda\}\otimes\{1^p\}$ ($p = 1, 2, \cdots$) contains the rectangular Young diagram $\{k^{k-1}\}$ and $\{k^{k+1}\}$, which we can see in the above sequences of Young diagrams. In the case of $\lambda = \{21\}$, we have $p_k = 1/3\cdot k(k-1)(k+1)$ and hence the last term of $\{21\}\otimes[p_k] = \{21\}\otimes\{p_k\}$ is equal to $\{k^{(k-1)(k+1)}\}$, which we can also verify in the above table.

As a next problem, we must capture the last term of $\{\lambda\}\otimes[p]$ for intermediate p, satisfying

$$(*) \qquad \dim \{{}^t\lambda\}_{k-1} < p \le \dim \{{}^t\lambda\}_k ,$$

i.e., we must find a rule of "box filling" of rectangles. We express this process (*) by the sequence of adding row vectors ($\in \mathbf{R}^k$) stated above, and denote it by the symbol $A^k(\lambda)$. (Note that $A^k(\lambda)$ is defined only in the case $k \ge \lambda_1$ where $\lambda = \{\lambda_1,\cdots,\lambda_m\}$, and the first term of $A^{\lambda_1}(\lambda)$ is the vector expressing the transpose of λ, i.e., the vector consisting of the number of each column of λ.)



For example, in the case of $\lambda = \{3\}$, we have

$$A^3(3) = (1,1,1),$$
$$A^4(3) = (1,1,0,1) + (1,0,1,1) + (0,1,1,1),$$
$$A^5(3) = (1,1,0,0,1) + (1,0,1,0,1) + (0,1,1,0,1)$$
$$+ (1,0,0,1,1) + (0,1,0,1,1) + (0,0,1,1,1).$$

Similarly, in the case of $\lambda = \{21\}$, we have

$$A^2(21) = (2,1) + (1,2),$$
$$A^3(21) = (2,0,1) + (1,1,1) + (1,1,1) + (0,2,1)$$
$$+ (1,0,2) + (0,1,2),$$

$$A^4(21) = (2,0,0,1) + (1,1,0,1) + (1,1,0,1) + (0,2,0,1)$$
$$+ (1,0,1,1) + (1,0,1,1) + (0,1,1,1) + (0,1,1,1)$$
$$+ (0,0,2,1) + (1,0,0,2) + (0,1,0,2) + (0,0,1,2).$$

Of course, the symbol "+" in these expressions does not have the actual meaning, and the adding order is essentially important. (We remark that in the previous paper [5], this process is denoted by the symbols $A_1^{k-1}(\lambda) \sim A_t^{k-1}(\lambda)$ where t is the depth of λ. In particular, the index k is shifted by one in the expression here.) In the following, we give the inductive formula expressing this adding process $A^k(\lambda)$ in terms of $A^j(\nu)$ $(j < k)$, where ν is a subdiagram contained in λ.

For this purpose, we first introduce the order in the set $(N \cup \{0\})^k$ as follows:

$$(a_1, \cdots, a_k) < (b_1, \cdots, b_k) \text{ if and only if}$$
$$a_k = b_k,\ a_{k-1} = b_{k-1},\ \cdots,\ a_{i+1} = b_{i+1},\ a_i < b_i \text{ for some } i.$$

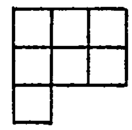
For example, we have

$$(2,1,1,0) < (1,2,1,0) < (1,1,2,0) < (2,1,0,1) < (1,2,0,1)$$
$$< (2,0,1,1) < (1,1,1,1) < (0,2,1,1) < (1,0,2,1) < (0,1,2,1)$$
$$< (1,1,0,2) < (1,0,1,2) < (0,1,1,2).$$

Next, we denote by t the depth of the Young diagram λ, and for $0 \le i \le t$, we denote by $\lambda(i)$ the set of Young diagrams consisting $|\lambda|-i$ boxes which are obtained by deleting at most one box from each row of λ.

For example, in the case of $\lambda = \{3^2 1\}$, we have

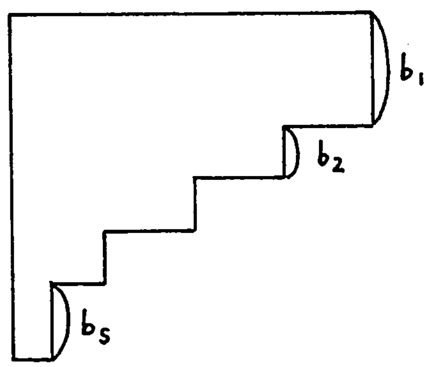
$$\lambda(0) = \{ \{3^2 1\} \},$$
$$\lambda(1) = \{ \{3^2\}, \{321\} \},$$
$$\lambda(2) = \{ \{32\}, \{2^2 1\} \},$$
$$\lambda(3) = \{ \{2^2\} \}.$$

$\lambda =$ 

Digressive Remark. We denote by $^{\#}X$ the number of elements of the set X. Then, we have

$$^{\#}\lambda(i) = {}^{\#}\{ (a_1,\cdots,a_s) \mid a_1+\cdots+a_s = i,\ 0 \le a_j \le b_j,\ a_j \in \mathbf{Z} \},$$

where s is the number of steps of λ and b_j the depth of j-th step.

$\lambda =$ 

Using this fact, we can easily show that the equality

$$^{\#}\lambda(i) = {}^{\#}\lambda(t-i)$$

holds for $0 \le i \le t$. In the special case $\lambda = \{n,n-1,\cdots,2,1\}$, this equality corresponds to the famous formula $\binom{n}{i} = \binom{n}{n-i}$ on binomial coefficients.

Now, under these notations, we state the following conjecture on the fundamental property of $A^k(\lambda)$.

CONJECTURE 2.3. (1) The row vectors in $A^k(\lambda)$ is ordered from small to large with respect to the above order.

(2) The k-th (= last) component of each vector in $A^k(\lambda)$ varies from 1 to t, and the total step number of $A^k(\lambda)$ having the form $(\overbrace{*,\cdots,*,i}^{k})$ $(1 \le i \le t)$ is equal to

$$\sum_{\nu\in\lambda(i)} \dim \{{}^t\nu\}_{k-1} .$$

As in the previous paper [5], we express this value as ℓ_i.

Example. $\lambda = \{21^2\}$. In this case, from the above table and [5; p.146 ~ 147], we have

$$A^2(21^2) = (3,1) + (2,2) + (1,3), \qquad \ell_1 = \ell_2 = \ell_3 = 1,$$

$$\begin{aligned} A^3(21^2) = {} & (3,0,1) + (2,1,1) + (2,1,1) \\ & + (1,2,1) + (1,2,1) + (0,3,1) && \ell_1 = 6 \\ & + (2,0,2) + (1,1,2) + (1,1,2) + (0,2,2) && \ell_2 = 4 \\ & + (1,0,3) + (0,1,3), && \ell_3 = 2 \end{aligned}$$

$$\begin{aligned} A^4(21^2) = {} & (3,0,0,1) + (2,1,0,1) + (2,1,0,1) + (1,2,0,1) \\ & + (1,2,0,1) + (0,3,0,1) + (2,0,1,1) + (2,0,1,1) \\ & + (1,1,1,1) + (1,1,1,1) + (1,1,1,1) + (0,2,1,1) \\ & + (0,2,1,1) + (1,0,2,1) + (1,0,2,1) + (0,1,2,1) \\ & + (0,1,2,1) + (0,0,3,1) && \ell_1 = 18 \\ & + (2,0,0,2) + (1,1,0,2) + (1,1,0,2) + (0,2,0,2) \\ & + (1,0,1,2) + (1,0,1,2) + (0,1,1,2) + (0,1,1,2) \\ & + (0,0,2,2) && \ell_2 = 9 \\ & + (1,0,0,3) + (0,1,0,3) + (0,0,1,3) && \ell_3 = 3. \end{aligned}$$

The vectors are actually arranged from small to large. In addition, from the above formula, we have

$$\begin{aligned} \ell_1 &= \dim \{3\}_{k-1} + \dim \{21\}_{k-1} \\ &= 1/6\cdot k(k-1)(k+1) + 1/3\cdot k(k-1)(k-2) = 1/2\cdot k(k-1)^2, \end{aligned}$$

$$\ell_2 = \dim\{2\}_{k-1} + \dim\{1^2\}_{k-1} = (k-1)^2,$$

$$\ell_3 = \dim\{1\}_{k-1} = k-1.$$

We can directly check that these values just give the desired step numbers ℓ_i denoted above.

Digressive Remark. From CONJECTURES 2.2 and 2.3, the integers $\{\ell_i\}$ must satisfy the following equalities:

$$\ell_1 + \ell_2 + \cdots + \ell_t = p_k - p_{k-1},$$

$$\ell_1 + 2\ell_2 + \cdots + t\ell_t = \frac{|\lambda|}{k} p_k .$$

(The first equality indicates the number of total steps from the rectangle $\{(k-1)^{|\lambda| p_{k-1}/(k-1)}\}$ to the next $\{k^{|\lambda| p_k/k}\}$, and the second equality indicates the number of boxes in the k-th column.) Substituting the exact value of ℓ_i in CONJECTURE 2.3 to these equalities, we obtain the following interesting identities on $\dim\{\lambda\}_k$, after some modification of the expressions:

$$\sum_{\nu} \dim\{\nu\}_{k-1} = \dim\{\lambda\}_k ,$$

$$\sum_{\nu} \frac{|\nu|}{k-1}\cdot\dim\{\nu\}_{k-1} = \frac{|\lambda|}{k}\cdot\dim\{\lambda\}_k .$$

Here, ν runs all over the Young diagrams obtained by deleting at most one box from each "column" of λ. For example, in the case of $\lambda = \{n\}$, we have

$$\sum_{p=0}^{n} \dim\{p\}_{k-1} = \sum_{p=0}^{n} \binom{p+k-2}{p} = \binom{n+k-1}{n} = \dim\{n\}_k,$$

and

$$\sum_{p=1}^{n} \frac{p}{k-1}\cdot\dim\{p\}_{k-1} = \sum_{p=1}^{n} \binom{p+k-2}{p-1} = \binom{n+k-1}{n-1} = \frac{n}{k}\dim\{n\}_k,$$

which are both well-known formulas on binomial coefficients. Similarly, in the case of $\lambda = \{1^n\}$, the above identities correspond to

$$\binom{k-1}{n} + \binom{k-1}{n-1} = \binom{k}{n},$$

and

$$\frac{n}{k-1}\binom{k-1}{n} + \frac{n-1}{k-1}\binom{k-1}{n-1} = \frac{n}{k}\binom{k}{n},$$

which are also well-known. The first identity for general λ can be proved directly by using the branching rule of $GL(k-1,\mathbf{C}) \subset GL(k,\mathbf{C})$ (cf. [18; p.143]). But we do not know the proof of the second identity for general λ at present. In view of the second identity, the integer

$$\frac{|\lambda|}{k}\cdot \dim \{\lambda\}_k$$

frequently appeared in our argument seems to be one of the fundamental value naturally associated with the partition λ.

Now, we return to our problem. Under these preliminaries, we express the inductive structure of $A^k(\lambda)$ in the following form, which may be considered as a principal conjecture in this section.

CONJECTURE 2.4. $$A^k(\lambda) = \sum_{i=1}^{t} \sum_{\substack{\nu\in\lambda(i)\\ j<k}}' (A^j(\nu), \overbrace{0,\cdots\cdots,0}^{k-j-1}, i) ,$$

where Σ' implies that the vectors are rearranged from small to large with respect to the order defined above.

Example. We consider the case $\lambda = \{3\}$, $k = 5$. In this case, since $t = 1$ and $\lambda(1) = \{\ \{2\}\ \}$, we have

$$A^5(3) = \sum_{\substack{\nu\in\lambda(1)\\ j<5}}{}' (A^j(\nu),\overbrace{0,\cdots,0}^{4-j},1)$$

$$= \sum_{j<5}{}' (A^j(2),0,\cdots,0,1)$$

$$= (1,1,0,0,1) + (1,0,1,0,1) + (0,1,1,0,1)$$

$$+ (1,0,0,1,1) + (0,1,0,1,1) + (0,0,1,1,1).$$

(In the last modification, we used the data

$$A^2(2) = (1,1), \quad A^3(2) = (1,0,1) + (0,1,1),$$

$$A^4(2) = (1,0,0,1) + (0,1,0,1) + (0,0,1,1),$$

which can be obtained in view of the table stated before.)

Next, as another example, we consider $A^3(21)$. In this case, since $\lambda(1) = \{ \{2\}, \{1^2\} \}$ and $\lambda(2) = \{ \{1\} \}$, we have

$$A^3(21) = \sum_{\substack{\nu\in\lambda(1)\\ j<3}}{}' (A^j(\nu),\overbrace{0,\cdots,0}^{2-j},1) + \sum_{\substack{\nu\in\lambda(2)\\ j<3}}{}' (A^j(\nu),\overbrace{0,\cdots,0}^{2-j},2)$$

$$= \sum_{j<3}{}' \{(A^j(2),0,\cdots,0,1) + (A^j(1^2),0,\cdots,0,1)\}$$

$$+ \sum_{j<3}{}' (A^j(1),0,\cdots,0,2)$$

$$= (2,0,1) + (1,1,1) + (1,1,1) + (0,2,1)$$

$$+ (1,0,2) + (0,1,2).$$

(Note that $A^2(2) = (1,1)$, $A^1(1^2) = (2)$, $A^2(1^2) = (1,1) + (0,2)$, $A^1(1) = (1)$, $A^2(1) = (0,1)$.) We advise to the readers that they check the property in CONJECTURE 2.4 for other cases.

Finally, to complete the inductive process, we must give the initial data. We put

$$A^k(1) = (\overbrace{0,\cdots,0}^{k-1},1) \qquad (k \geq 1),$$

and for the empty diagram ϕ, we define $A^k(\phi)$ only in the case $k = 0$, which is equal to the empty set ϕ. (This empty initial data is needed only for the case $\lambda = \{1^n\}$. In fact, from this data, we know that the last vector in $A^k(1^n)$ is $(\overbrace{0,\cdots,0}^{k-1},n)$.) In terms of these initial data, we can theoretically calculate the last term of $\{\lambda\}\otimes[p]$ in the inductive way for any partition λ. In § 5, we exhibit many examples based on this formula. The last term in § 5 with the symbol $\square$ implies that it is calculated by the procedure stated in this section. (Of course, the calculations will become complicated as $|\lambda|$ and p become large. In actual calculations in § 5, we used computers.)

3. Conjecture on the last term (part II)

In this section, we give the second type conjecture on the last term of $\{\lambda\}\otimes\{\mu\}$, stating that it can be essentially determined by the latter part of λ if μ satisfies some inequality which depends on the decomposition of λ. In particular, from this conjecture, we may say that there exist "purely fundamental" plethysms among fundamental plethysms $\{\lambda\}\otimes[p]$, by which the last terms of all plethysms can be calculated. In addition, combining with the conjecture on the last term of $\{m\}\otimes[p]$ $(p \leq m+1)$, we can obtain interesting closed formula on the last term of $\{\lambda\}\otimes\{\mu\}$ in case μ satisfies some inequality (CONJECTURE 3.4).

First, for two partitions λ and μ, we put

$$\mu_{[\lambda]} = \begin{cases} {}^t\mu & \text{if } |\lambda| = \text{even} \\ \mu & \text{if } |\lambda| = \text{odd} \end{cases} .$$

(Note that in the case $\mu = \{p\}$, we have $\mu_{[\lambda]} = [p]$.) Then, we have the following conjecture.

CONJECTURE 3.1. We express λ and $\mu_{[\lambda]}$ as

$$\lambda = \{a^x, \cdots, b^y, b^z, c^w, \cdots, d^u\}$$
$$(a > \cdots > b > c > \cdots > d > 0, \quad y \geq 1, \quad z \geq 0)$$

and

$$\mu_{[\lambda]} = \{e, f, \cdots\} \qquad (e \geq f \geq \cdots > 0),$$

respectively. We decompose λ into two parts $\lambda = \{\lambda_1, \lambda_2\}$, where

$$\lambda_1 = \{a^x, \cdots, b^y\} \quad \text{and} \quad \lambda_2 = \{b^z, c^w, \cdots, d^u\}.$$

Then the last term of $\{\lambda\}\otimes\{\mu\}$ is equal to

$$\{\lambda_1\}^{|\mu|} + \text{last term of } \{\lambda_2\}\otimes\{(\mu_{[\lambda]})_{[\lambda_2]}\}$$

$$= \{a^{|\mu|x}, \cdots, b^{|\mu|y}\} + \text{last term of } \{\lambda_2\}\otimes \begin{cases} \{\mu\} & \text{if } |\lambda_1| = \text{even} \\ \{{}^t\mu\} & \text{if } |\lambda_1| = \text{odd} \end{cases}$$

if and only if μ satisfies the inequality

$$e \leq \dim \{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b}.$$

(In the above addition, we must rearrange the numbers from large to small if necessary. In addition, we put $\dim \{{}^t\phi\}_b = 1$ for the empty partition ϕ. Remark that the partition $\{c^w, \cdots, d^u\}$ coincides with λ_2 if and only if the last number of λ_1 and the first number of λ_2 are different. In this case, we have clearly $\binom{b+z}{b} = 1$.)

Example. (1) $\{21^2\}\otimes\{21^2\}$. We decompose $\lambda = \{21^2\}$ as $\lambda_1 = \{2\}$ and $\lambda_2 = \{1^2\}$. Then, we have $\dim\{{}^t\{1^2\}\}_2 \times \binom{2+0}{2} = \dim\{2\}_2 = 3$, and the partition $\mu_{[\lambda]} = \{31\}$ satisfies the inequality $e \le 3$. Hence, from the above conjecture, the last term of $\{21^2\}\otimes\{21^2\}$ is equal to

$$\begin{aligned} &\{2^{|\mu|}\} + \text{last term of } \{1^2\}\otimes\{21^2\} \\ &= \{2^4\} + \{2^3 1^2\} = \{2^7 1^2\}. \end{aligned}$$

(2) $\{21^3\}\otimes\{32\}$. In this case, we decompose $\lambda = \{21^3\}$ as $\lambda_1 = \{21\}$ and $\lambda_2 = \{1^2\}$. Then, we have $\dim\{{}^t\phi\}_1 \times \binom{1+2}{1} = 3$, and the partition $\mu_{[\lambda]} = \{32\}$ satisfies the inequality $e \le 3$. Hence, the last term of $\{21^3\}\otimes\{32\}$ is equal to

$$\begin{aligned} &\{2^{|\mu|}1^{|\mu|}\} + \text{last term of } \{1^2\}\otimes\{2^2 1\} \\ &= \{2^5 1^5\} + \{2^4 1^2\} = \{2^9 1^7\}. \end{aligned}$$

In view of the table in [5], we know that the above conjecture actually holds for any decompositions of λ up to total degree ≤ 18. And up to total degree ≤ 27, the conjectural results obtained by the formulas in § 2 and § 4 all satisfy CONJECTURE 3.1. (We checked this result by computers.)

As a special case of CONJECTURE 3.1, we have

CONJECTURE 3.2. Under the same notations as above, the last term of the fundamental plethysm $\{\lambda\}\otimes[p]$ is equal to

$$\{a^{px},\cdots,b^{py}\} + \text{last term of } \{\lambda_2\}\otimes[p]$$

if and only if $p \le \dim\{{}^t\{c^w,\cdots,d^u\}\}_b \times \binom{b+z}{b}$.

(We remark that two $[p]$'s in $\{\lambda\}\otimes[p]$ and $\{\lambda_2\}\otimes[p]$ may be different according as the parity of $|\lambda|$ and $|\lambda_2|$.)

For example, the last term of $\{521\}\otimes[p] = \{521\}\otimes\{1^p\}$ is equal to

$$\{5^p\} + \text{last term of } \{21\}\otimes[p] \ (= \{21\}\otimes\{p\})$$

if and only if $p \le \dim\{21\}_5 = 40$.

As another example, we consider the case $\{42^2 1\}\otimes[3] = \{42^2 1\}\otimes\{3\}$. There exist three decompositions of $\lambda = \{42^2 1\}$. For each case, we have

$$\lambda_1 = \{4\}, \quad \lambda_2 = \{2^2 1\}, \quad \dim\{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b} = 60,$$

$$\lambda_1 = \{42\}, \quad \lambda_2 = \{21\}, \quad \dim\{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b} = 6,$$

$$\lambda_1 = \{42^2\}, \quad \lambda_2 = \{1\}, \quad \dim\{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b} = 2,$$

and hence only first two decompositions satisfy the inequality $p \le \dim\{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b}$. For these two cases, we have

$$\{a^{px}, \cdots, b^{py}\} + \text{last term of } \{\lambda_2\}\otimes[p] = \{4^3 3 2^5 1^2\},$$

while the third decomposition gives

$$\{a^{px}, \cdots, b^{py}\} + \text{last term of } \{\lambda_2\}\otimes[p] = \{4^3 3 2^6\},$$

which is not a correct last term. (See the table in § 5.)

There exist many decompositions of λ into two parts $\{\lambda_1, \lambda_2\}$. Using the above notations, we define the integer p_λ by

$$p_\lambda = \max \dim\{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b},$$

where $\{\lambda_1, \lambda_2\}$ runs all over the decompositions of λ, and we say that $\{\lambda\}\otimes[p]$ is "purely fundamental" if $p > p_\lambda$. Clearly, from the above conjecture, the determination of the last term of fundamental plethysm $\{\lambda\}\otimes[p]$ satisfying $p \le p_\lambda$ can be reduced to that of purely fundamental case with smaller $|\lambda|$. (In the table in § 5, purely fundamental plethysms are marked by the symbol **.)

Concerning the explicit value of p_λ, we have the following conjecture.

CONJECTURE 3.3. We express λ as

$$\lambda = \{a^x, b^y, \cdots, c^z\} \quad (a > b > \cdots > c > 0).$$

Then we have

$$p_\lambda = \dim \{{}^t\{b^y, \cdots, c^z\}\}_a \times \binom{a+x-1}{a} .$$

For example, we have $p_{\{m\}} = 1$, $p_{\{1^m\}} = m$, $p_{\{21^{m-2}\}} = m-1$, etc. We can directly check that for partitions λ satisfying $|\lambda| \leq 15$, CONJECTURE 3.3 actually holds. The maximum value is attained by the decomposition $\lambda_1 = \{a\}$ and $\lambda_2 = \{a^{x-1}, b^y, \cdots, c^z\}$. (In fact, considered as a function of the depth of λ_1, it seems that the above value $\dim \{{}^t\{c^w, \cdots, d^u\}\}_b \times \binom{b+z}{b}$ is a strictly decreasing function.)

Now, in view of the table in [5], the last term of $\{m\}\otimes[p]$ seems to be

$$\{(m+1)^{p-1}, m-p+1\}$$

if $p \leq m+1$. If this conjecture is correct, then combining with CONJECTURE 3.1, we can easily show that the last term of $\{a^x, b^y, \cdots, c^z, d^w\}\otimes[p]$ is equal to

$$\begin{aligned} &\{a^{px}, b^{py}, \cdots, c^{pz}, d^{p(w-1)}\} + \text{last term of } \{d\}\otimes[p] \\ &= \{a^{px}, b^{py}, \cdots, c^{pz}, d^{p(w-1)}\} + \{(d+1)^{p-1}, d-p+1\} \end{aligned}$$

$$= \{a^{px}, b^{py}, \cdots\cdots, c^{pz}, (d+1)^{p-1}, d^{p(w-1)}, d-p+1\}$$

if $p \leq d+1$. (We used the decomposition $\lambda_1 = \{a^x, b^y, \cdots\cdots, c^z, d^{w-1}\}$ and $\lambda_2 = \{d\}$. Note that we have $\dim \{{}^t\phi\}_d \times \binom{d+1}{d} = d+1 \geq p$ in the case $w \geq 2$, and $\dim \{{}^td\}_c = \binom{c}{d} \geq d+1 \geq p$ in the case $w = 1$.) Hence, by CONJECTURE 2.1, we have the following conjecture.

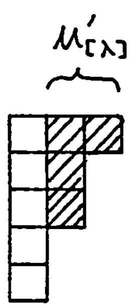
CONJECTURE 3.4 (cf. [5; p.127 ~ 128]). We express λ and $\mu_{[\lambda]}$ as

$$\lambda = \{a^x, b^y, \cdots\cdots, c^z, d^w\} \quad (a > b > \cdots > c > d > 0),$$

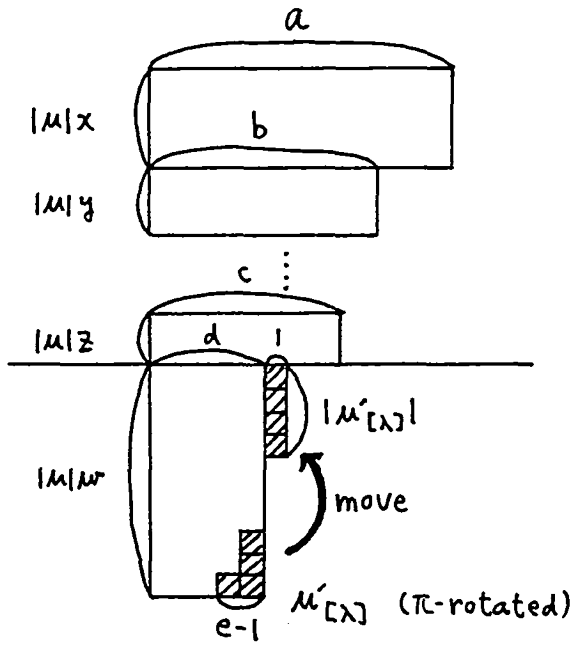
and

$$\mu_{[\lambda]} = \{e, f, \cdots\} \quad (e \geq f \geq \cdots > 0).$$

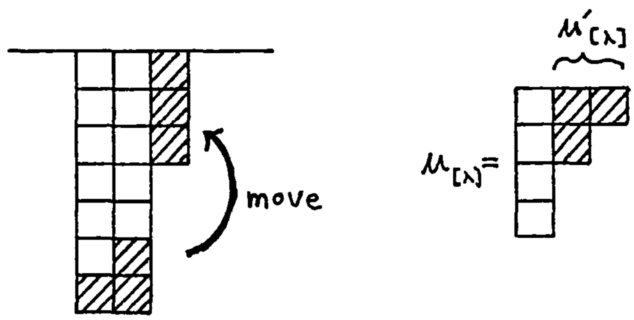
In addition, let $\mu'_{[\lambda]}$ be the partition obtained by deleting the left column of $\mu_{[\lambda]}$.

$\mu_{[\lambda]} =$ 

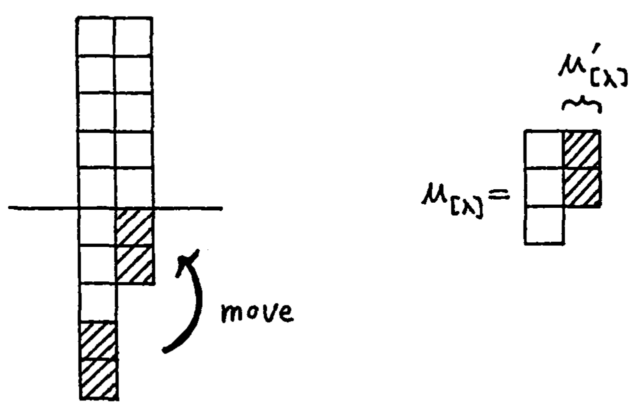
Under these notations, if μ satisfies the inequality $e \leq d+1$, the last term of $\{\lambda\}\otimes\{\mu\}$ is given by the following figure. Namely, in the diagram $\{a^{|\mu|x}, b^{|\mu|y}, \cdots\cdots, c^{|\mu|z}, d^{|\mu|w}\}$, we move the π-rotated diagram of $\mu'_{[\lambda]}$ in the right bottom to the right column in the shape of $\{1^{|\mu'_{[\lambda]}|}\}$.



Example. In the case $\{2\}\otimes\{421\}$, we have $\mu_{[\lambda]} = \{321^2\}$ and $\mu'_{[\lambda]} = \{21\}$ (d = 2, e = 3).



Hence the last term is equal to $\{3^3 2^2 1\}$. In the case $\{21\}\otimes\{2^2 1\}$, we have $\mu_{[\lambda]} = \{2^2 1\}$ and $\mu'_{[\lambda]} = \{1^2\}$ (d = 1, e = 2).



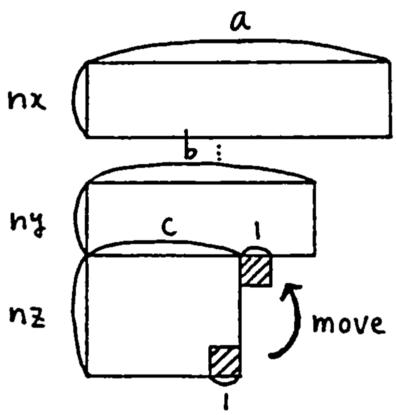
Hence the last term is equal to $\{2^7 1\}$.

This conjecture is very useful because many plethysms satisfy the condition "$e \leq d+1$". In the table in § 5, we marked the symbol $\boxdot$ on the last term of $\{\lambda\}\otimes\{\mu\}$ which satisfies this condition $e \leq d+1$. For these plethysms, this conjecture actually holds up to total degree ≤ 18, and coincides with the results obtained by other conjectures up to total degree ≤ 27.

Applying CONJECTURE 3.4 to special types of μ, we have several formulas on the last term of $\{\lambda\}\otimes\{\mu\}$. In the following, we list up some of them. (They are already stated in [5; p.151 ~ 153].)

CONJECTURE 3.5. Assume λ is expressed in the form $\{a^x, \cdots, b^y, c^z\}$ $(a > \cdots > b > c > 0)$. Then the last terms of $\{\lambda\}\otimes\{n-1,1\}$ $(|\lambda| = \text{even})$ and $\{\lambda\}\otimes\{21^{n-2}\}$ $(|\lambda| = \text{odd})$ are

$$\{a^{nx}, \cdots, b^{ny}, c+1, c^{nz-2}, c-1\}.$$



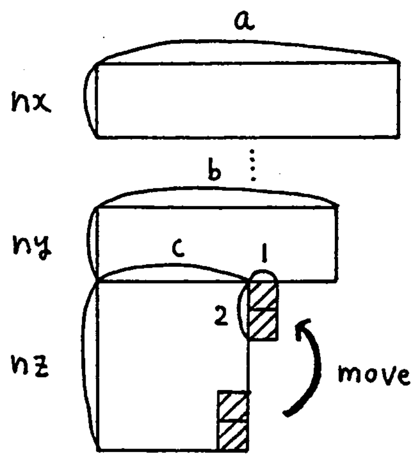
In particular, the last term of $\{\lambda\}\otimes\{21\}$ is

$$\{a^{3x}, \cdots, b^{3y}, c+1, c^{3z-2}, c-1\}$$

for any λ.

CONJECTURE 3.6. Under the same notations as above, the last terms of $\{\lambda\}\otimes\{n-2,2\}$ $(|\lambda| = \text{even})$ and $\{\lambda\}\otimes\{2^2 1^{n-4}\}$ $(|\lambda| = \text{odd})$ are given by

$$\{a^{nx}, \ldots, b^{ny}, (c+1)^2, c^{nz-4}, (c-1)^2\}.$$



In particular, the last term of $\{\lambda\}\otimes\{2^2\}$ is

$$\{a^{4x}, \ldots, b^{4y}, (c+1)^2, c^{4z-4}, (c-1)^2\}$$

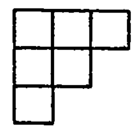
for any λ.

4. Conjecture on the last term (part III)

In this section, as the third type conjecture, we give the explicit formula of the last term of fundamental plethysms $\{\lambda\}\otimes[p]$. The final formula is summarized in CONJECTURE 4.1, by which we can calculate the last term directly (i.e., without the inductive argument as in § 2). But instead, it requires some complicated combinatorial calculations associated with the Young diagram λ. As special cases, we have relatively simple closed formulas on the last terms of $\{m\}\otimes[p]$ and $\{1^m\}\otimes[p]$, which are expressed in terms of the expansion of p by binomial coefficients (CONJECTURE 4.2).

First, we prepare some notations. We remind that $\lambda(i)$ ($i = 0 \sim$ depth of λ) is the set of Young diagrams consisting $|\lambda|-i$ boxes that are obtained by deleting at most one box from each row of λ. We extend this concept in the following way. We denote by $\lambda(i,j)$ the set of Young diagrams consisting of elements of $\mu(j)$, where μ runs all over the diagrams of $\lambda(i)$, counting with multiplicity. For example, in the case $\lambda = \{321\}$, we have

$$\lambda(1) = \{ \{2^2 1\}, \{31^2\}, \{32\} \}.$$

$\lambda =$ 

Hence, we have

$$\begin{aligned}\lambda(1,1) &= \{2^2 1\}(1) \cup \{31^2\}(1) \cup \{32\}(1) \\ &= \{ \{21^2\}, \{2^2\} \} \cup \{ \{21^2\}, \{31\} \} \cup \{ \{2^2\}, \{31\} \} \\ &= \{ 2\{21^2\}, 2\{2^2\}, 2\{31\} \}.\end{aligned}$$

In the same way, we have

$$\begin{aligned}\lambda(1,2) &= \{2^2 1\}(2) \cup \{31^2\}(2) \cup \{32\}(2) \\ &= \{ \{1^3\}, \{21\} \} \cup \{ \{21\}, \{3\} \} \cup \{ \{21\} \} \\ &= \{ \{1^3\}, 3\{21\}, \{3\} \}.\end{aligned}$$

Next, we define the set of Young diagrams $\lambda(\ell_1,\cdots,\ell_s)$ inductively by

$$\lambda(\ell_1,\cdots,\ell_s) = \lambda((\ell_1,\cdots,\ell_{s-1}),\ell_s).$$

Since two sets $\lambda(i,j)$ and $\lambda(j,i)$ just coincides, the set $\lambda(\ell_1,\cdots,\ell_s)$ does not depend on the order of $\ell_1 \sim \ell_s$. In the following, as an example, we give the list of $\lambda(\ell_1,\cdots,\ell_s)$ for the Young diagram $\lambda = \{321\}$:

$\lambda(1) = \{ \{2^2 1\}, \{31^2\}, \{32\} \}$,

$\lambda(2) = \{ \{21^2\}, \{2^2\}, \{31\} \}$, $\lambda(1,1) = \{ 2\{21^2\}, 2\{2^2\}, 2\{31\} \}$,

$\lambda(3) = \{ \{21\} \}$, $\lambda(2,1) = \{ \{1^3\}, 3\{21\}, \{3\} \}$,

$\lambda(1,1,1) = \{ 2\{1^3\}, 6\{21\}, 2\{3\} \}$,

$\lambda(3,1) = \{ \{1^2\}, \{2\} \}$, $\lambda(2,2) = \{ 2\{1^2\}, 2\{2\} \}$,

$\lambda(2,1,1) = \{ 4\{1^2\}, 4\{2\} \}$, $\lambda(1,1,1,1) = \{ 8\{1^2\}, 8\{2\} \}$,

$\lambda(3,2) = \{ \{1\} \}$, $\lambda(3,1,1) = \{ 2\{1\} \}$,

$\lambda(2,2,1) = \{ 4\{1\} \}$, $\lambda(2,1,1,1) = \{ 8\{1\} \}$,

$\lambda(1,1,1,1,1) = \{ 16\{1\} \}$.

Next, we must express the step number p in $\{\lambda\}\otimes[p]$ as a sum of integers in the following form:

$$p = f_1(k_1) + f_2(k_2) + \cdots + f_{|\lambda|}(k_{|\lambda|}) + r$$

$(k_1 \geq k_2 \geq \cdots \geq k_{|\lambda|} \geq 0)$. Here, f_i is a polynomial with degree $|\lambda|-i+1$ determined by the combinatorial property of the set of integers $\{k_1,\cdots,k_{i-1}\}$ as follows.

First, we define the polynomial f_1 by

$$f_1(k) = \dim \{{}^t\lambda\}_k ,$$

where $\{\mu\}_k$ denotes the representation space of $GL(k,\mathbf{C})$ corresponding to the partition μ (cf. § 2). Then, there exists the integer k_1 satisfying the inequality

$$f_1(k_1) \leq p < f_1(k_1+1) .$$

Next, we define the polynomial f_2 by

$$f_2(k) = \sum_{\nu\in\lambda(1)} \dim \{{}^t\nu\}_k .$$

Then, as above, there exists the integer k_2 satisfying

$$f_2(k_2) \le p - f_1(k_1) < f_2(k_2+1) \ .$$

In the same way, we define the polynomial f_3 by

$$f_3(k) = \sum_{\nu} \dim \{{}^t\nu\}_k \ ,$$

where ν runs all over the Young diagrams contained in the set

$$\begin{cases} \lambda(1,1) & \text{in the case } k_1 \ne k_2 \\ \lambda(2) & \text{in the case } k_1 = k_2 \end{cases} .$$

In terms of this polynomial, we define the integer k_3 by

$$f_3(k_3) \le p - f_1(k_1) - f_2(k_2) < f_3(k_3+1) \ .$$

We repeat this procedure. At the i-th step ($i \le |\lambda|$), we put

$$f_i(k) = \sum_{\nu} \dim \{{}^t\nu\}_k \ .$$

Here, if the set of integers $\{k_1, \cdots, k_{i-1}\}$ is expressed in the form

$$\{\overbrace{c_1,\cdots,c_1}^{\ell_1},\overbrace{c_2,\cdots,c_2}^{\ell_2},\cdots\cdots,\overbrace{c_s,\cdots,c_s}^{\ell_s}\} \quad (c_p \ne c_q,\ \ell_1+\cdots+\ell_s=i-1),$$

then the diagram ν appearing in the right hand summation runs all over the elements of $\lambda(\ell_1,\cdots,\ell_s)$. (If the set $\lambda(\ell_1,\cdots,\ell_s)$ is empty, we put $f_i(k) = 0$.) Finally, we put

$$r = p - f_1(k_1) - \cdots - f_{|\lambda|}(k_{|\lambda|}) \ .$$

Note that the last polynomial $f_{|\lambda|}$ is of degree 1, and is expressed as $f_{|\lambda|}(k) = qk$ for some integer q because $\dim \{1\}_k = k$. Hence, we have

$$r \equiv p - f_1(k_1) - \cdots - f_{|\lambda|-1}(k_{|\lambda|-1}) \pmod q,$$

and the integer r satisfies the inequality $0 \le r < q$.

Under these notations, we have the following conjecture, giving the explicit formula of the last term. Perhaps, this is the most fundamental conjecture in this paper.

CONJECTURE 4.1. For $1 \le i \le |\lambda|$, we put

$$a_i = f_{i+1}(k_{i+1}) + \cdots + f_{|\lambda|}(k_{|\lambda|}) + r ,$$
$$b_i = (|\lambda|-i+1)f_i(k_i)/k_i .$$

(We consider $b_i = 0$ in the case $f_i(k_i) = k_i = 0$.) Then, the last term of the fundamental plethysm $\{\lambda\}\otimes[p]$ is given by

$$\{(k_1+1)^{a_1},\ k_1^{b_1-a_1},\ (k_2+1)^{a_2},\ k_2^{b_2-a_2},\dots,\ (k_{|\lambda|}+1)^{r},\ k_{|\lambda|}^{q-r}\},$$

where $f_{|\lambda|}(k) = qk$. (Hence, we have $a_{|\lambda|} = r$ and $b_{|\lambda|} = q$.) In this expression, we must rearrange the order of numbers from large to small if $k_i = k_j$ for some i and j. In addition, if the exponent is negative, it must be canceled with another term having the same base. (See the example below.)

Remark. In this expression, if $p = f_1(k_1) + \cdots + f_i(k_i)$ for some i, then we have $f_{i+1}(k_{i+1}) = \cdots = f_{|\lambda|}(k_{|\lambda|}) = r = 0$, and

$$a_i = a_{i+1} = \cdots = b_{i+1} = b_{i+2} = \cdots = 0.$$

(We consider $b_{i+1} = 0$ even if $k_{i+1} = 0$ etc.) Hence, the last term is equal to

$$\{(k_1+1)^{a_1},\ k_1^{b_1-a_1},\dots,\ (k_{i-1}+1)^{a_{i-1}},\ k_{i-1}^{b_{i-1}-a_{i-1}},\ k_i^{b_i}\}$$

in this case. In particular, if $p = p_k$ (= dim $\{^t\lambda\}_k$), then we have $k_1 = k$, $f_2(k_2) = \cdots = f_{|\lambda|}(k_{|\lambda|}) = r = 0$, and hence the last term is equal to $\{k^{b_1}\} = \{k^{|\lambda|p_k/k}\}$, which is nothing but

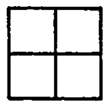
CONJECTURE 2.2.

Example. $\lambda = \{2^2\}$. In this case, we have

$\lambda(1) = \{ \{21\} \}$,

$\lambda(2) = \{ \{1^2\} \}$, $\qquad \lambda(1,1) = \{ \{2\}, \{1^2\} \}$,

$\lambda(1,1,1) = \{ 2\{1\} \}$, $\qquad \lambda(2,1) = \{ \{1\} \}$.

$\lambda =$ 

Hence, by putting $k = k_1$, $\ell = k_2$, $m = k_3$, $n = k_4$, we have

$$f_1(k) = \dim \{2^2\}_k = 1/12 \cdot k^2(k-1)(k+1),$$

$$f_2(\ell) = \dim \{21\}_\ell = 1/3 \cdot \ell(\ell-1)(\ell+1),$$

$$f_3(m) = \begin{cases} \dim \{1^2\}_m + \dim \{2\}_m = m^2 & (k \neq \ell) \\ \dim \{2\}_m = 1/2 \cdot m(m+1) & (k = \ell) \end{cases},$$

$$f_4(n) = \begin{cases} 2n & (k,\ \ell,\ m \text{ are mutually distinct}) \\ n & (\text{just two of } k,\ \ell,\ m \text{ coincide}) \\ 0 & (k = \ell = m) \end{cases}.$$

(Actually, in the last polynomial $f_4(n)$, the case $k = \ell = m$ does not occur, as the following table shows.) In terms of these polynomials, the step number p (≤ 50) is decomposed as a sum $p = f_1(k) + f_2(\ell) + f_3(m) + f_4(n) + r$ in the following way:

p	k	ℓ	m	n	r	$f_1(k)$	$f_2(\ell)$	$f_3(m)$	$f_4(n)$	r
1	2	1	0	0	0	1	0	0	0	0
2	2	1	1	0	0	1	0	1	0	0
3	2	2	0	0	0	1	2	0	0	0
4	2	2	1	0	0	1	2	1	0	0

p	k	ℓ	m	n	r	$f_1(k)$	$f_2(\ell)$	$f_3(m)$	$f_4(n)$	r
5	2	2	1	1	0	1	2	1	1	0
6	3	1	0	0	0	6	0	0	0	0
7	3	1	1	0	0	6	0	1	0	0
8	3	2	0	0	0	6	2	0	0	0
9	3	2	1	0	0	6	2	1	0	0
10	3	2	1	0	1	6	2	1	0	1
11	3	2	1	1	0	6	2	1	2	0
12	3	2	2	0	0	6	2	4	0	0
13	3	2	2	1	0	6	2	4	1	0
14	3	3	0	0	0	6	8	0	0	0
15	3	3	1	0	0	6	8	1	0	0
16	3	3	1	1	0	6	8	1	1	0
17	3	3	2	0	0	6	8	3	0	0
18	3	3	2	1	0	6	8	3	1	0
19	3	3	2	2	0	6	8	3	2	0
20	4	1	0	0	0	20	0	0	0	0
21	4	1	1	0	0	20	0	1	0	0
22	4	2	0	0	0	20	2	0	0	0
23	4	2	1	0	0	20	2	1	0	0
24	4	2	1	0	1	20	2	1	0	1
25	4	2	1	1	0	20	2	1	2	0
26	4	2	2	0	0	20	2	4	0	0
27	4	2	2	1	0	20	2	4	1	0
28	4	3	0	0	0	20	8	0	0	0
29	4	3	1	0	0	20	8	1	0	0
30	4	3	1	0	1	20	8	1	0	1

p	k	ℓ	m	n	r	$f_1(k)$	$f_2(\ell)$	$f_3(m)$	$f_4(n)$	r
31	4	3	1	1	0	20	8	1	2	0
32	4	3	2	0	0	20	8	4	0	0
33	4	3	2	0	1	20	8	4	0	1
34	4	3	2	1	0	20	8	4	2	0
35	4	3	2	1	1	20	8	4	2	1
36	4	3	2	2	0	20	8	4	4	0
37	4	3	3	0	0	20	8	9	0	0
38	4	3	3	1	0	20	8	9	1	0
39	4	3	3	2	0	20	8	9	2	0
40	4	4	0	0	0	20	20	0	0	0
41	4	4	1	0	0	20	20	1	0	0
42	4	4	1	1	0	20	20	1	1	0
43	4	4	2	0	0	20	20	3	0	0
44	4	4	2	1	0	20	20	3	1	0
45	4	4	2	2	0	20	20	3	2	0
46	4	4	3	0	0	20	20	6	0	0
47	4	4	3	1	0	20	20	6	1	0
48	4	4	3	2	0	20	20	6	2	0
49	4	4	3	3	0	20	20	6	3	0
50	5	1	0	0	0	50	0	0	0	0

Using this table, we have the following conjecture on the last term of $\{2^2\}\otimes[p] = \{2^2\}\otimes\{1^p\}$. (The result is actually correct for $1 \leq p \leq 12$ in view of the table in [5].)

$p = 1 \quad \{2^2\}$	$p = 2 \quad \{32^21\}$	$p = 3 \quad \{3^22^3\}$
$p = 4 \quad \{3^421^2\}$	$p = 5 \quad \{3^61^2\}$	$p = 6 \quad \{3^8\}$
$p = 7 \quad \{43^721\}$	$p = 8 \quad \{4^23^62^3\}$	$p = 9 \quad \{4^33^62^21^2\}$
$p = 10 \quad \{4^43^62^21^2\}$	$p = 11 \quad \{4^53^62^21^2\}$	$p = 12 \quad \{4^63^62^3\}$
$p = 13 \quad \{4^73^721\}$	$p = 14 \quad \{4^83^8\}$	$p = 15 \quad \{4^{10}3^61^2\}$
$p = 16 \quad \{4^{12}3^421^2\}$	$p = 17 \quad \{4^{14}3^22^3\}$	$p = 18 \quad \{4^{16}32^21\}$
$p = 19 \quad \{4^{18}2^2\}$	$p = 20 \quad \{4^{20}\}$	$p = 21 \quad \{54^{19}21\}$
$p = 22 \quad \{5^24^{18}2^3\}$	$p = 23 \quad \{5^34^{17}32^21^2\}$	$p = 24 \quad \{5^44^{16}3^22^21^2\}$
$p = 25 \quad \{5^54^{15}3^32^21^2\}$	$p = 26 \quad \{5^64^{14}3^42^3\}$	$p = 27 \quad \{5^74^{13}3^621\}$
$p = 28 \quad \{5^84^{12}3^8\}$	$p = 29 \quad \{5^94^{12}3^71^2\}$	$p = 30 \quad \{5^{10}4^{12}3^621^2\}$
$p = 31 \quad \{5^{11}4^{12}3^52^21^2\}$	$p = 32 \quad \{5^{12}4^{12}3^42^4\}$	$p = 33 \quad \{5^{13}4^{12}3^42^31\}$
$p = 34 \quad \{5^{14}4^{12}3^42^21^2\}$	$p = 35 \quad \{5^{15}4^{12}3^42^21\}$	$p = 36 \quad \{5^{16}4^{12}3^42^2\}$
$p = 37 \quad \{5^{17}4^{12}3^5\}$	$p = 38 \quad \{5^{18}4^{13}3^31\}$	$p = 39 \quad \{5^{19}4^{14}32\}$
$p = 40 \quad \{5^{20}4^{15}\}$	$p = 41 \quad \{5^{22}4^{13}1^2\}$	$p = 42 \quad \{5^{24}4^{11}21^2\}$
$p = 43 \quad \{5^{26}4^92^3\}$	$p = 44 \quad \{5^{28}4^732^21\}$	$p = 45 \quad \{5^{30}4^53^22^2\}$
$p = 46 \quad \{5^{32}4^33^4\}$	$p = 47 \quad \{5^{34}4^23^31\}$	$p = 48 \quad \{5^{36}43^22\}$
$p = 49 \quad \{5^{38}3^2\}$	$p = 50 \quad \{5^{40}\}$.	

For example, in the case $p = 4$, the formula in CONJECTURE 4.1 formally gives the result $\{3^32^{-1}32^21^2\}$, and we consider that it is equal to $\{3^421^2\}$ by the rule stated at the end of CONJECTURE 4.1.

The above algorithm uniquely determines the values k_i and r for each p. But in general, the decomposition of p of the form

$$p = f_1(k_1) + \cdots + f_{|\lambda|}(k_{|\lambda|}) + r$$

is not uniquely determined under the conditions

$$k_1 \geq \cdots \geq k_{|\lambda|} \geq 0, \quad q > r \geq 0 \quad (f_{|\lambda|}(k) = qk)$$

only. For example, in the case $\lambda = \{2^2\}$ and $p = 3$, besides the above, we have another expression $(k, \ell, m, n, r) = (2,1,1,1,0)$. In this case, $(f_1(k), f_2(\ell), f_3(m), f_4(n)) = (1,0,1,1)$, and substituting these values to the formula in CONJECTURE 4.1, we obtain $\{3^2 2^2 1^{-2} 211\} = \{3^2 2^3\}$, which gives the same result as above.

As other typical examples, we consider the cases $\lambda = \{m\}$ and $\{1^m\}$. If $\lambda = \{m\}$, then we have

$$\lambda(\overbrace{1,\cdots,1}^{i}) = \{\ \{m-i\}\ \} \quad \text{and} \quad f_i(k) = \binom{k}{m-i+1}.$$

Similarly, if $\lambda = \{1^m\}$, we have

$$\lambda(\ell_1,\cdots,\ell_s) = \{\ \{1^{m-\ell_1-\cdots-\ell_s}\}\ \} \quad \text{and} \quad f_i(k) = \binom{k+m-i}{m-i+1}.$$

For both cases, we have $r = 0$ because $f_m(k) = k$. Hence, from CONJECTURE 4.1, we have

CONJECTURE 4.2. (1) $\{m\}\otimes[p]$. We express p in the form

$$p = \binom{p_1}{m} + \binom{p_2}{m-1} + \cdots + \binom{p_{m-1}}{2} + \binom{p_m}{1},$$

where $p_1 > p_2 > \cdots > p_m \geq 0$. Then, the last term of $\{m\}\otimes[p]$ is equal to

$$\{(p_1+1)^{a_1},\ p_1^{\,b_1-a_1},\ (p_2+1)^{a_2},\ p_2^{\,b_2-a_2},\ \cdots\cdots,$$
$$(p_{m-1}+1)^{a_{m-1}},\ p_{m-1}^{\,b_{m-1}-a_{m-1}},\ p_m\},$$

where $a_i = \binom{p_{i+1}}{m-i} + \cdots + \binom{p_m}{1}$, and $b_i = \binom{p_i - 1}{m-i}$.

(2) $\{1^m\}\otimes[p]$. We express p in the form

$$p = \binom{p_1+m-1}{m} + \binom{p_2+m-2}{m-1} + \cdots + \binom{p_{m-1}+1}{2} + \binom{p_m}{1},$$

where $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. Then, the last term of $\{1^m\}\otimes[p]$ is equal to

$$\{(p_1+1)^{a_1},\ p_1^{b_1-a_1},\ (p_2+1)^{a_2},\ p_2^{b_2-a_2},\ \ldots\ldots,$$
$$(p_{m-1}+1)^{a_{m-1}},\ p_{m-1}^{b_{m-1}-a_{m-1}},\ p_m\},$$

where $a_i = \binom{p_{i+1}+m-i-1}{m-i} + \cdots + \binom{p_m}{1}$, and $b_i = \binom{p_i+m-i}{m-i}$.

(As above, we must rearrange the order of numbers in this formula if necessary.)

For example, in the case $\lambda = \{1^4\}$ and $p = 100$, we have

$$100 = \binom{5+3}{4} + \binom{4+2}{3} + \binom{4+1}{2} + \binom{0}{1},$$

and from this expression, we have

$$(p_1, p_2, p_3, p_4) = (5, 4, 4, 0),$$
$$(a_1, a_2, a_3) = (30, 10, 0),$$
$$(b_1, b_2, b_3) = (56, 15, 5).$$

Hence, from the above conjecture (2), the last term of $\{1^4\}\otimes[100]$ $= \{1^4\}\otimes\{1^{100}\}$ is equal to

$$\{6^{30}5^{26}5^{10}4^5 5^0 4^5 0\} = \{6^{30}5^{36}4^{10}\}.$$

(But, unfortunately we cannot check it by actual calculations of $\{1^4\}\otimes\{1^{100}\}$ on the lack of the memory space of computers.)

Note that in the above conjecture, the decomposition of p is uniquely determined, which is the famous expansion of positive integers by binomial coefficients, frequently appeared in the field of combinatorics. (For example, see [37; p.55] etc.) Hence, our decomposition $p = f_1(k_1) + \cdots + f_{|\lambda|}(k_{|\lambda|}) + r$ is a natural generalization of this expansion, corresponding to the given Young diagram λ.

We checked that the formula in CONJECTURE 4.1 actually holds for fundamental plethysms $\{\lambda\}\otimes[p]$ with total degree ≤ 18 by using computers. In addition, up to total degree ≤ 27, it gives the same conjectural results listed up in § 5, which we calculated by using the conjectures in § 2 and § 3. In the final section (§ 6), we exhibit a table of the first and the last terms of fundamental plethysms $\{\lambda\}\otimes[p]$ that are obtained by applying the formulas in CONJECTURES 1.2 and 4.1. The author hopes that the readers will check these conjectures by their own methods (or computers), if it is possible.

Appendix

After I wrote this note, I found the following interesting paper which gives a range of Young diagrams appearing in $\{\lambda\}\otimes\{\mu\}$.

M.Yang, An algorithm for computing plethysm coefficients, Discrete Math. **180** (1998), 391-402.

The main results of this paper may be stated as follows (cf. p.395, Theorem 3.1):

Theorem. Assume $|\gamma| = |\lambda||\mu|$, and let $c(\gamma)$ (resp. $r(\gamma)$) be the depth (resp. width) of γ. If the partition $\{\gamma\}$ appears in the plethysm $\{\lambda\}\otimes\{\mu\}$, then we have

$$r(\gamma) \le |\mu| r(\lambda) \quad \text{and} \quad c(\gamma) \le |\mu| c(\lambda).$$

If our CONJECTURE 1.2 (p.6) is correct, then combined with the conjugate formula (cf. [19; p.220], [26, p.136]) we can improve the above results to the following form.

CONJECTURE A. Under the same notations as in the above theorem, we have

$$r(\gamma) \le \begin{cases} |\mu| r(\lambda) & c(\lambda) \ge 2 \\ |\mu|(r(\lambda) - 1) + r(\mu) & c(\lambda) = 1 \end{cases}$$

$$c(\gamma) \le \begin{cases} |\mu| c(\lambda) & r(\lambda) \ge 2 \\ |\mu|(c(\lambda) - 1) + \begin{cases} r(\mu) & |\lambda| = \text{even} \\ c(\mu) & |\lambda| = \text{odd} \end{cases} & r(\lambda) = 1 . \end{cases}$$

In addition, there exists a partition $\{\gamma\}$ in $\{\lambda\}\otimes\{\mu\}$ satisfying the above equalities.

Note that the first term of $\{\lambda\}\otimes\{\mu\}$ gives the maximum of $r(\gamma)$, but the last term does not in general give the maximum of $c(\gamma)$, and we use the conjugate formula to obtain the above estimate.

Example. For the plethysm $\{1^3\}\otimes\{n\}$, we have

$$r(\gamma) \leq n \quad \text{and} \quad c(\gamma) \leq 2n+1.$$

For the case $\{1^4\}\otimes\{n\}$, we have

$$r(\gamma) \leq n \quad \text{and} \quad c(\gamma) \leq 4n.$$

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